Estimation of the Parameters of the

Logistic Distribution*

by

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1. Introduction

This paper investigates the estimation of the parameters (both location and scale) of the logistic distribution using sample quantiles and order statistics. Three kinds of estimators have been considered; (1) Best linear unbiased estimators based on sample quantiles; (2) Unbiased linear asymptotically best estimators of Blom; (3) Asymptotically best, asymptotically unbiased linear estimators of Jung. All these methods of estimation are asymptotically efficient and one of the purposes of this investigation is to determine how good they are when compared to the best linear unbiased estimators in terms of their relative efficiency.

Let $(x_1, x_2, \ldots, x_n)$ be a sample from a logistic distribution with p.d.f. of x given by

$$(1.1) \qquad \frac{a}{\sigma} f\left(\frac{x-\mu}{\sigma/a}\right) = \frac{a}{\sigma} \frac{e^{-a(x-\mu)/\sigma}}{[1 + e^{-a(x-\mu)/\sigma}]^2}, \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

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where

$$a = \pi/\sqrt{3}$$

The c.d.f. $F(x)$ is then defined by

$$F(x) = \frac{1}{1 + e^{-a(x-\mu)/\sigma}} \tag{1.2}$$

Section 2 contains a discussion of the estimators based on sample quantiles. When σ is known the optimum symmetric spacing of the quantiles used in the estimation of μ has been obtained, for any number k of quantiles. When μ is known, the optimum spacing of the quantiles for estimating σ has been derived for $k = 3, 4$.

Section 3 contains a brief discussion of the approximations to the best linear unbiased estimators suggested by Blom (1957) and Jung (1956), and compares these two sets of estimators in terms of their relative efficiency with respect to the best linear unbiased estimators.

2. Quantile Estimators

The quantile estimators are based on a fixed number of sample quantiles when the total sample size is very large. Such a method of estimation would thus be useful when the experimenter has a very large sample, but would like to estimate the parameters with a few selected observations which he has the freedom to choose. Such a situation could arise in life-testing experiments when the observations do arise in a certain order and it is possible for the experimenter to select a few quantiles, the choice of the number of quantiles and the spacings between the quantiles being left to the experimenter. Expression for the quantile estimators of the parameters are discussed in this section. The scale parameter is assumed to be known, the optimum spacing of the quantiles for estimating the location parameter μ has been obtained for any number of quantiles. If the location parameter μ is assumed to be known, the optimum spacing of the quantiles for estimating σ has been determined for the number of quantiles = 3,4.

Let $x_{(n_1)} \leq x_{(n_2)} \leq \cdots \leq x_{(n_k)}$ be the k order statistics in a sample of size n from the logistic distribution (1.1). The following expressions will be needed and are defined as follows

(2.1) $$\lambda_i = \lim_{n \to \infty} \frac{n_i}{n}, \ \lambda_o \doteq 0, \ \lambda_{k+1} \doteq 1, \quad i = 1, 2, \ldots, k$$

(2.2) $$\lambda_i = \int_{-\infty}^{u_i} f(t)dt, \quad u_i = \log_e(\lambda_i/(1-\lambda_i)), \quad i = 1, 2, \ldots, k$$

(2.3) $$f_i = f(u_i) = \lambda_i(1-\lambda_i), \quad f_o \doteq f_{k+1} \doteq 0, \ i = 1, 2, \ldots, k$$

(2.4) $$X = \sum_{i=1}^{k+1} (1-\lambda_i-\lambda_{i-1})(\lambda_i(1-\lambda_i)x_{(n_i)} - \lambda_{i-1}(1-\lambda_{i-1})x_{(n_{i-1})})$$

(2.5) $$Y = \sum_{i=1}^{k+1} [(\lambda_i(1-\lambda_i)\log_e(\lambda_i/(1-\lambda_i)) - \lambda_{i-1}(1-\lambda_{i-1})\log_e(\lambda_{i-1}/(1-\lambda_{i-1}))(\lambda_i(1-\lambda_i)x_{(n_i)} - \lambda_{i-1}(1-\lambda_{i-1})x_{(n_{i-1})})]+[\lambda_i-\lambda_{i-1}]$$

(2.6) $$K_1 = \sum_{i=1}^{k+1} (\lambda_i-\lambda_{i-1})(1-\lambda_i-\lambda_{i-1})^2$$

(2.7) $$K_2 = \sum_{i=1}^{k+1} \frac{(\lambda_i(1-\lambda_i)\log_e(\lambda_i/(1-\lambda_i)) - \lambda_{i-1}(1-\lambda_{i-1})\log_e(\lambda_{i-1}/(1-\lambda_{i-1})))^2}{\lambda_i - \lambda_{i-1}}$$

$$(2.8)\quad K_3 = \sum_{i=1}^{k+1} (1-\lambda_i-\lambda_{i-1})(\lambda_i(1-\lambda_i)\log_e(\lambda_i/(1-\lambda_i)) - \lambda_{i-1}(1-\lambda_{i-1})\log_e(\lambda_{i-1}/(1-\lambda_{i-1})))\ .$$

Three different cases will be discussed.

(A) Estimation of μ, (σ known)

From the general expressions for the estimators derived by Ogawa (1951), the best linear unbiased estimator μ^* of μ and its variance are given by

$$(2.9)\qquad \mu^* = \frac{X}{K_1} - \frac{\sigma K_3}{K_1}$$

$$(2.10)\qquad V(\mu^*) = \frac{\sigma^2}{n} \cdot \frac{1}{K_1}$$

(2.9) and (2.10) give the estimator of μ^* and its variance for a fixed number k of quantiles and for fixed values of λ_i. For a fixed number of quantiles, the following theorem gives the values of λ_i's which will minimize $V(\mu^*)$.

Theorem 1. For a fixed number k of sample quantiles, the spacing of the quantiles for which $V(\mu^*)$ is minimal, is symmetric and is given by $\lambda_i = i/(k+1)$.

Since minimizing $V(\mu^*)$ would be equivalent to maximizing K_1, then by maximizing K_1 w.r.t. λ_i's ($i = 1, 2, \ldots, k$), the optimum λ_i's will be obtained as the solutions of the system of equations

$$(2.11)\qquad \lambda_{i+1} - \lambda_i = \lambda_i - \lambda_{i-1} \quad i = 1, 2, \ldots, k\ .$$

The above set of equations is satisfied for $\lambda_i = i/(k+1)$. To prove that this set of λ_i's does maximize K_1, consider the matrix $D = (\partial^2 K_1/\partial\lambda_i\partial\lambda_j)$ evaluated at $\lambda_i = i/(k+1)$. Let Δ_{k-p} be the determinant of the matrix obtained from D by deleting the last p rows and p columns. Then

$$\Delta_\alpha = -2(\Delta_{\alpha-1}) - \Delta_{\alpha-2} = (-1)^\alpha(\alpha+1) .$$

It follows that the matrix D is negative definite. Hence K_1 is maximized when $\lambda_i = i/(k+1)$. Now for $\lambda_i = i/(k+1)$ and for $i = 1, 2, \ldots, k$, it follows

(2.12)
$$\lambda_i + \lambda_{k-i+1} = 1$$
$$u_i + u_{k-i+1} = 0$$
$$f_i = f_{k-i+1} .$$

Clearly, (2.12) implies that the spacing is symmetric and $K_3 = 0$. Thus the best linear unbiased estimator with optimum spacings is given by

(2.13)
$$\mu^* = \frac{6}{k(k+1)(k+2)} \sum_{i=1}^{k} i(k+1-i)x_{(n_i)}$$

(2.14)
$$V(\mu^*) = \frac{3\sigma^2(k+1)^2}{a^2 nk(k+2)} .$$

The Cramer-Rao lower bound for the variance of an unbiased estimator for μ and σ is known is given by $3\sigma^2/(a^2n)$. Hence the relative efficiency is given by

(2.15) $$\frac{\text{C.R. lower bound}}{V(\mu^*)} = \frac{k(k+2)}{(k+1)^2} .$$

The above relative efficiency increases with k, its minimum (for $k > 0$) being 0.75.

Estimation of μ for censored samples

In practice there often arise situations when some of the observations are missing. Suppose that the (r_1-1) smallest and the (r_2-1) largest observations are not available. Then imposing the following restriction on k

(2.16) $$k \leq \min\left[\frac{r_2}{n-r_2} , \frac{n-r_1}{r_1}\right]$$

the best linear unbiased estimator for μ can be obtained from (2.13).

(B) Estimation of σ when μ is known

In this case it has been proved by Ogawa that the best linear unbiased estimator for σ, for a fixed number k of sample quantiles is

(2.17) $$\sigma^* = Y/K_2 - \mu\ K_3/K_2$$

(2.18) $$V(\sigma^*) = \sigma^2/(nK_2)$$

K_2 and K_3 being defined by (2.7), (2.8).

As in the previous case, the problem of deriving the optimum spacing of the sample quantiles arises. The optimum spacing for σ^* is obtained by maximizing K_2. Now

$$\frac{\partial K_2}{\partial \lambda_i} = 0, \quad \text{for} \quad i = 1, 2, \ldots, k$$

yields

$$(2.19) \qquad \left[\frac{Q_i}{\lambda_i-\lambda_{i-1}} - \frac{Q_{i+1}}{\lambda_{i+1}-\lambda_i}\right]\left[2\left\{(1-2\lambda_i)\log_e\left(\frac{\lambda_i}{1-\lambda_i}\right)+1\right\} - \frac{Q_i}{\lambda_i-\lambda_{i-1}} - \frac{Q_{i+1}}{\lambda_{i+1}-\lambda_i}\right] = 0, \quad (i = 1, 2, \ldots, k)$$

where

$$(2.20) \qquad Q_i = \lambda_i(1-\lambda_i)\log_e\left(\frac{\lambda_i}{1-\lambda_i}\right) - \lambda_{i-1}(1-\lambda_{i-1})\log_e\left(\frac{\lambda_{i-1}}{1-\lambda_{i-1}}\right) .$$

For a subclass of the class of all distribution function, Tischendorf (1955) has derived necessary conditions for the spacings that makes K_2 maximum. It can be verified that the logistic distribution belongs to this subclass. This necessary condition for logistic distribution is

$$(2.21) \qquad 2\left\{(1-2\lambda_i)\log_e\left(\frac{\lambda_i}{1-\lambda_i}\right)+1\right\} - \frac{Q_i}{\lambda_i-\lambda_{i-1}} - \frac{Q_{i+1}}{\lambda_{i+1}-\lambda_i} = 0, \quad i = 1, 2, \ldots, k .$$

From (2.20) and (2.21) it is clear that the optimum spacing can be obtained by solving (2.21). However, it is not possible to solve (2.21) explicitly for λ_i. Besides the system of equations (2.21) may also possess multiple roots which further entails a choice of the proper λ_i's. A slight simplification of the problem is effected by considering only symmetric spacings.

For $k = 2$, and using symmetric quantiles,

$$K_2 = \frac{2\lambda_1(1-\lambda_1)^2(\log_e\{\lambda_1/(1-\lambda_1)\})^2}{1-2\lambda_1} \quad . \tag{2.22}$$

For this case, equation (2.21) becomes

$$\frac{1 - 3\lambda_1 + 4\lambda_1^2}{1-2\lambda_1} \log_e \left(\frac{\lambda_1}{1-\lambda_1} \right) + 2 = 0 \; . \tag{2.23}$$

By solving (2.23), it was found that $\lambda_1 = .103$ is a solution of (2.23) and it was verified that K_2 does have an absolute maximum at $\lambda_1 = .103$.

The estimator σ^* and its variance are given by

$$\sigma^* = .4192(x_{([.897n]+1)} - x_{([.103n]+1)}) \tag{2.24}$$

$$V(\sigma^*) = 1.0227 \frac{\sigma^2}{n} \quad . \tag{2.24}$$

It can be shown that the Cramer-Rao lower bound for the variance of an unbiased estimator δ of σ is ,

$$V(\delta) \geq \frac{9\sigma^2}{n(3+\pi^2)} \quad . \tag{2.26}$$

[The details are lengthy and have been omitted.] Hence the relative efficiency of σ^* as compared with the Cramer-Rao bound is 68.38°/o .

For $k = 3$, the assumption that the spacing of the quantiles is symmetric gives λ_2 to be equal to .5, and the coefficient of $x_{([\lambda_2 n]+1)}$ in the estimator of σ is zero. In general for $k = 2m + 1$, the condition of symmetric spacing reduces the coefficient of $x_{([\lambda_{m+1} n]+1)}$ in the estimator of σ to zero. Thus for $k = 3$, K_2 is the same function of λ_1 as given in (2.22) so that the estimator σ^* and its variance is again given by (2.24) and (2.25), respectively.

(C) Estimation of both μ and σ

In the case where both μ and σ are unknown, the estimators are given by

$$\mu^* = \frac{1}{\Delta}(K_2 X - K_3 Y) \tag{2.27}$$

$$\sigma^* = \frac{1}{\Delta}(-K_3 X + K_1 Y) \tag{2.28}$$

where

$$\Delta = K_1 K_2 - K_3^2 .$$

$$\mathrm{Var}(\mu^*) = \frac{\sigma^2}{n}\frac{K_2}{\Delta}, \quad V(\sigma^*) = \frac{\sigma^2}{n}\frac{K_1}{\Delta}, \tag{2.29}$$

$$\mathrm{Cov}(\mu^*, \sigma^*) = -\frac{\sigma^2}{n}\frac{K_3}{\Delta}$$

The problem of obtaining the optimum spacing in this case even under the simplifying assumption of symmetry of the spacings is complicated since minimizing the generalized variance leads to simultaneous equations which cannot be solved explicitly.

3. Blom's and Jung's Estimators

This section discusses Blom's and Jung's estimators for estimating μ and σ by linear functions of order statistics. Blom (1957) approximated the best linear unbiased estimators by estimators that are unbiased but do not necessarily have the minimum variance of all linear unbiased estimators. Jung (1955) approximated the best linear unbiased estimators by estimators that are "asymptotically unbiased and asymptotically best".

Since this investigation was carried out, the best linear unbiased estimators of the location and scale parameters using order statistics have been computed for sample size ≤ 25, and are given in Gupta, Qureishi and Shah (1965). However the estimators of Blom and Jung are relatively simple to compute and hence it is of interest to determine how good these asymptotically 'good' estimators are when compared to the best linear unbiased estimators in terms of their relative efficiency for moderate sample sizes. This investigation would thus be useful for estimating the parameters for sample sizes ≥ 25, and also as providing some indication of the general properties of these two kinds of estimators, and hence has been included.

The authors computed both these estimators for both μ and σ. Since the estimators of Jung are biased they were modified by multiplying them by an appropriate constant. The following were the main results.

(i) The modified estimator of Jung for estimating μ reduces to Blom's estimator μ' where

$$\mu' = \sum_{i=1}^{n} \frac{6i(n+1-i)}{n(n+1)(n+2)} \cdot x_{(i)}$$

The approximate variance of μ' is given by

$$V(\mu') \simeq \frac{3\sigma^2(n+1)^2}{a^2 n(n+2)^2}$$

(ii) Jung's estimation of σ modified to make it unbiased did not reduce to Blom's estimator and its exact variance for $n \leq 25$ was found to be uniformly lower than the exact variance of the corresponding estimator of Blom. Jung's estimator for σ, $\hat{\sigma}$ is given by

$$\hat{\sigma} = \frac{9\pi}{n(n+1)^2(3+\pi^2)\sqrt{3}} \cdot \sum_{\nu=1}^{n} [-(n+1)^2 + 2\nu(n+1) + 2\nu(n+1-\nu)\log_e(\nu/n+1-\nu)] \; x_{(\nu)} \; .$$

Blom's estimator of σ, σ' is

$$\sigma' = \sum_{i=1}^{n} \alpha_i x_{(i)}$$

where

$$\alpha_i = \frac{ai(n+1-i)(c_i - c_{i-1})}{d(n+1)^2} \, , \quad i \leq n$$

$$c_i = \frac{i(n+1-i)}{(n+1)^2} \mu_1(i,n) - \frac{(i+1)(n-i)}{(n+1)^2} \mu_1(i+1,n), \quad i \leq n$$

$$\mu_1(i,n) = -\frac{1}{a} \sum_{j=i}^{n-i} \frac{1}{j} = -\mu_1(n-i+1,n), \quad n - i > i - 1$$

$$d = \sum_{i=0}^{n} c_i^2 \; .$$

Table I gives the coefficients of the order statistics in Blom's estimator of μ. The last column gives the exact variance of the estimator. These exact variances were calculated by using the variances and covariances of the order statistics given in Shah (1965) and Gupta, Shah and Qureishi (1965). The relative efficiency of this estimator when compared with the best linear unbiased estimator is given in Table III for selected values of n.

Table II gives the coefficients of the order statistics in the modified Jung's estimator of σ, since this estimator has smaller variance than Blom's estimator. Again the last column gives the exact variances of these estimators.

Although Blom's estimator of σ did have higher variance than the corresponding estimator of Jung (modified), yet it does have fairly high relative efficiency when compared with the best linear unbiased estimator, and was found to be at least 96 percent for $n \leq 25$. Table III gives the relative efficiency of Blom's estimator of σ, and the relative efficiency of Jung's estimator (modified) of σ for selected values of n.

From this table it is obvious that Blom's and Jung's estimators, besides being simple to compute, have very high relative efficiency even for moderate values of n, so that for $n > 25$, one could expect these estimators to be almost as efficient as the best linear unbiased estimators of Lloyd.

Table I

Coefficients of the ith order statistic in the unbiased nearly best estimator of μ, the mean of the logistic distribution, using Blom's method.
Coefficient of $x_{(n-i+1)}$ = coefficient of $x_{(i)}$.

n \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	$\frac{\text{Variance}}{\sigma^2}$
5	.1429	.2286	.2571											.1927
6	.1071	.1786	.2143											.1594
7	.0833	.1429	.1786	.1905										.1358
8	.0667	.1167	.1500	.1667										.1182
9	.0545	.0970	.1273	.1455	.1515									.1047
10	.0455	.0818	.1091	.1273	.1364									.0939
11	.0385	.0699	.0944	.1119	.1224	.1259								.0852
12	.0330	.0604	.0824	.0989	.1099	.1154								.0779
13	.0286	.0527	.0725	.0879	.0989	.1055	.1077							.0717
14	.0250	.0464	.0643	.0786	.0893	.0964	.1000							.0665
15	.0221	.0412	.0574	.0706	.0809	.0882	.0926	.0941						.0620
16	.0196	.0368	.0515	.0637	.0735	.0809	.0858	.0882						.0580
17	.0175	.0330	.0464	.0578	.0671	.0743	.0795	.0826	.0836					.0546
18	.0158	.0298	.0421	.0526	.0614	.0684	.0737	.0772	.0789					.0515
19	.0143	.0271	.0383	.0481	.0564	.0632	.0684	.0722	.0744	.0752				.0487
20	.0130	.0247	.0351	.0442	.0519	.0584	.0636	.0675	.0701	.0714				.0463
21	.0119	.0226	.0322	.0407	.0480	.0542	.0593	.0632	.0661	.0678	.0683			.0440
22	.0109	.0208	.0296	.0375	.0445	.0504	.0553	.0593	.0623	.0642	.0652			.0420
23	.0100	.0191	.0274	.0348	.0413	.0470	.0517	.0557	.0587	.0609	.0622	.0626		.0401
24	.0092	.0177	.0254	.0323	.0385	.0438	.0485	.0523	.0554	.0577	.0592	.0600		.0385
25	.0085	.0164	.0236	.0301	.0359	.0410	.0455	.0492	.0523	.0547	.0564	.0574	.0578	.0369

Table III

Table giving the relative efficiency of Blom's estimator of μ and Jung's estimator (modified) of σ; relative efficiency is with respect to the best linear unbiased estimators.

n / Rel. Eff.	5	7	10	15	20	25
Blom's Estimator of μ	.991	.993	.996	.997	.998	.999
Jung's Estimator (modified) of σ	.998	.998	.999	1.000	1.000	1.000

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