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On Linear Functions of Ordered Correlated Normal Random Variables

by

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1. Introduction and Summary

Let $X_1, X_2, \ldots, X_n$ be jointly normally distributed random variables with $EX_i = 0$, $EX_i^2 = 1$ and $EX_iX_j = \rho_{ij}$, $i \neq j = 1,2,\ldots,n$. Let $X_{(k)}$ be the kth order statistic when X_i's are arranged in an increasing order as follows:

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(k)} \leq \cdots \leq X_{(n)}. \tag{1.1}$$

Many problems of statistical inference, notably the ones in multiple decisions and life testing involve the use of ordered X_i's. In an earlier paper Gupta, Pillai and Steck (1964) considered the distribution of linear functions of $X_{(i)}$'s and gave closed form results for the case when the random variables are equally correlated. Also in the same paper closed form expressions for the distribution of the range $W = X_{(n)} - X_{(1)}$ were obtained for $n=3$ and 4 for the general case. In this paper we study the characteristic functions of individual order statistics and also the linear functions for the bivariate and trivariate cases for the general correlation matrix. Formulae for the expected values of $X_{(i)}$, $X_{(i)}^2$ and $X_{(i)}X_{(j)}$ and the first and second moments of a linear function of the $X_{(i)}$'s are obtained. The joint distribution of the range and the mid-range is given in a closed form in the trivariate case, from which the distributions of the mid-range, and the mid-range-range

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ratio are derived, again in closed forms. Best linear unbiased estimators of the common mean of three correlated normal variables have been obtained and tabulation of the coefficients made for different sets of values of ρ_{ij}'s. Applications in the fields of life testing and time series analysis are discussed.

2. The Bivariate Case

In the bivariate case, the characteristic function is given by:

$$(2.1) \qquad \phi_{X_{(1)},X_{(2)}}(t_1,t_2) = E(e^{it_1 X_{(1)}+it_2X_{(2)}})=2E(e^{it_1X_1+it_2X_2}/X_1\leq X_2).$$

Hence

$$(2.2) \qquad \phi_{X_{(1)},X_{(2)}}(t_1,t_2) = \frac{1}{\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{X_2} \exp(it_1X_1+ it_2X_2) \cdot \exp\left[-\frac{X_1^2+X_2^2-2\rho X_1X_2}{2(1-\rho^2)}\right] dX_1dX_2$$

Now we state a lemma which simplifies (2.2).

Lemma 1. For any real number α and β

$$(2.3) \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Phi(\alpha X + \beta)\, e^{-X^2/2}\, dX = \Phi\left(\frac{\beta}{\sqrt{1+\alpha^2}}\right)$$

where $\Phi(\cdot)$ denotes the c.d.f. of the random variable $N(0,1)$.

After completing squares, performing integrations on X_1 and using the lemma (2.3), one obtains from (2.2), the following explicit closed form for the characteristic function.

$$(2.3a) \qquad \phi_{X_{(1)},X_{(2)}}(t_1,t_2) = 2\exp\left(-\frac{t_1^2+2\rho t_1t_2+t_2^2}{2}\right) \Phi\left(\sqrt{\frac{1-\rho}{2}}\, i(t_2-t_1)\right).$$

Note that the formula (2.3a) gives the answer for the characteristic function in an explicit form and has been obtained directly. The formula (5) in the paper by Owen and Steck (1962) after evaluation of the characteristic function of the two ordered random variables (arising from independent case) reduces to the right hand side of (2.3a).

It follows from (2.3a) that

$$\phi_{X_{(2)}}(t_2) = 2\exp\left(-\frac{t_2^2}{2}\right) \Phi\left(\sqrt{\frac{1-\rho}{2}}\, it_2\right) \tag{2.4}$$

and by substituting $t_2 = t_1$, we get the characteristic function for $X_{(1)}$. Note that the characteristic function of a linear function of $X_{(1)}$ and $X_{(2)}$ can be obtained from (2.3a) by substituting $t_i = a_i t (i = 1,2)$ and is given by

$$\phi_{a_1X_{(1)}+a_2X_{(2)}}(t) = 2\exp\left[-\frac{t^2}{2}(a_1^2+2\rho a_1a_2+a_2^2)\right] \Phi\left(\sqrt{\frac{1-\rho}{2}}\, it(a_2-a_1)\right). \tag{2.5}$$

From (2.5) one can obtain all the moments of $a_1X_{(1)} + a_2X_{(2)}$ and verify the usual formulae for $a_1 = a_2 = 1$ i.e. the case of the midrange. The characteristic function for $a_1 = a_2 = 1$ is

$$\phi_{X_{(1)}+X_{(2)}}(t) = \exp\left[-t^2(1+\rho)\right], \tag{2.6}$$

and hence midrange follows the normal distribution $N(0, 2(1+\rho))$.

From (2.5) we obtain the density function $f(y)$ of $Y = a_1X_{(1)} + a_2X_{(2)}$ as

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_y(t)\, e^{-ity}\, dy \tag{2.7}$$

which after some simplification gives

$$f(y) = \sqrt{\frac{2}{\pi\xi}} \exp(-\frac{y^2}{2\xi}) \Phi(\eta y) \tag{2.8}$$

where

$$\xi = a_1^2 + a_2^2 + 2\rho a_1 a_2 \text{ and } \eta = [\sqrt{2(1-\rho)} (a_2 - a_1)] / [(1+\rho)(a_2 + a_1)^2].$$

By substituting $a_1 = 0$ and $a_2 = 1$ in (2.8), we obtain the density function for $X_{(2)}$, which is,

$$f(X_{(2)}) = \sqrt{\frac{2}{\pi}} \exp(-\frac{X_{(2)}^2}{2}) \Phi(\frac{\sqrt{2(1-\rho)}}{(1+\rho)} X_{(2)}) \tag{2.9}$$

which is a special case of the distribution of the maximum of several equally correlated normal random variables and is discussed and tabulated by Gupta (1963). Now we discuss the evaluation of the density function of the statistic $a_1X_{(1)} + a_2X_{(2)} - X_0$ where X_0 is $N(0,1)$ distributed independently of X_1 and X_2. This statistic arises in connection with some multiple decision problems (See Gupta (1956) and Gupta and Sobel (1957)).

A convolution of the earlier derived distribution of $a_1X_{(1)} + a_2X_{(2)}$ with X_0 gives the following result for the density $g(Z)$ of $Z = a_1X_{(1)} + a_2X_{(2)} - X_0$

$$g(Z) = \frac{2}{\sqrt{2\pi(1+\xi)}} \exp[-\frac{Z^2}{2(1+\xi)}] \Phi(\frac{Z}{\sqrt{(1+\xi)\{1+\xi(1+\eta^2)\}}}). \tag{2.10}$$

3. Trivariate Case

(3.1) $$E\left[e^{i(t_1X_{(1)} + t_2X_{(2)} + t_3X_{(3)})}\right]$$

$$= \sum_{i,j,k} \iiint_{X_i \le X_j \le X_k} e^{i(t_1X_1 + t_2X_2 + t_3X_3)} \frac{e^{-(\underline{X}'A^{-1}\underline{X})/2}}{|A|^{1/2}(2\pi)^{3/2}} dX_1 dX_2 dX_3$$

where the (i,j,k) represents a permutation of the positive integers $(1,2,3)$ and where the summation extends over all the 6 permutations of (i,j,k). We shall evaluate the one term of (3.1) corresponding $X_1 \le X_2 \le X_3$. We transform from X_1, X_2, X_3 $(X_1 \le X_2 \le X_3)$ to

(3.1a)
$$W = X_3 - X_1$$
$$M = X_3 + X_1$$
$$U = X_2 + (A^{32}X_3 + A^{12}X_1)/A^{22},$$

where A^{12}, A^{22}, etc. are elements of A^{-1} (a 3 x 3 matrix) whose positions are denoted by respective upper suffixes. Then

(3.2) $$I = \iiint_{X_1 \le X_2 \le X_3} e^{i(t_1X_1 + t_2X_2 + t_3X_3)} \frac{e^{-(\underline{X}'A^{-1}\underline{X})/2}}{(2\pi)^{3/2}|A|^{1/2}} dX_1 dX_2 dX_3$$

$$= \frac{1}{2|A|^{1/2}(2\pi)^{3/2}} \int_0^{\infty} dW \int_{-\infty}^{\infty} dM \int_{C_0M+d_1W}^{C_0M+d_2W} \exp\left[it_1\left(\frac{M-W}{2}\right) + it_3\left(\frac{M+W}{2}\right)\right.$$

$$\left. + it_2(U + a_0M + bW)\right] \exp\left[-\frac{W^2}{4(1-\rho_{13})} - \frac{M^2}{4(1+\rho_{13})} - \frac{A^{22}U^2}{2}\right] dU$$

where

(3.3)
$$\begin{cases} a_0 = -\dfrac{A^{12}+A^{32}}{2A^{22}} = +\dfrac{\rho_{12}+\rho_{23}}{2(1+\rho_{13})}\ ; \ b = \dfrac{A^{12}-A^{32}}{2A^{22}} = -\dfrac{\rho_{12}-\rho_{23}}{2(1-\rho_{13})} \\[2ex] c_0 = \dfrac{1}{2} - a_0 = \dfrac{1+\rho_{13}-\rho_{12}-\rho_{23}}{2(1+\rho_{13})}\ ; \quad d_1 = -\dfrac{1}{2} - b = \dfrac{-1+\rho_{13}+\rho_{12}-\rho_{23}}{2(1-\rho_{13})} \quad d_2 = \dfrac{1}{2} - b = \dfrac{1-\rho_{13}+\rho_{12}-\rho_{23}}{2(1-\rho_{13})} \\[2ex] |A| = 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{23}\rho_{13} . \end{cases}$$

After some simplification (completing the squares) and integrating out U, we get

(3.4)
$$I = \frac{\exp\left[-t_2^2/A^{22}\right]}{4\pi \cdot |A|^{1/2}\sqrt{A^{22}}} \int_0^{\infty} \psi_2(W)dW \int_{-\infty}^{\infty} \left[\Phi\left(\sqrt{A^{22}}\,(c_0M+d_2W) - \frac{it_2}{\sqrt{A^{22}}}\right) - \Phi\left(\sqrt{A^{22}}\,(c_0M + d_1W) - \frac{it_2}{\sqrt{A^{22}}}\right)\right] \cdot \psi_1(M)dM$$

where

(3.5)
$$\psi_1(M) = \exp\left[-\frac{M^2}{4(1+\rho_{13})} + \frac{iM}{2}(t_1 + t_3 + 2a_0t_2)\right]$$

$$\psi_2(W) = \exp\left[-\frac{W^2}{4(1-\rho_{13})} + \frac{iW}{2}(-t_1 + t_3 + 2bt_2)\right]$$

Now integrating out M by applying the lemma (2.3), we obtain

(3.6)
$$I = \frac{1}{2}\sqrt{\frac{1}{(1-\rho_{13})\pi}}\,\exp\left[-\frac{t_2^2}{A^{22}} - \frac{(1+\rho_{13})}{4}(t_1+t_3+2a_0t_2)^2\right] I_1$$

where

$$(3.7)\quad I_1 = \int_0^\infty \exp\left[-\frac{W^2}{4(1-\rho_{13})} + \frac{iW}{2}(-t_1+t_3+2bt_2)\right]\left[\Phi(d_2) - \Phi(d_1)\right]dW$$

where

$$(3.8)\qquad \Phi(d_i) = \Phi\left(\frac{C_0\sqrt{A^{22}}\,(1+\rho_{13})i(t_1+t_3+2a_0t_2) - \frac{it_2}{\sqrt{A^{22}}} + \sqrt{A^{22}}\,d_iW}{\sqrt{1+2A^{22}(1+\rho_{13})C_0^2}}\right).$$

Thus, formula (3.6) provides the characteristic function $\phi(t_1,t_2,t_3)$ for $X_1 \leq X_2 \leq X_3$. By permuting the indices 1,2,3 in (3.6), we can obtain the other cases. Note (3.6) involves an integral I_1.

It should be pointed out that the characteristic function for the linear function $a_1X_{(1)} + a_2X_{(2)} + a_3X_{(3)}$ is given by

$$(3.8a)\qquad \phi_{a_1X_{(1)}+a_2X_{(2)}+a_3X_{(3)}}(t) = \phi_{X_1,X_2,X_3}(a_1t,\ a_2t,\ a_3t).$$

Special Case

If we are interested in $\phi_{X_{(1)}}(t_1)$ we can find this by substituting $t_2 = t_3 = 0$ in the 6 terms of $\phi_{X_{(1)},X_{(2)},X_{(3)}}(t_1,t_2,t_3)$. Thus

$$(3.8b)\qquad \phi_{X_{(1)}}(t_1) = \sum_{i,j,k} I^{i,j,k}(t_1,0,0)$$

where

(3.9) $I^{i,j,k}(t_1,t_2,t_3)$ = integral similar to I of (3.6) which corresponds to the permutation (i,j,k).

Now we write one term in (3.8b), viz.

$$(3.10)\quad I^{1,2,3}(t_1,0,0) = \frac{1}{2}\sqrt{\frac{1}{(1-\rho_{13})\pi}}\ \exp\left[-\frac{1+\rho_{13}}{4}t_1^2\right] I_1^{1,2,3}(t_1,0,0)$$

where

$$(3.11)\quad I_1^{1,2,3}(t_1,0,0) = \int_0^\infty \exp\left[-\frac{W^2}{4(1-\rho_{13})} - \frac{it_1W}{2}\right]\left[\Phi\left(\frac{iC_0\sqrt{A^{22}}(1+\rho_{13})t_1+\sqrt{A^{22}}d_2W}{\sqrt{1+2A^{22}(1+\rho_{13})\ C_0^2}}\right]\right.$$

$$- \Phi\left(\frac{iC_0\sqrt{A^{22}}(1+\rho_{13})t_1 + \sqrt{A^{22}}d_1W}{\sqrt{1+2A^{22}(1+\rho_{13}C_0^2}}\right)\Big]\ dW$$

From (3.11), we obtain

$$(3.12)\qquad E(X_1|X_1 \leq X_2 \leq X_3) = \frac{\partial I^{1,2,3}(t_{1,0,0})}{i\,\partial t_1}\Big|_{t_1=0}$$

$$(3.13)\quad E(X_1|X_1 \leq X_2 \leq X_3) = -\frac{1}{2}\sqrt{\frac{1-\rho_{13}}{\pi}}\ I_3^{1,2,3} + \frac{1}{2}\sqrt{\frac{1}{\pi(1-\rho_{13})}}\ I_4^{1,2,3}$$

$$(3.14)\quad I_3^{1,2,3} = -\frac{1}{2}\ \theta(d_1)\left[\frac{1}{2(1-\rho_{13})} + \left\{\theta(d_1)\right\}^2\right]^{-\frac{1}{2}} + \frac{1}{2}\,\theta(d_2)\left[\frac{1}{2(1-\rho_{13})} + \left\{\theta(d_2)\right\}^2\right]^{-1/2}$$

$$\theta(d_i) = d_i\sqrt{\frac{A^{22}}{1+2A^{22}(1+\rho_{13})\ C_0^2}}\ ,\quad i = 1,2,$$

and

$$(3.15)\quad I_4^{1,2,3} = \frac{C_0(1+\rho_{13})}{2}\left[\frac{\theta(d_2)}{d_2}\left\{(\theta(d_2))^2 + \frac{1}{2(1-\rho_{13})}\right\}^{-\frac{1}{2}}\right.$$

$$\left. - \frac{\theta(d_1)}{d_1}\left\{(\theta(d_1))^2 + \frac{1}{2(1-\rho_{13})}\right\}^{-\frac{1}{2}}\right].$$

Again from (3.13), by permuting the indices 1,2,3 of ρ_{ij}, one obtains the other 5 terms.

Special Case

If $\rho_{ij} = \rho$, $i \neq j = 1,2,3$, we have

$$d_1 = -d_2 = -\frac{1}{2}, \; C_0 = \frac{1-\rho}{2(1+\rho)};$$

(3.16)

$$A^{11} = A^{22} = A^{33} = (1+\rho)/[(1-\rho)(1+2\rho)] \; ; \; A^{12} = A^{23} = A^{13} = -\rho/[(1-\rho)(1+2\rho)]$$

so that

(3.17)
$$I_4^{i,j,k} = 0 \text{ and } I_3^{i,j,k} = +\frac{1}{2} \text{ for all } i,j,k.$$

(3.18)
$$EX_{(1)} = 6E(X_1 | X_1 \leq X_2 \leq X_3) = -\frac{3}{2}\sqrt{\frac{1-\rho}{\pi}}$$

which checks with the result obtained directly for this case and further for $\rho = 0$ checks with the well-known values.

From (3.8a), following similar methods as for obtaining the $E(X_1 \mid X_1 \leq X_2 \leq X_3)$, we can obtain

(3.19)
$$E(a_1X_1 + a_2X_2 + a_3X_3 | X_1 \leq X_2 \leq X_3)$$

$$= \frac{1}{2}\sqrt{\frac{1-\rho_{13}}{\pi}} \; I_5^{1,2,3} + \frac{1}{2}\sqrt{\frac{1}{\pi(1-\rho_{13})}} \; I_6^{1,2,3}$$

where

$$I_5^{1,2,3} = (-a_1 + a_3 + 2ba_2) \; I_3^{1,2,3}$$

and

$$I_6^{1,2,3} = (a_1 + a_3 + 2a_0a_2 \frac{a_2}{C_0(1+\rho_{13})A^2}) \; I_4^{1,2,3}$$

From (3.19) we obtain

$$(3.20)\quad E(X_2|X_1\leq X_2\leq X_3) = b\sqrt{\frac{1-\rho_{13}}{\pi}}\, I_3^{1,2,3} + \frac{1}{2}\sqrt{\frac{1}{\pi(1-\rho_{13})}}\,\left(2a_0 - \frac{1}{C_0(1+\rho_{13})A^{22}}\right) I_4^{1,2,3}$$

and

$$(3.21)\quad E(X_3|X_1\leq X_2\leq X_3) = \frac{1}{2}\sqrt{\frac{1-\rho_{13}}{\pi}}\, I_3^{1,2,3} + \frac{1}{2}\sqrt{\frac{1}{\pi(1-\rho_{13})}}\, I_4^{1,2,3}\ .$$

Again, putting $-a_1 = a_3 = 1$ and $a_2 = 0$, we get

$$(3.22)\quad E(X_3 - X_1 \mid X_1\leq X_2\leq X_3) = \sqrt{\frac{1-\rho_{13}}{\pi}}\; I_3^{1,2,3}\ .$$

Similarly,

$$(3.23)\quad E(X_3 + X_1|X_1\leq X_2\leq X_3) = \sqrt{\frac{1}{\pi(1-\rho_{13})}}\; I_4^{1,2,3}\ .$$

Covariance between $X_{(j)}$ and $X_{(2)}$, $j = 1,3$.

From (3.8) proceeding as usual we obtain

$$(3.24)\quad E(X_jX_2|X_1\leq X_2\leq X_3,\ j = 1,3) = \psi_j$$

where

$$(3.25)\quad \psi_j = \frac{1}{2\pi}\left[a_0(1+\rho_{13}) + b\ \gamma_j(1-\rho_{13})\right] \text{arc tan}\left[\frac{\theta(d_2) - \theta(d_1)}{\sqrt{2(1-\rho_{13})}\left[\frac{1}{2(1-\rho_{13})} + \theta(d_2)\theta(d_1)\right]}\right]$$

$$+ \frac{1}{2\pi\sqrt{2(1-\rho_{13})}\ \left(\frac{1}{2(1-\rho_{13})} + [\theta(d_2)]^2\right)}\left[b\ \gamma_j(1-\rho_{13}) + \frac{bC_0(1+\rho_{13})\theta(d_2)}{d_2}\right.$$

$$\left. + \frac{\gamma_j\lambda_0}{2} - \frac{C_0[\theta(d_2)]^2(1+\rho_{13})\lambda_0}{d_2}\right]$$

$$- \frac{1}{2\pi\sqrt{2(1-\rho_{13})}\left(\frac{1}{2(1-\rho_{13})} + [\theta(d_1)]^2\right)} \Big[b\, \gamma_j(1-\rho_{13}) + bC_0(1+\rho_{13}) \frac{\theta(d_1)}{d_1}$$

$$+ \frac{\gamma_j \lambda_0}{2} - \frac{C_0[\theta(d_1)]^2(1+\rho_{13})\lambda_0}{d_1} \Big]$$

where λ_0 is obtained by putting $a_1 = a_3 = 0$ and $a_2 = 1$ in λ defined in (3.28) below. It should be noted that $\gamma_j = -1$ or $+1$ according as $j = 1$ or 3. Further, in the case of equal correlation, we obtain from (3.25)

$$(3.26) \qquad 6E(X_1X_2|X_1 \leq X_2 \leq X_3) = E(X_3X_2|X_1 \leq X_2 \leq X_3) = \rho + \frac{\sqrt{3}}{2\pi}(1-\rho).$$

Now we obtain $E[(a_1X_1 + a_2X_2 + a_3X_3)^2|X_1 \leq X_2 \leq X_3]$. Starting from (3.8a) and differentiating the function $E[e^{it(a_1X_1 + a_2X_2 + a_3X_3)}]$ twice partially with respect to t and putting $t = 0$, we obtain after simplification (details omitted)

$$(3.27) \qquad E[(a_1X_1 + a_2X_2 + a_3X_3)^2|X_1 \leq X_2 \leq X_3]$$

$$= \frac{1}{\pi}[\beta + \delta^2(1-\rho_{13})] \text{ arc tan } \left[\frac{\theta(d_2) - \theta(d_1)}{\sqrt{2(1-\rho_{13})}\left[\frac{1}{2(1-\rho_{13})} + \theta(d_2)\theta(d_1)\right]} \right]$$

$$+ \frac{1}{\pi}\sqrt{\frac{1-\rho_{13}}{2}} \cdot \frac{1}{[2\{\theta(d_2)\}^2(1-\rho_{13})+1]} [2(1-\rho_{13})\delta^2\, \theta(d_2) + \lambda(2\delta - \lambda\theta(d_2))]$$

$$- \frac{1}{\pi}\sqrt{\frac{1-\rho_{13}}{2}} \cdot \frac{1}{[2\{\theta(d_1)\}^2(1-\rho_{13})+1]} [2(1-\rho_{13})\, \delta^2\theta(d_1) + \lambda(2\delta - \lambda\theta(d_1))]$$

where

$$\beta = \frac{2a_2^2 + (1+\rho_{13})A^{22}(a_1+a_3+2a_0a_2)^2}{4A^{22}}$$

(3.28) $$\delta = (-a_1 + a_3 + 2ba_2)/2$$

and

$$\lambda = \frac{C_0 \sqrt{A^{22}}\,(1+\rho_{13})(a_1+a_3+2a_0a_2) - (a_2/\sqrt{A^{22}})}{\sqrt{1 + 2A^{22}(1+\rho_{13})\,C_0^2}}$$

where $a_0, C_0 b, d_1, d_2$ are defined in (3.3), $\theta(d_1)$ and $\theta(d_2)$ are defined by (3.14) and where $A^{22} = (1-\rho_{13}^2)/|A|$. As a particular case take $a_2 = a_3 = 0$, $a_1 = 1$ and $\rho_{ij} = \rho$ (all i,j, $i \neq j$), then we obtain

(3.29) $$EX_{(1)}^2 = 6E(X_1^2 | X_1 \leq X_2 \leq X_3) = 1 + (1-\rho)[\,1 + \sqrt{3}/(2\pi)\,]$$

which agrees with the well-known result for this case (see, for example, Owen and Steck (1962)).

4. Distributions of the Range, Mid-range and the Ratio of Mid-range to the Range

The distribution of the range for $n = 3$ and 4 normal random variables for the general case has been obtained by Gupta, Pillai and Steck (1964). Now we shall obtain closed-form expressions for the distribution of the ratio of mid-range to the range.

For the case of $n = 3$ correlated normal random variables, it is easy to show that the joint density function of W, M, U (defined in (3.1a) is given by

(4.1) $$f(W,M,U) = \frac{\exp\left[-\frac{W^2}{4(1-\rho_{13})} - \frac{M^2}{4(1+\rho_{13})} - \frac{A^{22}}{2}U^2\right]}{2(2\pi)^{3/2}\,|A|^{1/2}}\,.$$

Integrating out U in (4.1) and writing $W_1 = \sqrt{A^{22}}\ W/2$ and

$z_1 = \sqrt{A^{22}}\ M(1 + \rho_{13} - \rho_{12} - \rho_{23})/2(1 + \rho_{13})$ we obtain

$$(4.2)\quad f(W_1,z_1) = C\exp\left[-\frac{W_1^2}{(1-\rho_{13})A^{22}} - \frac{Dz_1^2}{2}\right]\left[\Phi\,(z_1+W_1(a+1)) - \Phi(z_1+W_1(a-1))\right]$$

where

$$(4.3)\qquad a = (\rho_{12}-\rho_{23})/(1-\rho_{13}) \text{ and } D = 2\left[A^{22}(1+\rho_{13})\left(1 - \frac{\rho_{12}+\rho_{23}}{1+\rho_{13}}\right)^2\right]^{-1}$$

and

$$C = \frac{1}{\pi}\sqrt{D/(2A^{22}(1-\rho_{13}))}\ .$$

From (4.2) we obtain (by putting $z_1/W_1 = U_1$ and $W_1 = W_1$ and integrating out W_1), the density function of U_1 is

$$(4.4)\qquad f(U_1) = \frac{C}{\lambda_1}\left\{\frac{\Phi\left(\frac{U_1}{\sqrt{1+\lambda_1/(a+1)^2}}\right)}{\sqrt{1+\lambda_1/(a+1)^2}}\exp\left[-\frac{U_1^2}{2}\left(1 - \frac{1}{1+\lambda_1/(a+1)^2}\right)\right] - \frac{\Phi\left(\frac{-U_1}{\sqrt{1+\lambda_1/(a-1)^2}}\right)}{\sqrt{1+\lambda_1/(a-1)^2}}\exp\left[-\frac{U_1^2}{2}\left(1 - \frac{1}{1+\lambda_1/(a-1)^2}\right)\right]\right\}$$

where

$$\lambda_1 = DU_1^2 + 2/(A^{22}(1-\rho_{13}))\ .$$

Equation (4.4) provides the density function of

$$(4.5)\qquad U_1 = \frac{M}{W}\,\frac{(1+\rho_{13}-\rho_{12}-\rho_{23})}{(1+\rho_{13})}\ .$$

Again from (4.2), we obtain the distribution of z_1 by making use of the following lemma.

Lemma 2. For any real numbers α_1 and β_1

$$(4.6)\quad \frac{1}{\sqrt{2\pi}} \int_0^\infty \Phi(\alpha_1 + \beta_1 x) e^{-x^2/2}\, dx = (\tfrac{1}{2})\ \Phi(\alpha_1/\sqrt{1+\beta_1^2}) + (\text{arc tan } \beta_1)/2\pi - V(\alpha_1/\sqrt{1+\beta_1^2},\ \alpha_1\beta_1/\sqrt{1+\beta_1^2})$$

where $V(h,q)$ is the V-function of Nicholson (1943). Applying lemma 2 to (4.2) for integration with respect to W_1, we get

$$(4.7)\quad f_2(z_1) = (\sqrt{D}\ e^{-Dz_1^2/2}/\sqrt{8\pi})\Big\{\Phi(z_1/\sqrt{1+B_1^2}) - \Phi(z_1/\sqrt{1+B_2^2}) + \Big[(\text{arc tan } B_1) - (\text{arc tan } B_2)\Big]/\pi - 2V(z_1/\sqrt{1+B_1^2},\ z_1B_1/\sqrt{1+B_1^2}) + 2V(z_1/\sqrt{1+B_2^2},\ z_1B_2/\sqrt{1+B_2^2})\Big\},$$

where $B_1 = (a+1)\sqrt{(1-\rho_{13})A^{22}/2}$ and $B_2 = (a-1)\sqrt{(1-\rho_{13})A^{22}/2}$.

5. Best Linear Unbiased Estimators of the Common Mean of the Three Correlated Normal Random Variables

Now we consider the use of the results in the previous Sections for the construction of the best linear unbiased estimators of the common mean μ of three correlated normal random variables. Let X_1', X_2', X_3' be normal random variables with the common mean μ, common variance unity and $E(X_i' X_j') = \rho_{ij} + \mu^2$. Then we are interested in finding the linear function based on $X'_{(i)}$ which is an unbiased estimator of μ and which has the smallest variance in the class of all linear unbiased estimators. Let $\Sigma a_i X'_{(i)}$ be the estimator that we are looking for. Then the condition for unbiasedness, implies

$$\Sigma a_i = 1, \quad \Sigma a_i \, EX_{(i)} = 0. \tag{5.1}$$

It should be pointed out that (5.1) gives two necessary conditions for the unbiasedness of an estimator based on any number of random variables. Now we have to minimize the variance of $\Sigma a_i X'_{(i)}$ which is the same as the variance of $\Sigma a_i X_{(i)}$ and can be computed by using the formula (3.27) for all the six permutations of the subscripts 1,2,3. Thus one minimizes the variance of $\Sigma a_i X_{(i)}$ subject to the condition $\Sigma a_i = 1$, and the condition that the sum of the terms obtained by interchanging the subscripts of ρ_{ij} in (3.19) equals zero. The equations that determine the coefficients corresponding to the minimum value are given in the Appendix. Table I gives the values of a_i's for selected set of values of $\rho_{12}, \rho_{13}, \rho_{23}$. The computations were carried out on IBM 7090. It should be pointed out that the selected values of ρ_{ij} have to satisfy certain linear restrictions.

6. Applications

(i) Test of equality of the means of a multivariate population involving a common ρ.

Let $X^{(j)}_{(1)} \le X^{(j)}_{(2)} \le \dots \le X^{(j)}_{(n)}$, j=1,2,...,k, be the n ordered random variables which have come from a multivariate normal population with mean vector $(\mu_1, \mu_2, \ldots, \mu_n)$ and a common correlation coefficient ρ. Suppose we are interested in testing the hypothesis H against the alternative A where

$$H: \ \mu_1 = \mu_2 = \ldots = \mu_n, \ A: \ \text{not } H, \tag{6.1}$$

then the following simple test can be applied. Let $W_j = X_{(n)}^{(j)} - X_{(1)}^{(j)}$ and $W_{max} = \max_j W_j$. Test: Reject H if $\frac{W_{max}}{\sum_{j=1}^{k} W_j} \geq C(\alpha, n, k)$ where $C(\alpha, n, k)$ is a constant which depends on the size of the test and on n and k. In fact, $C(\alpha, n, k)$ is such that

$$P\left\{\frac{W_{max}}{\sum_{j=1}^{k} W_j} \geq C(\alpha, n, k) \,|\, \mu_1 = \ldots = \mu_k\right\} = \alpha \ . \tag{6.2}$$

It should be pointed out that the fact that the distribution of $W_{max}/(\sum_{j=1}^{k} W_j)$ is independent of ρ has already been shown in the paper by Gupta, Pillai and Steck (1964).

It should be pointed out that the above forms an important application to the situations in life testing where the observations are ordered and we can perform the test without knowing the unordered random samples from the multivariate normal population for which the common correlation coefficient is unknown.

(ii) Confidence Interval for the Common ρ in a Multivariante Normal Population with Common Méan and Common Known Standard Deviation.

It is clear that if $X_{(n)}$ and $X_{(1)}$ are the largest and smallest of n equally correlated random variables with a common mean, and common standard deviation say unity, then one or two-sided 100 α percent confidence bounds for ρ are obtained from

$$\frac{X_{(n)}-X_{(1)}}{C_2} \leq \sqrt{1-\rho} \leq \frac{X_{(n)}-X_{(1)}}{C_1} , \tag{6.3}$$

where C_1 and C_2 are the percentiles of the distribution of the range of n independent and identically distributed normal random variables. It may be pointed out that in life testing where the observations are naturally ordered, use of (6.3) gives a confidence interval statement for ρ even when the unordered sample is not known.

(iii) Test of equality of two oscillations in non-overlapping intervals of equal length in time series.

Tests based on the ratio of differences between peaks and troughs from two non-overlapping intervals of equal length might be indicative of the nature of the oscillations involved in the time series. If we assume the variables to follow a multivariate normal distribution within intervals of equal length, this test would amount to the one based on the ratio of two independent ranges in the correlated case. (The distribution of this ratio for common mean and common standard deviation can be obtained as a series of beta functions using the distribution of range developed by the authors as a series of gamma functions which is available in Mimeograph series No. 13 Department of Statistics, Purdue University. For the case of means not necessarily equal the distribution has not been worked out.) To establish the equality of two oscillations, in addition, tests based on the differences of mid-ranges may have to be carried out. However, since tests based on the differences of mid-ranges cannot be easily performed, the ratio of mid-range to the range can be used to test any assigned value of the average of two population means (corresponding to the peak and trough) and this test can be repeated for each interval. In all the discussions above, it is assumed that the variables have a common standard deviation, otherwise adjustments have to be made at various stages. Also the assumption of a multivariate normal distribution means that the set of correlation coefficients are known, or otherwise have to be estimated possibly through the use of serial correlation coefficients. However, if the correlation coefficients are neither known nor evaluated, some simulation process could be adopted for the performance of the tests.

The authors are grateful to the referee for bringing to their attention the result in Lemma 2. They also wish to thank Mrs. Louise Lui, Statistical Laboratory, Purdue University, for the excellent programming of the material in the Appendix of this paper for computing Table 1 on IBM 7094 computer, Purdue University's Computer Sciences Center.

Appendix

The condition $\Sigma a_i \, EX_{(i)} = 0$ given in (5.1) reduces to the following equation by using (3.19) and $\Sigma a_i = 1$.

(A.1) $$k_1 a_1 = k_2 a_2 + k_3$$

where

$$k_1 = \sqrt{\frac{1-\rho_{13}}{\pi}}\, I_3^{(1)} + \sqrt{\frac{1-\rho_{12}}{\pi}}\, I_3^{(2)} + \sqrt{\frac{1-\rho_{23}}{\pi}}\, I_3^{(3)}$$

$$k_2 = -\Big\{ \sqrt{\frac{1-\rho_{13}}{\pi}}\, d_{21}\, I_3^{(1)} + \sqrt{\frac{1-\rho_{12}}{\pi}}\, d_{22}\, I_3^{(2)} + \sqrt{\frac{1-\rho_{23}}{\pi}}\, d_{23}\, I_3^{(3)}$$

$$+ \left[\pi(1-\rho_{13})\right]^{-1/2} + \left[C_{01} + (2A^{22}C_{01}(1+\rho_{13}))^{-1}\right] I_4^{(1)}$$

$$+ \left[\pi(1-\rho_{12})\right]^{-1/2} \left[C_{02} + (2A^{33}C_{02}(1+\rho_{12}))^{-1}\right] I_4^{(2)}$$

$$+ \left[\pi(1-\rho_{23})\right]^{-1/2} \left[C_{03} + ((2A^{11}C_{03}(1+\rho_{23}))^{-1}\right] I_4^{(3)} \Big\}$$

$$k_3 = \frac{1}{2}\Big[\sqrt{\frac{1-\rho_{13}}{\pi}}\, I_3^{(1)} + \sqrt{\frac{1-\rho_{12}}{\pi}}\, I_3^{(2)} + \sqrt{\frac{1-\rho_{23}}{\pi}}\, I_3^{(3)}$$

$$+ \left[\pi(1-\rho_{13})\right]^{-1/2} I_4^{(1)} + \left[\pi(1-\rho_{12})\right]^{-1/2} I_4^{(2)} + \left[\pi(1-\rho_{23})\right]^{-1/2} I_4^{(3)} \Big].$$

where $I_3^{(1)}$ is a simpler notation for $I_3^{1,2,3}$ in (3.14), $d_{11}, d_{21}, C_{01}, \theta(d_{11})$ and $\theta(d_{21})$ being the same as $d_1, d_2, C_0, \theta(d_1)$ and $\theta(d_2)$ respectively in (3.14). $I_3^{(2)}$ is obtained from $I_3^{(1)}$ by interchanging in the latter ρ_{13} and ρ_{12} and so also d_{12} from d_{11}, d_{22} from d_{21}, C_{02} from C_{01} etc. Similarly $I_4^{(1)}$ is a briefer notation for $I_4^{1,2,3}$ in (3.15). $I_4^{(2)}$ is obtained from $I_4^{(1)}$ by interchange of ρ_{13} and ρ_{12}. Further, $I_3^{(3)}$ is obtained from $I_3^{(1)}$ by interchanging in the latter ρ_{13} and ρ_{23} and similarly $I_4^{(3)}$ from $I_4^{(1)}$. d_{13}, d_{23}, C_{03} etc. are obtained in a similar manner.

The value of a_2 corresponding to the minimum variance is given by the equation

$$\ell_1 a_2 = \ell_2 \tag{A.2}$$

where

$$\ell_1 = 2\Big[A_1\Big(\frac{(A^{22})^{-1}}{2} + C_{01}^2(1+\rho_{13})\Big) + A_2\Big(\frac{(A^{33})^{-1}}{2} + C_{02}^2(1+\rho_{12})\Big)$$

$$+ A_3\Big(\frac{(A^{11})^{-1}}{2} + C_{03}^2(1+\rho_{23})\Big) + B_1'\Big(d_{21} + \frac{k_2}{k_1}\Big)^2 + B_2'\Big(d_{22} + \frac{k_2}{k_1}\Big)^2 + B_3'\Big(d_{23} + \frac{k_2}{k_1}\Big)^2$$

$$+ C_1 \frac{\theta(d_{11})}{d_{11}}\Big(2C_{01}^2(1+\rho_{13}) + \frac{1}{A^{22}}\Big)\Big(d_{21} + \frac{k_2}{k_1}\Big) + C_2 \frac{\theta(d_{12})}{d_{12}}\Big(2C_{02}^2(1+\rho_{12}) + \frac{1}{A^{33}}\Big) \times$$

$$\times\Big(d_{22} + \frac{k_2}{k_1}\Big) + C_3 \frac{\theta(d_{13})}{d_{13}}\Big(2C_{03}^2(1+\rho_{23}) + \frac{1}{A^{11}}\Big)\Big(d_{23} + \frac{k_2}{k_1}\Big)$$

$$+ D_1 \frac{\theta^2(d_{11})}{d_{11}^2}\Big(2C_{01}^2(1+\rho_{13}) + \frac{1}{A^{22}}\Big)^2 + D_2 \frac{\theta^2(d_{12})}{d_{12}^2}\Big(2C_{02}^2(1+\rho_{12}) + \frac{1}{A^{33}}\Big)^2$$

$$+ D_3 \frac{\theta^2(d_{13})}{d_{13}^2}\Big(2C_{03}^2(1+\rho_{23}) + \frac{1}{A^{11}}\Big)^2\Big]$$

$$\ell_2 = A_1(1+\rho_{13})\ C_{01} + A_2(1+\rho_{12})\ C_{02} + A_3(1+\rho_{23})\ C_{03}$$

$$+ \left(1 - \frac{2k_3}{k_1}\right)\left\{B_1'\left(d_{21} + \frac{k_2}{k_1}\right) + B_2'\left(d_{22} + \frac{k_2}{k_1}\right) + B_3'\left(d_{23} + \frac{k_2}{k_1}\right)\right\}$$

$$+ C_1\left\{C_{01}\left(d_{21} + \frac{k_2}{k_1}\right)(1+\rho_{13})\ \frac{\theta(d_{11})}{d_{11}} + \frac{1}{2}\left(1 - \frac{2k_3}{k_1}\right)\left(2C_{01}^2(1+\rho_{13})\ \frac{\theta(d_{11})}{d_{11}} + \frac{\theta(d_{11})}{d_{11}A^{22}}\right)\right\}$$

$$+ C_2\left\{(1+\rho_{12})\ C_{02}\left(d_{22} + \frac{k_2}{k_1}\right)\frac{\theta(d_{12})}{d_{12}} + \frac{1}{2}\left(1 - \frac{2k_3}{k_1}\right)\left(2C_{02}^2(1+\rho_{12})\ \frac{\theta(d_{12})}{d_{12}} + \frac{\theta(d_{12})}{d_{12}A^{33}}\right)\right\}$$

$$+ C_3\left\{(1+\rho_{23})\ C_{03}\left(d_{23} + \frac{k_2}{k_1}\right)\ \frac{\theta(d_{13})}{d_{13}} + \frac{1}{2}\left(1 - \frac{2k_3}{k_1}\right)\left(2C_{03}^2(1+\rho_{23})\ \frac{\theta(d_{13})}{d_{13}} + \frac{\theta(d_{13})}{d_{13}A^{11}}\right)\right\}$$

$$+ 2\left\{D_1C_{01}\ \frac{\theta^2(d_{11})}{d_{11}^2}\ (1+\rho_{13})\left(2C_{01}^2(1+\rho_{13}) + \frac{1}{A^{22}}\right) + D_2C_{02}\frac{\theta^2(d_{12})}{d_{12}^2}\ (1+\rho_{12}) \times\right.$$

$$\left.\times\left(2C_{02}^2(1+\rho_{12}) + \frac{1}{A^{33}}\right) + D_3C_{03}\ \frac{\theta^2(d_{13})}{d_{13}^2}\ (1+\rho_{23})\left(2C_{03}^2(1+\rho_{23}) + \frac{1}{A^{11}}\right)\right\}$$

and where

$$A_1 = \frac{1}{\pi}\ \text{arc tan}\left[\frac{\theta(d_{21}) - \theta(d_{11})}{\left[\frac{1}{2(1-\rho_{13})} + \theta(d_{21})\theta(d_{11})\right]\sqrt{2(1-\rho_{13})}}\right],$$

$$B_1' = \left\{(1-\rho_{13})A_1 + \frac{\sqrt{2}}{\pi}\ (1-\rho_{13})^{3/2}\left[\frac{\theta(d_{21})}{\left\{2\theta^2(d_{21})(1-\rho_{13})+1\right\}} - \frac{\theta(d_{11})}{\left\{2\theta^2(d_{11})(1-\rho_{13})+1\right\}}\right]\right\},$$

$$C_1 = \frac{1}{\pi}\sqrt{2(1-\rho_{13})}\left\{\frac{1}{\{2\theta^2(d_{21})(1-\rho_{13})+1\}} - \frac{1}{\{2\theta^2(d_{11})(1-\rho_{13})+1\}}\right\},$$

$$D_1 = \frac{1}{\pi}\sqrt{\frac{1-\rho_{13}}{2}}\left\{\frac{\theta(d_{11})}{2\theta^2(d_{11})(1-\rho_{13})+1} - \frac{\theta(d_{21})}{2\theta^2(d_{21})(1-\rho_{13})+1}\right\}$$

Finally, interchange ρ_{13} and ρ_{12} in A_1 to get A_2 and ρ_{13} and ρ_{23} in A_1 to get A_3. B_2' and B_3' are obtained in a similar manner from B_1', C_2 and C_3 from C_1, and D_2 and D_3 from D_1.

Table 1

Values* of a_i's (i=1,2,3) corresponding to the minimum variance for given values of ρ_{ij}'s.

ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3	ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3
-.9	-.5	.1	.4299	.2873	.2828	-.7	.7	-.1	1.0204	-.6742	.6538
-.9	-.5	.7	.6175	-.1348	.5173	-.7	.9	-.9	.8423	-1.0081	1.1658
-.9	-.3	-.1	.4259	.2745	.2996	-.5	-.9	.1	.4145	.2426	.3429
-.9	-.3	.3	.4874	.2140	.2986	-.5	-.9	.5	.4759	.1479	.3762
-.9	-.1	-.3	.4283	.2468	.3249	-.5	-.7	-.1	.3761	.3030	.3208
-.9	-.1	.5	.7431	-.3505	.6074	-.5	-.7	.7	.4819	.1803	.3377
-.9	.1	-.5	.4371	.2034	.3594	-.5	-.5	-.3	.3501	.3301	.3198
-.9	.1	-.1	.5131	.1316	.3553	-.5	-.5	.3	.4179	.2901	.2921
-.9	.1	.3	.7587	-.3260	.5673	-.5	-.5	.9	.4905	.2424	.2670
-.9	.3	-.5	.4846	.1244	.3909	-.5	-.3	-.5	.3367	.3266	.3367
-.9	.3	.1	.7884	-.3451	.5567	-.5	-.3	.1	.3957	.3126	.2917
-.9	.5	-.7	.4982	.0639	.4379	-.5	-.3	.9	.6287	-.3132	.6845
-.9	.5	-.1	.8467	-.4186	.5719	-.5	-.1	-.7	.3387	.2877	.3736
-.9	.7	-.9	.4996	.0009	.4996	-.5	-.1	.1	.4109	.2933	.2958
-.9	.7	-.5	.7397	-.2730	.5333	-.5	-.1	.5	.4803	.2322	.2874
-.9	.9	-.7	1.1658	-1.0081	.8423	-.5	.1	-.9	.3594	.2034	.4371
-.7	-.7	.1	.4060	.2985	.2955	-.5	.1	-.3	.3851	.2714	.3435
-.7	-.7	.7	.4723	.2512	.2765	-.5	.1	.3	.4893	.1931	.3175
-.7	-.7	.9	.4917	.2422	.2661	-.5	.3	-.9	.3909	.1244	.4846
-.7	-.5	-.1	.3862	.3177	.2961	-.5	.3	-.1	.4564	.1873	.3563
-.7	-.5	.9	.6187	-.3001	.6814	-.5	.3	.5	.8163	-.4455	.6291
-.7	-.3	-.3	.3750	.3145	.3105	-.5	.5	-.9	.4279	.0204	.5517
-.7	-.3	.5	.4819	.2361	.2820	-.5	.5	.5	0	0	1
-.7	-.3	.7	.6257	-.1436	.5179	-.5	.7	-.9	.5333	-.2730	.7397
-.7	-.1	-.5	.3736	.2877	.3387	-.5	.7	-.3	.5483	-.0003	.4520
-.7	-.1	-.1	.4184	.2718	.3098	-.5	.7	.1	1.0259	-.6835	.6576
-.7	-.1	.7	.7707	-.5040	.7334	-.5	.9	-.7	.8136	-.9542	1.1406
-.7	.1	-.7	.3834	.2331	.3835	-.5	.9	-.1	-5.1259	9.0862	-2.9603
-.7	.1	.1	.5013	.1610	.3377	-.3	-.9	-.1	.3929	.2379	.3692
-.7	.3	-.7	.4154	.1692	.4154	-.3	-.9	.3	.4644	.1286	.4069
-.7	.3	-.1	.5215	.1034	.3751	-.3	-.7	-.3	.3479	.3041	.3479
-.7	.3	.3	.7979	-.3773	.5795	-.3	-.7	.5	.4713	.1465	.3821
-.7	.5	-.9	.4379	.0638	.4982	-.3	-.5	-.5	.3198	.3301	.3501
-.7	.5	-.3	.5433	.0400	.4167	-.3	-.5	.1	.3848	.2930	.3221
-.7	.7	-.9	.4659	-.0227	.5568	-.3	-.5	.7	.4785	.1789	.3426

*For $\rho_{ij} = \rho (i \neq j = 1,2,3)$, $a_1 = a_2 = a_3 = 1/3$.

Table 1 (Contd.)

ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3	ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3
-.3	-.3	-.7	.3105	.3145	.3750	-.1	.5	.1	.4369	.1951	.3680
-.3	-.3	.1	.3770	.3173	.3057	-.1	.5	.7	.9636	-.7794	.8158
-.3	-.3	.9	.4890	.2429	.2682	-.1	.7	-.7	.6538	-.6742	1.0204
-.3	-.1	-.9	.3249	.2468	.4283	-.1	.7	-.1	.4616	.0768	.4616
-.3	-.1	.1	.3802	.3197	.3000	-.1	.7	.5	1.0843	-.8164	.7321
-.3	-.1	.9	.6415	-.3300	.6885	-.1	.9	-.5	-2.9603	9.0862	-5.1259
-.3	.1	-.9	.3519	.1783	.4698	-.1	.9	.3	8.1657	-11.5549	4.3892
-.3	.1	-.1	.3715	.3011	.3274	.1	-.9	-.5	.3429	.2426	.4145
-.3	.1	.5	.4787	.2261	.2951	.1	-.9	.3	.6347	-.2756	.6409
-.3	.3	-.9	.3905	.7625	.5332	.1	-.7	-.7	.2955	.2985	.4061
-.3	.3	-.1	.4036	.2445	.3520	.1	-.7	-.1	.3875	.2000	.4125
-.3	.3	.3	.4925	.1739	.3336	.1	-.7	.5	.6568	-.3395	.6827
-.3	.5	-.9	.4972	-.1915	.6943	.1	-.5	-.9	.2828	.2873	.4299
-.3	.5	-.1	.4576	.1504	.3920	.1	-.5	-.1	.3538	.2617	.3845
-.3	.5	.5	.8932	-.5492	.6560	.1	-.5	.7	.7003	-.4985	.7982
-.3	.7	-.7	.5286	-.2680	.7394	.1	-.3	-.9	.2954	.2505	.4541
-.3	.7	-.1	.5405	.0185	.4410	.1	-.3	.1	.3624	.2752	.3624
-.3	.7	.3	1.0421	-.7180	.6758	.1	-.3	.9	.9105	-1.5755	1.6650
-.3	.9	-.5	.7838	-.8939	.1115	.1	-.1	-.9	.3243	.1784	.4973
-.3	.9	-.1	1.0896	-.8427	.7531	.1	-.1	.1	.3412	.3176	.3412
-.1	-.9	-.3	.3692	.2379	.3929	.1	-.1	.7	.4668	.1778	.3555
-.1	-.9	.5	.6403	-.3088	.6685	.1	.1	-.9	.4772	-.1353	.6581
-.1	-.7	-.5	.3208	.3030	.3761	.1	.1	-.1	.3145	.3237	.3619
-.1	-.7	.3	.4570	.1290	.4140	.1	.1	.9	.4834	.2450	.2715
-.1	-.5	-.7	.2961	.3177	.3862	.1	.3	-.9	.5567	-.3450	.7884
-.1	-.5	.1	.3845	.2617	.3538	.1	.3	-.1	.3304	.2839	.3856
-.1	-.5	.9	.8430	-1.4005	1.5575	.1	.3	.7	.4670	.2546	.2785
-.1	-.3	-.9	.2996	.2745	.4259	.1	.5	-.7	.5819	-.4358	.8539
-.1	-.3	-.1	.3387	.3226	.3387	.1	.5	.1	.3775	.2450	.3775
-.1	-.3	.7	.4738	.1778	.3484	.1	.5	.9	1.5250	-2.9382	2.4133
-.1	-.1	-.9	.3201	.2209	.4590	.1	.7	-.5	.6576	-.6835	1.0259
-.1	-.1	.1	.3570	.3275	.3155	.1	.7	.1	.4407	.1185	.4407
-.1	-.1	.9	.4867	.2436	.2696	.1	.7	.7	1.2195	-1.1603	.9408
-.1	.1	-.9	.3553	.1316	.5131	.1	.9	-.3	-22.5518	64.0420	-40.4902
-.1	.1	-.1	.3398	.3205	.3398	.1	.9	.5	4.5413	-5.9373	2.3960
-.1	.1	.9	.6587	-.3527	.6940	.3	-.9	-.5	.3528	.2039	.4433
-.1	.3	-.7	.3751	.1034	.5215	.3	-.9	-.1	.6409	-.2756	.6347
-.1	.3	-.1	.3617	.2766	.3617	.3	-.7	-.7	.2892	.2825	.4283
-.1	.3	.5	.4774	.2157	.3069	.3	-.7	-.3	.6576	-.3152	.6576
-.1	.5	-.9	.5719	-.4186	.8467	.3	-.5	-.9	.2776	.2673	.4551

Table 1 (Contd.)

ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3	ρ_{12}	ρ_{13}	ρ_{23}	a_1	a_2	a_3
.3	-.5	-.1	.3631	.2156	.4214	.5	.5	-.5	0	0	1
.3	-.5	.5	.6839	-.3872	.7032	.5	.5	.1	.3270	.1958	.4773
.3	-.3	-.9	.2986	.2140	.4874	.5	.5	.9	.4681	.2539	.2780
.3	-.3	-.1	.3256	.2769	.3976	.5	.7	-.1	.7321	-.8164	1.0842
.3	-.3	.7	.7380	-.5675	.8295	.5	.9	.1	2.3960	-5.9373	4.5413
.3	-.1	-.7	.3066	.2055	.4879	.5	.9	.7	1.0259	-.6835	.6576
.3	-.1	.1	.3332	.2910	.3759	.7	-.9	-.9	.2752	.2493	.4755
.3	-.1	.9	1.0315	-1.8878	1.8563	.7	-.9	-.5	.5526	-.1218	.5692
.3	.1	-.5	.3175	.1931	.4893	.7	-.7	-.9	.2624	.2532	.4843
.3	.1	.1	.3121	.3247	.3632	.7	-.7	-.1	.7777	-.4547	.6770
.3	.1	.7	.4556	.1800	.3644	.7	-.5	-.9	.5173	-.1348	.6175
.3	.3	-.7	.5795	-.3773	.7979	.7	-.5	.1	.7982	-.4985	.7003
.3	.3	-.1	.3106	.2656	.4238	.7	-.3	-.7	.5179	-.1436	.6257
.3	.3	.9	.4781	.2478	.2741	.7	-.3	.3	.8295	-.5675	.7380
.3	.5	-.5	.6040	-.4708	.8668	.7	-.1	-.7	.7334	-.5040	.7707
.3	.5	.1	.3390	.2482	.4128	.7	-.1	-.1	.2822	.2631	.4547
.3	.5	.9	.7239	-.4400	.7161	.7	-.1	.5	.8854	-.6973	.8119
.3	.7	.1	.4024	.0879	.5098	.7	.1	-.5	.7478	-.5553	.8075
.3	.7	.7	.7517	-.2960	.5442	.7	.1	.1	.2853	.2714	.4433
.3	.9	-.1	4.3892	-11.5549	8.1657	.7	.1	.7	1.0264	-1.0527	1.0264
.3	.9	.5	1.0234	-.6864	.6630	.7	.3	-.3	.7711	-.6346	.8635
.5	-.9	-.7	.3176	.2234	.4590	.7	.3	.7	.6203	-.2405	.6203
.5	-.9	-.1	.6685	-.3088	.6402	.7	.5	-.1	.8158	-.7794	.9636
.5	-.7	-.9	.2679	.2704	.4617	.7	.5	.9	.7003	-.4985	.7982
.5	-.7	.1	.6827	-.3395	.6568	.7	.7	.3	.5442	-.2960	.7517
.5	-.5	-.9	.2779	.2388	.4833	.7	.7	.9	.4433	.2714	.2853
.5	-.5	-.3	.3244	.2310	.4446	.7	.9	.9	1.2195	-1.1603	.9407
.5	-.5	.5	0	0	1	.9	-.9	-.9	.2652	.2421	.4927
.5	-.3	-.9	.4877	-.1190	.6313	.9	-.7	-.7	.2660	.2422	.4917
.5	-.3	-.1	.3289	.2376	.4335	.9	-.5	-.5	.2670	.2424	.4905
.5	-.3	.5	.7372	-.4743	.7372	.9	-.3	-.3	.2682	.2429	.4890
.5	-.1	-.5	.2874	.2322	.4803	.9	-.1	-.5	1.5950	-1.5320	.9370
.5	-.1	-.1	.2914	.2884	.4202	.9	-.1	-.1	.2696	.2436	.4867
.5	-.1	.3	.4071	.1506	.4422	.9	-.1	.3	1.8563	-1.8878	1.0315
.5	.1	-.7	.6156	-.3880	.7725	.9	.1	.5	2.3105	-2.6337	1.3231
.5	.1	.1	.2970	.3018	.4012	.9	.3	-.1	1.9207	-2.0968	1.1761
.5	.1	.9	1.3231	-2.6337	2.3105	.9	.3	.3	.2741	.2478	.4781
.5	.3	-.5	.6291	-.4455	.8163	.9	.5	.5	.2780	.2539	.4681
.5	.3	.1	.2892	.2973	.4135	.9	.7	.9	1.0264	-1.0527	1.0264
.5	.3	.9	.6491	-.4010	.7519	.9	.9	.7	.9408	-1.1603	1.2195

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