On the Distribution of the Largest Characteristic Root of a Matrix in Multivariate Analysis

by

K. C. Sreedharan Pillai

Purdue University

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 22

September 1964

On the Distribution of the Largest Characteristic Root of a Matrix in Multivariate Analysis*

by

K. C. Sreedharan Pillai

Purdue University

1. Introduction

The cumulative distribution function of the largest characteristic root of a matrix in multivariate analysis has been studied by Pillai (1954, 1956a, 1957, 1960 and 1964) with a view to obtaining an approximation to this c.d.f. useful for computing the upper percentage points. The approach, so far, has been to approximate the c.d.f. of the largest root for each value of the number, s, of non-null roots from two to seven. In this paper, general expressions are given for the first time, approximating at the upper end the c.d.f. of the largest of s non-null characteristic roots. Further, these expressions are used to compute the upper 5 and 1 per cent points of the largest root for $s = 8$, 9 and 10.

It may be pointed out that Roy (1945, 1953, 1957) has shown that tests of certain hypotheses in multivariate analysis and associated confidence interval estimation can be based on the extreme characteristic roots. The monotonic character of the power functions of two multivariate tests using the largest characteristic root has been demonstrated by several authors (Roy and Mikhail, 1961; Das Gupta, Anderson and Mudholkar, 1964; Anderson and Das Gupta, 1964).

2. C.D.F. of the Largest Root

The joint distribution of s non-null characteristic roots of a matrix in multivariate analysis given by Fisher (1939), Girshick (1939), Hsu (1939) and Roy (1939) can be expressed in the form

$$f(\theta_1,\theta_2,\ldots,\theta_s) = C(s,m,n) \prod_{i=1}^{s} \theta_i^m (1-\theta_i)^n \prod_{i>j} (\theta_i - \theta_j) \qquad (1)$$

$$(0 < \theta_1 \leq \ldots \leq \theta_s < 1),$$

*This research was supported by Contract NONR-1100(26) with the Office of Naval Research. Reproduction in whole or in part permitted for any purpose of the United States Govt.

where

$$C(s,m,n) = \frac{\pi^{\frac{1}{2}s} \prod_{i=1}^{s} \Gamma\left(\frac{2m+2n+s+i+2}{2}\right)}{\prod_{i=1}^{s} \Gamma\left(\frac{2m+i+1}{2}\right) \Gamma\left(\frac{2n+i+1}{2}\right) \Gamma\left(\frac{i}{2}\right)} \tag{2}$$

and the parameters m and n are defined differently for various situations as described by Pillai (1955, 1957).

It has been shown (1954, 1956b) that the c.d.f. of the largest root, θ_s, can be presented in the following determinantal form:

$$P_r(\theta_s \leq x) = C(s,m,n) \begin{vmatrix} \int_0^x \theta_s^{m+s-1}(1-\theta_s)d\theta_s \ldots & \int_0^x \theta_s^m(1-\theta_s)^n \, d\theta_s \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \int_0^{\theta_2} \theta_1^{m+s-1}(1-\theta_1)^n \, d\theta_1 \ldots & \int_0^{\theta_2} \theta_1^m(1-\theta_1)^n \, d\theta_1 \end{vmatrix} . \tag{3}$$

In order to overcome the difficulty of integrating each of the $s!$ multiple integrals in the expansion of the determinant in (3), Pillai (1954, 1956b) suggested a reduction formula and gave exact expressions for the c.d.f. of the largest root in terms of incomplete beta functions or functions of incomplete beta functions for values of s from 2 to 10. In addition, Pillai (1954, 1956a) suggested a method of approximating the c.d.f. of the largest root for computing the upper percentage points and these two methods are applied below in conjunction in order to obtain the general expression approximating at the upper end the c.d.f. of the largest of s non-null characteristic roots.

3. Approximation to the C.D.F. of the Largest Root

Now denote by $V(x;\, q_s,\, q_{s-1},\ldots,\, q_1;\, n)$ the determinant of the type on the right side of (3) with the powers of θ's in the integrands in each row from left to right being $q_s,\, q_{s-1},\ldots,\, q_1$ respectively (the q's need not necessarily differ by unity). It has been shown (1954, 1956b) that

$$V(x;\, q_s,\, q_{s-1},\ldots,q_1;\, n) = (q_s + n + 1)^{-1}(A^{(s)} + B^{(s)} + q_s C^{(s)}) \tag{4}$$

where

$$A^{(s)} = -\, I_0(x;\, q_s,\, n+1)\, V(x;\, q_{s-1},\ldots,\, q_1;\, n),$$

$$B^{(s)} = 2 \sum_{j=s-1}^{1} (-1)^{s-j-1}\, I(x;q_s + q_j,\, 2n+1)\, V(x;q_{s-1},\ldots,q_{j+1},q_{j-1},\ldots,q_1;n),$$

$$C^{(s)} = V(x;\, q_s-1,\, q_{s-1},\ldots,\, q_1;\, n),$$

$$I_0(x;\, q_s,\, n+1) = x^{q_s}\, (1-x)^{n+1},$$

and

$$I(x;\, q,\, r) = \int_0^x x_1^q\, (1-x_1)^r\, dx_1.$$

It may be noted that $C^{(s)}$ vanishes if $q_s = q_{s-1} + 1$.

Now apply (4) to the right side of (3). We get

$$P_r(\theta_s \leq x) = \left[\frac{C(s,m,n)}{m+n+s}\right]\left[-I_0(x;m+s-1,n+1)\, V(x;m+s-2,\ldots,m;n) + B_1^{(s)}\right] \tag{5}$$

where

$$B_1^{(s)} = 2 \sum_{j=s-1}^{1} (-1)^{s-j-1} I(x;2m+s+j-2,2n+1)\ V(x;m+s-2,\ldots,m+j,m+j-2,\ldots,m;n) \cdot \quad (6)$$

Now the approximation suggested by Pillai (1954, 1956a) for approximating the upper end of the c.d.f. of the largest root neglects terms involving $(1-x)^{2n+1}$ and higher powers of $(1-x)$ in the development of the c.d.f. obtained by re-repeated application of (4) to the V-function in the c.d.f. (3). Following this method it can be easily seen from (5) that the coefficient of $-I_0(x;m+s-1,n+1)$, denoted by k_{m+s-1}, is given by

$$k_{m+s-1} = \frac{C(s,m,n)}{m+n+s}\ V(0;m+s-2,\ldots,m;n) \qquad (7)$$

Note that k_{m+s-1} can also be written as

$$k_{m+s-1} = \frac{C(s,m,n)}{(m+n+s)\ C(s-1,m,n)} \ . \qquad (8)$$

Now consider $B_1^{(s)}$ term by term and apply (4) to each term separately.

$$\begin{aligned} B_1^{(s)} = 2 \Big[& I(x;2m+2s-3,2n+1)\ V(x;m+s-3,\ldots,m;n) \\ & -I(x;2m+2s-4,2n+1)\ V(x;m+s-2,m+s-4,\ldots,m;n) \\ & +I(x;2m+2s-5,2n+1)\ V(x;m+s-2,m+s-3,m+s-5,\ldots,m;n) \\ & -I(x;2m+2s-6,2n+1)\ V(x;m+s-2,m+s-3,m+s-4,m+s-6,\ldots,m;n) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +(-1)^{s-2} I(x;2m+s-1,2n+1)\ V(x;m+s-2,\ldots,m+1;n) \Big] \cdot \end{aligned} \qquad (9)$$

Now apply (4) to each term of (9). It is easy to see that the coefficient of $-I_0(x;m+s-2,2n+1)$ in (9) is given by

$$\left(\frac{2}{m+n+s-1}\right)\Big[-I(0;2m+2s-4,2n+1)\ V(0;m+s-4,\ldots,m;n)$$
$$+I(0;2m+2s-5,2n+1)\ V(0;m+s-3,m+s-5,\ldots,m;n)$$
$$-I(0;2m+2s-6,2n+1)\ V(0;m+s-3,m+s-4,m+s-6,\ldots,m;n)$$
$$+\cdots\cdots\cdots$$
$$+(-1)^{s-2}\ I(0;2m+s-1,2n+1)\ V(0;m+s-3,\ldots\ldots,m+1;n)\Big]. \qquad (10)$$

Now using (4) we can see that (10) equals

$$(m+n+s-1)^{-1}\Big[-(m+n+s)\ V(0;m+s-1,m+s-3,\ldots,m;n) + (m+s-1)\ V(0;m+s-2,\ldots,m;n)\Big] \qquad \ldots(11)$$

Thus from (5) and (11), the coefficient of $I_0(x;m+s-2,n+1)$ is given by

$$k_{m+s-2} = (m+n+s-1)^{-1}\Big[C(s,m,n)\ V(0;m+s-1,m+s-3,\ldots,m;n) - (m+s-1)k_{m+s-1}\Big]. \qquad (12)$$

Now let us obtain the coefficient of $-I_0(x;m+s-3,n+1)$. From the first term on the right side of (9) after applying (4) we get for the coefficient of $-I_0(x;m+s-3,n+1)$

$$\frac{2I(0;2m+2s-3,2n+1)}{(m+n+s-2)}\ V(0;m+s-4,\ldots,m;n). \qquad (13)$$

From the second term on the right side of (9) we get

$$\frac{-2(m+s-2)I(0;2m+2s-4,2n+1)}{(m+n+s-1)(m+n+s-2)}\ V(0;m+s-4,\ldots,m;n). \qquad (14)$$

Similarly, we obtain from the third term

$$\frac{2I(0;2m+2s-5,2n+1)}{(m+n+s-2)(m+n+s-1)} \Big[-2I(0;2m+2s-7,2n+1)\ V(0;m+s-6,\dots,m;n)$$

$$+2I(0;2m+2s-8,2n+1)\ V(0;m+s-5,m+s-7,\dots,m;n)$$

$$-\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+(-1)^{s-4}\ 2I(0;2m+s-2,2n+1)\ V(0;m+s-5,\dots\dots,m+1;n \Big]. \quad (15)$$

Now again using (4), (15) can be written as

$$\frac{2I(0;2m+2s-5,2n+1)}{(m+n+s-2)} \Big[-V(0;m+s-2,m+s-5,m+s-6,\dots,m;n)$$

$$+ \left(\frac{(m+s-2)}{m+n+s-1}\right) V(m+s-3,m+s-5,\dots\dots,m;n) \Big]. \quad (16)$$

Similarly from the fourth term the coefficient is obtained as

$$\frac{-2I(0;2m+2s-6,2n+1)}{(m+n+s-2)} \Big[-V(0;m+s-2,m+s-4,m+s-6,\dots,m;n)$$

$$+ \left(\frac{m+s-2}{m+n+s-1}\right) V(0;m+s-3,m+s-4,m+s-6,\dots,m;n) \Big]. \quad (17)$$

Proceeding in a similar manner we obtain the coefficients in the other terms. !

last term gives

$$\frac{(-1)^{s-2}2I(0;2m+s-1,2n+1)}{(m+n+s-2)} \Big[-V(0;m+s-2,m+s-4,\dots,m+1;n)$$

$$+\left(\frac{m+s-2}{m+n+s-1}\right) V(0;m+s-3,m+s-4,\dots,m+1;n) \Big]. \quad (18)$$

Now combining (13) and the first terms in (16)-(18) (including other similar terms obtainable from (9)) one obtains

$$\left(\frac{m+n+s}{m+n+s-2}\right) V(0;m+s-1,m+s-2,m+s-4,\ldots,m;n). \qquad (19)$$

Further, combining (14) and the second terms in (16) - (18) (including other similar terms obtainable from (9)) we get

$$-\left[\frac{(m+s-2)}{(m+n+s-2)(m+n+s-1)}\right]\left[(m+n+s)\ V(0;m+s-1,m+s-3,\ldots,m;n)\right.$$
$$\left.-(m+s-1)\ V(0;m+s-2,m+s-3,\ldots,m;n)\right]. \qquad (20)$$

Now from (19) and (20) and (5), the coefficient of $-I_0(x;m+s-3,n+1)$ is given by

$$k_{m+s-3} = (m+n+s-2)^{-1}\left[C(s,m,n)\ V(0;m+s-1,m+s-2,m+s-4,\ldots,m;n)\right.$$
$$\left.-\ (m+s-2)\ k_{m+s-2}\right]. \qquad (21)$$

Proceeding on similar lines we can in general show that the coefficient of $(-1)^i\ I_0(0;m+s-i,n+1)$ is given by

$$(m+n+s-i+1)^{-1}\left[C(s,m,n)\ V(0;m+s-1,m+s-2,\ldots,m+s-i+1,m+s-i-1,\ldots m;n)\right.$$
$$\left.-(m+s-i+1)\ k_{m+s-i+1}\right] \qquad (22)$$

where $k_{m+s=0}$.

Now it may be noted that i in (22) goes from 1 to s-1. But the approximation will contain one more term which should be considered separately for even and odd values of s.

(i) s even. In this case repeated application of (4) to the V-function in the c.d.f. (3) would leave a final term independent of x which would be equal to $P_r(\theta_s \leq 1)$ and therefore, is unity. Hence the approximation to the c.d.f. for even values of s is given by

$$1 + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i} I_0(x;m+s-i,n+1). \tag{23}$$

(ii) s odd. When s is odd the remaining term is $\frac{I(x;m,n)}{\beta(m+1,n+1)}$ which is obtained as follows:

Apply (4) repeatedly to the V-function in the c.d.f. (3) and after the last stage of reduction, a set of simple incomplete eta functions are obtained which cannot be neglected and the sum of these terms involving x will be of the form

$$b_0 I(x;m,n) + b_1 I(x;m+1,n) + \ldots + b_{\frac{s-1}{2}} I(x;m+\frac{s-1}{2}, n) \tag{24}$$

where the b's are obtained by pooling together all the terms involving the respective I integrals after the last stage of reduction. Now since

$$I(x^1;q,r) = (q+r+1)^{-1} [-I_0(x^1,q,r+1) + qI(x^1;q-1,r] \tag{25}$$

and since

$$b_0\ \beta(m+1,n+1) + b_1\ \beta(m+2,n+1) + \ldots + b_{\frac{s-1}{2}}\ \beta(m+\frac{s+1}{2}, n+1) = 1 \tag{26}$$

(obtained by putting x=1 in (24) which equals $P_r(\theta_s \leq 1) = 1$), for the coefficient of $I(x;m,n)$ in (24) after integration by parts of the remaining integrals using (25), we get

$$I(x;m,n)\left[b_0 + b_1 \frac{(m+1)}{(m+n+2)} + b_2 \frac{(m+1)(m+2)}{(m+n+2)(m+n+3)} + \ldots + b_{\frac{s-1}{2}} \frac{(m+1)\ldots(m+\frac{s-1}{2})}{(m+n+2)\ldots(m+n+\frac{s+1}{2})}\right]$$

$$= I(x;m,n)/\beta(m+1,n+1). \qquad (27)$$

It may be pointed out that integration by parts in (24) using (25) would also give I_0 terms which, in fact, are already pooled together in the respective I_0 terms whose coefficients are given by (22). Hence when s is odd, approximation to the c.d.f. is given by

$$P_r(\theta_s \leq x) = \frac{I(x;m,n)}{\beta(m+1,n+1)} + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i} I_0(x;m+s-i,n+1). \qquad (28)$$

Now it may be noted that for evaluating (22) we should know the value of the V-determinants involved therein and it has been already shown (Pillai and Mijares (1959)) that

$$V(0;m+s-1,m+s-2,\ldots,m+s-i+1,m+s-i-1,\ldots,m;n)$$

$$= \left[\binom{s-1}{i-1} \prod_{j=1}^{i-1} \frac{(2m+s-j+1)}{(2m+2n+2s-j+1)}\right]/C(s-1,m,n) \qquad (29)$$

and thus (23) and (28) gives the desired approximation to the c.d.f. of the largest root at the upper end in a form simple to compute. It may be pointed out that the approximations to the c.d.f. obtained by Pillai for $s = 2$ to 7 (Pillai, 1954, 1956a, 1964; Pillai and Bantegui, 1959) are easily deduced from (23) or (28) by putting the appropriate value of s.

4. Upper Percentage Points

Pillai (1954, 1956, 1957, 1960) gave the upper 5°/o and 1°/o points of the largest root for $s=2$ and $s=5$ using his approximation formulae. Sen (1957) computed similar upper percentage points for three roots, Ventura (1957) for four roots, both following Pillai's method. Pillai and Bantegui (1959) gave such percentage points for $s=6$. All these percentage points were given for values of $m=0(1)4$ and n varying from 5 to 1000. Further, Jacildo (1959) extended the tables for $s=2$ and $s=3$ for values of $m=5,7,10$ and 15 and the same range of values of n as before. Pillai (1960) has published all these percentage points for $s=2,3,4,5$ and 6. More recently, Pillai (1964) has given such percentage points for $s=7$ for values of $m=0(1)5,7,10$ and values of n as before.

Foster and Rees (1957) have tabulated the upper percentage points (80, 85, 90, 95 and 99) of the largest root for $s=2$, $m=-0.5,0(1)9$ and $n=1(1)19(5)49,59,79$. Foster (1957, 1958) has further extended these tables for values of $s=3$ and 4. The arguments they have used for tabulation are $\nu_1=2n+s+1$ and $\nu_2=2m+s+1$. Heck (1960) has given some charts of upper 5°/o, 2.5°/o and 1°/o points for $s=2(1)5$, $m=-\frac{1}{2}$, $0(1)10$ and $n \geq 5$.

Upper 5°/o and 1°/o points were computed for θ_8 and θ_{10} using (23) and θ_9 using (28) for values of $m=0(1)5,7,10$ and 15 and n ranging from 5 to 1000 as before. These are presented in Tables 1-6. The computation was carried out on IBM 7090 but a trial value was extrapolated from Pillai's tables (1960) and (1964), of percentage points for $s=2,3,4,5,6$ and 7 for each of $9 \times 16 = 144$ combinations of m and n in order to be fed into the machine. The values of x were computed such that $P_r(\theta_s \leq x)$, $(s=8,9,10)$ was within a unit difference in the sixth decimal which gave six places accuracy for the percentage points with the exception that for $s=9$ and 10, for $m=15$, $P_r(\theta_s \leq x)$ was allowed to be within five units difference in the fifth decimal. At the right bottom corner of each table a few values have been extrapolated since the computer failed to obtain those values. The extrapolated values are given only to four significant figures while all others are given to five places.

5. Error of Approximation

All the values given in Tables 1 to 6 are believed to be accurate within a unit in the last decimal quoted. A detailed examination of the error of approximation for earlier studies has been made by Pillai and Bantegui (1959). In the more recent study for $s=7$ (Pillai, 1964) the error of approximation has been further dealt with but in a different manner than in the previous studies, for example by comparing the frequency of differences between the trial and final values. By comparison, the trial values in general were even closer to the final values in the present computations since percentage points were available for more values of s for computing the trial values. It should be pointed out that the trial values for $s=9$ were computed after the final values of $s=8$ were available and similarly for $s=10$. An added feature of the present computations is that percentage points for $m=15$ are also obtained as well as a uniform accuracy of five significant digits is provided.

The author wishes to thank Mr. Peter E. Dress, Instructor in Statistics, Department of Statistics and Mrs. Louise Mao Lui, Statistical Laboratory, Purdue University, for the excellent programming of this material for the IBM 7090 Computer, Purdue University's Computer Science's Center.

REFERENCES

Anderson, T.W. and Das Gupta S. (1964). Monotonicity of the power functions of some tests of independence between two sets of variates. Ann. Math. Statist., 35, 206-208.

Das Gupta, S., Anderson, T.W. and Mudholkar, G.S. (1964). Monotonicity of the power functions of some tests of the multivariate linear hypothesis. Ann. Math. Statist., 35, 200-205.

Fisher, R.A. (1939). The sampling distribution of some statistics obtained from non-linear equations. Ann. Eugen., London, 9, 238-249.

Foster F.G. (1957, 1958). Upper percentage points of the generalized Beta distribution. Biometrika, 44, 441-453; 45, 492-503.

Foster, F.G. and Rees, D.H. (1957). Upper percentage points of the generalized Beta distribution. Biometrika, 44, 237-247.

Girshick, M.A. (1939). On the sampling theory of the roots of determinental equations, Ann. Math. Statist., 10, 203-224.

Heck, D.L. (1960). Charts of some upperpercentage points of the distribution of the largest characteristic root. Ann. Math. Statist., 31, 625-641.

Hsu, P.L. (1939). On the distribution of roots of certain determinantal equations. Ann. Eugen., 9, 250-258.

Jacildo, L. (1959). Further studies on the distributions of the largest of two and three roots. Unpublished Thesis. The Statistical Center, University of the Philippines.

Pillai, K.C.S. (1954). On some distribution problems in multivariate analysis, Mimeo. Series No. 88, Institute of Statistics, University of North Carolina.

Pillai, K.C.S. (1955). Some new test criteria in multivariate analysis. Ann. Math. Statist. 26, 117-121.

Pillai, K.C.S. (1956a). On the distribution of the largest or the smallest root of a matrix in multivariate analysis. Biometrika, 43, 122-127.

Pillai, K.C.S. (1956b). Some results useful in multivariate analysis. Ann. Math. Statist. 27, 1106-1114.

Pillai, K.C.S. (1957). Concise Tables for Statisticians, The Statistical Center University of the Philippines.

Pillai, K.C.S. (1960). Statistical Tables for Tests of Multivariate Hypotheses, The Statistical Center, University of the Philippines.

Pillai, K.C.S. (1964). On the distribution of the largest of seven roots of a matrix in multivariate analysis, Biometrika, 51, 270-275.

Pillai, K.C.S. and Mijares, T.A. (1959), On the moments of the trace of a matrix and approximations to its distribution, Ann. Math. Statist., 30, 1135-1140.

Pillai, K.C.S. and Bantegui, C.G. (1959). On the distribution of the largest of six roots of a matrix in multivariate analysis. Biometrika, 46, 237-240.

Roy, S.N. (1939). P-statistics or some generalizations in analysis of variance appropriate to multivariate problems. Sankhyā, 4, 381-396.

Roy, S.N. (1945). The individual sampling distribution of the maximum, the minimum and any intermediate of the p-statistics on the null hypotheses. Sankhyā, 7, 133-158.

Roy, S.N. (1953). On a heuristic method of test construction and its use in multivariate analysis. Ann. Math. Statist., 24, 220-238.

Roy, S.N. (1957). Some Aspects of Multivariate Analysis, New York, John Wiley and Sons.

Roy, S.N. and Mikhail, W.F. (1961). On the monotonic character of the power functions of two multivariate tests. Ann. Math. Statist., 32, 1145-1151.

Sen, P. (1957). On a multivariate test criterion and its applications, Unpublished Thesis. The Statistical Center, University of the Philippines.

Ventura, S.R. (1957). On the extreme roots of a matrix in multivariate analysis and associated tests, Unpublished Thesis. The Statistical Center, University of the Philippines.

Table 1. Upper 5°/o Points of the Largest Root for s = 8

n \ m	0	1	2	3	4	5	7	10	15
5	.87386	.88974	.90198	.91173	.91968	.92630	.93670	.94773	.95948
10	.72804	.75412	.77534	.79300	.80798	.82085	.84191	.86546	.89206
15	.61550	.64525	.67024	.69164	.71022	.72656	.75402	.78589	.82359
20	.53065	.56108	.58718	.60997	.63010	.64806	.67886	.71565	.76068
25	.46544	.49524	.52122	.54420	.56475	.58331	.61561	.65503	.70464
30	.41408	.44274	.46800	.49059	.51099	.52957	.56228	.60290	.65518
40	.33876	.36473	.38799	.40909	.42841	.44623	.47816	.51881	.57296
60	.24794	.26912	.28844	.30627	.32286	.33838	.36679	.40411	.45600
80	.19537	.21303	.22931	.24447	.25871	.27215	.29703	.33034	.37787
100	.16115	.17623	.19022	.20334	.21572	.22748	.24942	.27915	.32232
130	.12759	.13993	.15145	.16231	.17263	.18247	.20099	.22638	.26392
160	.10559	.11602	.12579	.13504	.14386	.15230	.16827	.19034	.22335
200	.085849	.094482	.10260	.11032	.11769	.12478	.13824	.15698	.18531
300	.058496	.064526	.070225	.075663	.080887	.085929	.095564	.109119	.1298*
500	.035726	.039482	.043047	.046463	.049755	.052946	.059074	.067768	.08119*
1000	.018105	.020038	.021878	.023645	.025355	.027016	.030219	.03480*	.04201*

*value extrapolated

Table 2. Upper 1°/o Points of the Largest Root for s = 8

n \ m	0	1	2	3	4	5	7	10	15
5	.91031	.92176	.93055	.93754	.94323	.94796	.95537	.96320	.97152
10	.77855	.80027	.81787	.83248	.84482	.85541	.87267	.89189	.91349
15	.66867	.69502	.71708	.73589	.75218	.76647	.79039	.81803	.85053
20	.58250	.61043	.63429	.65505	.67334	.68961	.71741	.75045	.79066
25	.51467	.54266	.56696	.58839	.60751	.62472	.65456	.69080	.73614
30	.46038	.48773	.51176	.53317	.55245	.56997	.60071	.63869	.68727
40	.37948	.40480	.42741	.44786	.46653	.48371	.51438	.55327	.60475
60	.28011	.30123	.32046	.33814	.35456	.36990	.39787	.43447	.48507
80	.22175	.23958	.25597	.27120	.28547	.29892	.32375	.35686	.40386
100	.18344	.19878	.21298	.22626	.23878	.25064	.27271	.30251	.34559
130	.14565	.15829	.17007	.18115	.19165	.20166	.22043	.24609	.28386
160	.12076	.13149	.14152	.15100	.16002	.16865	.18492	.20734	.24074
200	.098336	.10725	.11562	.12356	.13114	.13841	.15219	.17132	.20013
300	.067151	.073412	.079320	.084948	.090347	.095551	.105476	.119400	.1406*
500	.041086	.045003	.048715	.052266	.055685	.058994	.065337	.074316	.08822*
1000	.020850	.022872	.024794	.026638	.028420	.030148	.033477	.03824*	.04573*

*value extrapolated

Table 3. Upper 5°/o Points of the Largest Root for s = 9

n \ m	0	1	2	3	4	5	7	10	15
5	.89098	.90390	.91401	.92215	.92886	.93448	.94340	.95293	.96318
10	.75598	.77833	.79670	.81212	.82528	.83665	.85536	.87644	.90045
15	.64727	.67357	.69584	.71503	.73178	.74655	.77150	.80061	.83524
20	.56307	.59053	.61426	.63507	.65353	.67005	.69848	.73256	.77447
25	.49716	.52447	.54841	.56969	.58878	.60606	.63622	.67314	.71976
30	.44455	.47111	.49465	.51577	.53491	.55237	.58319	.62156	.67107
40	.36633	.39077	.41278	.43280	.45118	.46816	.49863	.53752	.58939
60	.27040	.29069	.30929	.32650	.34254	.35757	.38512	.42135	.47178
80	.21408	.23118	.24699	.26177	.27567	.28880	.31315	.34578	.39237
100	.17713	.19182	.20550	.21836	.23052	.24208	.26368	.29297	.33552
130	.14066	.15275	.16408	.17480	.18499	.19472	.21305	.23822	.27542
160	.11663	.12689	.13654	.14570	.15444	.16283	.17869	.20065	.23349
200	.094986	.10351	.11156	.11922	.12656	.13362	.14704	.16575	.19404
300	.064875	.070856	.076532	.081961	.087184	.092232	.10189	.11548	.1363*
500	.039699	.043440	.047005	.050427	.053733	.056939	.063104	.07184*	.08542*
1000	.020149	.022080	.023925	.025702	.027423	.029097	.03233*	.03694*	.04416*

*value extrapolated

Table 4. Upper 1°/o Points of the Largest Root for s = 9

n \ m	0	1	2	3	4	5	7	10	15
5	.92264	.93192	.93916	.94498	.94978	.95379	.96013	.96691	.97417
10	.80179	.82030	.83548	.84818	.85899	.86831	.88361	.90078	.92025
15	.69676	.71994	.73951	.75631	.77093	.78381	.80547	.83066	.86047
20	.61221	.63727	.65886	.67774	.69444	.70935	.73493	.76547	.80282
25	.54440	.56992	.59221	.61196	.62965	.64561	.67339	.70724	.74977
30	.48940	.51462	.53691	.55685	.57486	.59126	.62013	.65590	.70182
40	.40632	.43003	.45131	.47064	.48833	.50464	.53383	.57092	.62012
60	.30247	.32262	.34104	.35804	.37386	.38867	.41570	.45114	.50021
80	.24060	.25778	.27365	.28844	.30232	.31542	.33965	.37199	.41796
100	.19966	.21454	.22837	.24134	.25359	.26521	.28688	.31617	.35853
130	.15901	.17134	.18288	.19377	.20411	.21398	.23252	.25789	.29524
160	.13209	.14260	.15248	.16183	.17075	.17929	.19541	.21766	.25082
200	.10775	.11652	.12479	.13265	.14017	.14739	.16110	.18015	.20885
300	.073760	.079948	.085812	.091413	.096795	.10199	.11191	.12584	.1471*
500	.045220	.049108	.052807	.056355	.059777	.063093	.069460	.07848*	.09243*
1000	.022983	.024997	.026918	.028767	.030555	.032293	.03564*	.04042*	.04788*

*value extrapolated

Table 5. Upper 5°/o Points of the Largest Root for s = 10

n \ m	0	1	2	3	4	5	7	10	15
5	.90483	.91547	.92393	.93083	.93656	.94141	.94916	.95759	.96720
10	.77978	.79907	.81509	.82864	.84026	.85037	.86708	.88605	.90789
15	.67519	.69855	.71848	.73575	.75090	.76431	.78704	.81372	.84561
20	.59217	.61704	.63867	.65772	.67468	.68990	.71619	.74784	.78695
25	.52606	.55114	.57325	.59297	.61073	.62684	.65503	.68966	.73355
30	.47263	.49728	.51923	.53901	.55697	.57339	.60245	.63872	.68566
40	.39214	.41517	.43599	.45499	.47248	.48865	.51774	.55494	.60465
60	.29180	.31125	.32914	.34574	.36124	.37580	.40248	.43765	.48664
80	.23209	.24864	.26400	.27839	.29194	.30477	.32857	.36050	.40613
100	.19260	.20690	.22028	.23287	.24480	.25616	.27739	.30622	.34812
130	.15339	.16524	.17638	.18693	.19699	.20660	.22473	.24963	.28647
160	.12743	.13752	.14704	.15610	.16476	.17307	.18882	.21063	.24327
200	.10396	.11237	.12034	.12794	.13523	.14226	.15562	.17427	.2025*
300	.071167	.077098	.082744	.088157	.093372	.098418	.10808	.12169	.1426*
500	.043634	.047358	.050918	.054344	.057657	.060875	.067067	.07585*	.08948*
1000	.022179	.024107	.025955	.027739	.029470	.031155	.03441*	.03905*	.04632*

*value extrapolated

Table 6. Upper 1°/o Points of the Largest Root for s = 10

n \ m	0	1	2	3	4	5	7	10	15
5	.93258	.94020	.94624	.95116	.95524	.95869	.96419	.97016	.97680
10	.82149	.83740	.85058	.86170	.87122	.87948	.89311	.90854	.92620
15	.72134	.74183	.75927	.77433	.78751	.79916	.81886	.84189	.86931
20	.63873	.66133	.68093	.69814	.71344	.72714	.75072	.77900	.81378
25	.57137	.59468	.61519	.63344	.64982	.66465	.69054	.72222	.76217
30	.51602	.53932	.56002	.57862	.59547	.61085	.63798	.67172	.71517
40	.43131	.45356	.47362	.49189	.50867	.52416	.55195	.58733	.63440
60	.32368	.34292	.36058	.37692	.39216	.40644	.43257	.46688	.51445
80	.25867	.27524	.29059	.30495	.31844	.33120	.35482	.38640	.43132
100	.21530	.22974	.24321	.25588	.26786	.27924	.30049	.32925	.37088
130	.17197	.18401	.19532	.20602	.21619	.22592	.24420	.26925	.30617
160	.14313	.15344	.16315	.17238	.18119	.18963	.20560	.22765	.26054
200	.11696	.12559	.13375	.14153	.14898	.15615	.16977	.18871	.2173*
300	.080258	.086375	.092191	.097760	.10312	.10830	.11820	.13213	.1534*
500	.049301	.053160	.056844	.060386	.063808	.067128	.073509	.08258*	.09661*
1000	.025096	.027100	.029020	.030871	.032666	.034411	.03778*	.04260*	.05013*

* value extrapolated