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Introduction. The martingale convergence theorems were first utilized by Doob [2; p. 343] in giving a new proof of the Lebesgue differentiation theorem of functions of bounded variation on a real line. Later Chow [1] gave a proof of the Lebesgue differentiation theorem of interval functions of bounded variation by applying convergence theorems of partially ordered martingales. In 1959, Ward's differentiation theorem [8; p. 137, p. 141], among other things, have been generalized by Rutowitz [7] to cell functions by introducing the concept of the p -bordering property. In this paper, by following Doob's approach in [3], we are able to obtain a convergence theorem [Theorem I], which includes some martingale convergence theorems and extends a theorem of Rutowitz [7; Theorem II] to non-atomic basis. Theorem IV puts the above cited Ward's theorem into Martingale setting.

1. Definitions and notation.

Suppose that $(\Omega, \mathcal{F}, P)$ is a complete measure space with $P(\Omega) = 1$. A stochastic basis $(\mathcal{F}_\delta, \Delta)$ is a net, where Δ is a directed set, $\mathcal{F}_\delta$ is a sub-σ-algebra of $\mathcal{F}$ for each $\delta \in \Delta$, and $\mathcal{F}_\delta \subset \mathcal{F}_{\delta'}$ if $\delta < \delta'$. A stochastic process $(x_\delta, \mathcal{F}_\delta, \Delta)$ is a triple, where $(\mathcal{F}_\delta, \Delta)$ is a stochastic basis and x_δ is an $\mathcal{F}_\delta$-measurable function. P* is the outer measure induced by P and the integral $\int_A x$ will mean $\int_A x dP$. For a set A, the $\mathcal{F}_\delta$-cover of A is denoted by A_δ^* and the $\mathcal{F}$ -cover by A^*. A-B will be the proper difference of sets A and B, and I(A) the indicator (or characteristic) function of the set A. The function x_δ is sometimes written as $x(\delta)$. $||x||_q$ is the L_q-norm of x. For sets A and B, $A \in \mathcal{F}_\delta B$, if $A \subset B$ and $A \in \mathcal{F}_\delta$.

Definition 1. A stochastic basis is said to satisfy the Vitali condition

V_q for $1 \leq q \leq \infty$, if for every $\epsilon > 0$, every set A and every net (K_δ, Δ) of $\mathcal{F}_\delta$-sets such that $\limsup_\Delta K_\delta \supset A$ a.e., there exist $\delta_i > \delta$ for any given δ, and $\mathcal{F}_{\delta_i}$-sets $L_i \subset K_{\delta_i}$ so that

(1.1) $$P^*(A-B) < \epsilon$$

where $B = \bigcup_1^n L_i$, and that

(1.2) $$\left|\left| \sum_1^n I(L_i) - I(B)\right|\right|_q < \epsilon.$$

The conditions V_1 and V_∞ are called respectively the weak and the strong Vitali conditions. If Δ is a countable linearly ordered set, then any stochastic basis $(\mathcal{F}_\delta, \Delta)$ satisfies V_∞. The ordinary differentiation basis satisfies the strong Vitali condition V_∞ (See [1] or [4; p.209], in [1] V_∞ has been denoted by V_0), and the strong differentiation basis has the property V_1 (see [4; p.210]).

A stochastic basis is said to satisfy the Vitali condition V_q^*, if it satisfies the conditions of Definition 1, replacing lim sup by ess lim sup and A by A^*. Both definitions of V_q and V_q^* are due to Krickeberg ([5], [6]. He denotes V_q and V_q^* by V_q^* and V_q).

Definition 2. Let $b > 0$, $1 \leq q \leq \infty$ and $V = [\sup |x(\delta)| < b]$. $(x_\delta, \mathcal{F}_\delta, \Delta)$ is said to satisfy the condition $(A,b)_q$, if for every $\delta_0 \in \Delta$ there exists $0 < c < \infty$ such that for any given $\delta_1, \ldots, \delta_m$ in $\Delta(\delta > \delta_0)$ and $L_i \in V^*_{\delta_i} \mathcal{F}_{\delta_i}$, there are $\eta \geq \delta_i$ $(i = 1,2,\ldots,m)$ and $\mathcal{F}_\eta$-measurable functions $y' = y'(\eta)$, $y'' = y''(\eta)$ with $||y'||_q \leq c$, $||y''||_q \leq c$ so that there exist $\eta_{i,1} = \delta_i \leq \eta_{i,2} \leq \cdots \leq \eta_{i,k_i} = \eta$ and $\mathcal{F}_\eta$-measurable functions $x_i' = x_i'(\eta)$, $x_i'' = x_i''(\eta)$ satisfying for $i = 1,2,\ldots,n$ and $j = n+1, \ldots, m$

(1.3) $$x_i' = x(\eta) = x_j'' \text{ in } V, \quad x_i' \leq c \text{ in } L_i, \quad x_j'' \geq c \text{ in } L_j,$$

(1.4) $$\int_{L_i} x(\delta_i) \leq \int_{L_i A_i} y' + \int_{L_i - A_i} x_i' ,$$

(1.5) $$\int_{L_j} x(\delta_j) \geq \int_{L_j B_j} y'' + \int_{L_j - B_j} x_j'' ,$$

where $A_i = \left\{ \max_{k \leq k_i} x(\eta_{i,k}) \geq b \right\}$ and $B_j = \left\{ \min_{k \leq k_j} x(\eta_{j,k}) \leq -b \right\}$.

Definition 3. A stochastic process $(x_\delta, \mathcal{F}_\delta, \Delta)$ **is a martingale, if** $(\mathcal{F}_\delta, \Delta)$ is a stochastic basis, x_δ is integrable, and if for $\delta' \leq \delta$ $E(x_\delta | \mathcal{F}_{\delta'}) = x_{\delta'}$ a.e., where $E(x_\delta | \mathcal{F}_{\delta'})$ is the Radon-Nikodym derivative of the integral of x_δ relative to $\mathcal{F}_{\delta'}$.

If $(x_\delta, \mathcal{F}_\delta, \Delta)$ is a martingale and $\sup_\Delta ||x_\delta||_q \leq K < \infty$, then the condition $(A,b)_q$ is satisfied for every $b > 0$, by taking $\eta \geq \delta_i$ $(i = 1,2,\ldots,m)$, $y' = y'' = x(\eta)$, $\eta_{i,2} = \eta$, $x_i' = \min [x(\eta), b]$, $x_i'' = \max [x(\eta), -b]$, and $c > \max [b, K]$.

2. Martingale convergence theorems

Theorem 1. If $1 \leq q < \infty$, $p^{-1} + q^{-1} = 1$ and $(x_\delta, \mathcal{F}_\delta, \Delta)$ is a stochastic process satisfying the Vitali condition V_q, then x_δ converges a.e. where $\sup_\Delta |x_\delta| < b$, provided $(A,b)_p$ is satisfied for some $b > 0$.

Proof. Suppose that it is false and $\delta_0 \in \Delta$. Then there exist two real numbers $a < d$ and a set V with $P^*(V) > 0$ such that

(2.1) $$\sup_\Delta |x_\delta| < b, \quad \limsup_\Delta x_\delta > d > a > \liminf_\Delta x_\delta$$

on V. Put

(2.2) $$K_\delta = V_\delta^* (x_\delta > d).$$

Then $\limsup K_\delta \supset V$. By the Vitali condition V_q, for $1 > \epsilon > 0$ there exist $\delta_i > \delta_0$ and $L_i \in K_{\delta_i} \mathcal{F}_{\delta_i}$, $i = 1,\ldots, n$, such that

(2.3) $$P^*(V - A) < \epsilon, \ ||\sum_1^n I(L_i) - I(A)||_q < \epsilon,$$

where $A = \bigcup_1^n L_i$. Put

(2.4) $$H_\delta = AV^*_\delta\ (x_\delta < a).$$

By V_q again, for $\delta'_0 > \delta_i$, $i = 1,\ldots, n$, there exist $\delta_j > \delta'_0$ and $L_j \in H_{\delta_j}\mathcal{F}_{\delta_j}$, $j = n+1,\ldots,m$, such that

(2.5) $$P^*(AV - B) < \epsilon,\ \left|\left| \sum_{n+1}^{m} I(L_j) - I(E)\right|\right|_q < \epsilon,$$

where $B = \bigcup_{n+1}^{m} L_j$. By the condition $(A,b)_p$, there exist c, η, y', y'', x'_i x''_i and $\eta_{i,k}$ $(i = 1, \ldots, n, \ldots, m;\ k = 1, \ldots, k_i)$ satisfying the conditions in $(A,b)_p$. For each $i = 1, \ldots, n$, let s_i be the first $k \le k_i$ such that $x(\eta_{i,k}) \ge b$ if there is one, and $s_i = \infty$ otherwise. Then for $i = 1, \ldots, n$

(2.6) $$\int_{L_i} x(\delta_i) \le \int_{L_i(s_i < \infty)} y' + \int_{L_i(s_i = \infty)} x'_i,$$

$$d \sum_1^n P(L_i) \le \sum_1^n \int_{L_i(s_i < \infty)} y' + \sum_1^n \int_{L_i(s_i = \infty)} x'_i .$$

Choose δ_0 so large such that $P(V^*_{\delta_0} - V^*_\delta) < \epsilon$ for $\delta > \delta_0$. Then

$$\sum_1^n \int_{L_i[(s_i = \infty) - V^*_\eta]} x'_i \le c \sum_1^n P(L_i[(s_i = \infty) - V^*_\eta]) - c\, P(\bigcup_1^n L_i[(s_i = \infty) - V^*_\eta])$$

$$+ c\, P(\bigcup_1^n L_i[(s_i = \infty) - V^*_\eta])$$

$$\le c\left[\sum_1^n P(L_i) - P(A)\right] + c\, P(\bigcup_1^n L_i[(s_i = \infty) - V^*_\eta])$$

$$< c\epsilon + c\, P(A - V^*_\eta) \le c\epsilon + c\, P(V^*_{\delta_0} - V^*_\eta) < 2c\epsilon.$$

Hence

(2.7) $$\sum_1^n \int_{L_i[(s_i = \infty) - V^*_\eta]} x'_i < 2c\epsilon.$$

Since $q < \infty$, we can assume that δ_0 is so large that $P(V^*_{\delta_0} - V^*_\delta) < \epsilon^q$ for every $\delta > \delta_0$. Then

$$(2.8) \qquad \int_{V^*_{\delta_0} - V^*_\eta} |y'| \leq ||y'||_p \, \epsilon \leq c\epsilon.$$

Put $D = \bigcup_1^n L_i(s_i < \infty)$. Since $V^*_\eta \subset (s_i = \infty)$ for each i and $D \subset A \subset V^*_{\delta_0}$,

$$\int_D y' \leq \int_{A-V^*_\eta} |y'| \leq \int_{V^*_{\delta_0} - V^*_\eta} |y'| \leq c\epsilon.$$

By (2.3),

$$\sum_1^n \int_{L_i(s_i < \infty)} y' - \int_D y' \leq ||\sum_1^n I(L_i) - I(A)||_q ||y'||_p < c\epsilon.$$

Hence

$$(2.9) \qquad \sum_1^n \int_{L_i(s_i < \infty)} y' < 2c\epsilon.$$

From (2.9), (2.7) and (2.6),

$$(2.10) \qquad d \sum_1^n P(L_i) \leq 4c\epsilon + \sum_1^n \int_{L_i V^*_\eta} x_i'.$$

Similarly,

$$(2.11) \qquad a \sum_{n+1}^m P(L_j) \geq -4c\epsilon + \sum_{n+1}^m \int_{L_j V^*_\eta} x_j'' .$$

Put $L_1' = L_1$ and $L_i' = L_i - \bigcup_1^{i-1} L_k'$ for $i = 2,\ldots,n$ and $L'_{n+1} = L_{n+1}$ and $L_j' = L_j - \bigcup_{n+1}^{j-1} L_k'$ for $j = n+2,\ldots,m$. Define $z' = x_i'$ on each L_i' and $z'' = x_j''$ on each L_j'. Then

$$(2.12) \qquad \sum_1^n \int_{L_i V^*_\eta} x_i' \leq \int_{AV^*_\eta} z' + c\Big[\sum_1^n P(L_i) - P(A)\Big] \leq \int_{AV^*_\eta} z' + c\epsilon$$

Similarly,

(2.13) $$\sum_{n+1}^{m} \int_{L_j V^*_\eta} x''_j \geq \int_{BV^*_\eta} z'' - c\epsilon.$$

Hence

(2.14) $$\int_{AV^*_\eta} z' - \int_{BV^*_\eta} z'' \leq c P[(A-B)V^*_\eta] \leq c \; P(A-B)$$

$$\leq P^*(AV-B) + P^*(A-V) < 2\epsilon.$$

From (2.10)-(2.14), we have

(2.15) $$d \sum_{1}^{n} P(L_i) - a \sum_{n+1}^{m} P(L_j) < 12 \; c\epsilon.$$

Thus we completed the proof.

Theorem 2. Let $(\mathcal{F}_\delta, \Delta)$ satisfy the Vitali condition V_q and $(x_\delta, \mathcal{F}_\delta, \Delta)$ be a martingale with $\sup_\Delta ||x_\delta||_p < \infty$, where $p \geq 1$ and $p^{-1} + q^{-1} = 1$. Then x_δ converges a.e.

Proof. For $p = 1$, it follows immediately from Theorem 4.2 of [1] that $\lim_\Delta x_\delta$ exists a.e., and for $p > 1$ Theorem 1 states that $\lim_\Delta x_\delta$ exists a.e. where both $\lim_\Delta \sup x_\delta$ and $\lim_\Delta \inf x_\delta$ are finite. Hence we need only to prove that under the conditions of Theorem 2, both $\lim_\Delta \sup x_\delta$ and $\lim_\Delta \inf x_\delta$ are finite a.e.

Assume that $V = (\lim_\Delta \sup x_\delta = \infty)$ and $P^*(V) > a > 0$. Then by V_q, for any $0 < K < \infty$, $\epsilon > 0$, and $\delta_0 \in \Delta$, there exist $\delta_1, \delta_2, \ldots, \delta_m$ and $\mathcal{F}_{\delta_i}$-sets $L_i \subset [x(\delta_i) > K]$ such that $\delta_i > \delta_0$ and

(2.16) $$P(A) > a, \quad || \sum_{1}^{m} I(L_i) - I(A)||_q < \epsilon,$$

where $A = \bigcup_{1}^{m} L_i$. Take $\eta > \delta_i$ $(i = 1,2,\ldots,m)$. Then

$$K_a \leq \sum_{1}^{m} \int_{L_i} x(\delta_i) = \sum_{1}^{m} \int_{L_i} x(\eta) \leq \left\|\sum_{1}^{m} I(\,'_i) - I(A)\right\|_q \| x(\eta)\|_p + \| x(\eta)\|_p$$

$$\leq (1 + \epsilon)\| x(\eta)\|_p .$$

Hence we arrive at a contradiction and $P(V) = 0$. Similarly, $P(\liminf_{\Delta} x_\delta = -\infty) = 0$

From the previous proofs, immediately we have:

Corollary 1. Both Theorems 1 and 2 hold, if we replace V_q by V_q^*, sup by ess sup and convergence by essential convergence.

Corollary 1 completes a theorem due to Krickeberg [5; Theorem 3.5] on essential convergence of martingales of decreasing stochastic basis.

3. A convergence theorem of martingales generated by cell function.

Let $\mathcal{S}$ be a family of $\mathcal{F}$-sets with positive measures. Each element in is called a cell. A partition of a set $X \subset \Omega$ is a sequence of non-overlapping cells I_n with $\bigcup_1^\infty I_n = X$ and any cell meets at most a finite number of I_n. For a family $\mathcal{G}$ of cells, each cell in $\mathcal{G}$ is called a $\mathcal{G}$-cell. $A(\mathcal{G})$ will be the union of all $\mathcal{G}$-cells, $\mathcal{G}^u$ the family of cells which are finite unions of $\mathcal{G}$-cells, and for a set X, $\mathcal{G}X$ is the family of all $\mathcal{G}$-cells which are subsets of X. A complex $\mathcal{K}$ is a finite family of non-overlapping cells. For a complex $\mathcal{K}$, define $P(\mathcal{K}) = P(A(\mathcal{K}))$. For two families $\mathcal{G}$ and $\mathcal{H}$ of cells, if $\mathcal{G} \subset \mathcal{H}^u$, we say that $\mathcal{H}$ refines $\mathcal{G}$, or $\mathcal{H}$ is $\mathcal{G}$-fine, denoted by $\mathcal{G} < \mathcal{H}$. For two complexes $\mathcal{K}$ and $\mathcal{K}^b$, $\mathcal{K}^b$ is said to be a bordering complex of $\mathcal{K}$, if every $\mathcal{K}$-cell is contained in some $\mathcal{K}^b$-cell and no $\mathcal{K}^b$-cell is contained in $\mathcal{K}^u$ (or equivalently $A(\mathcal{K})$). For a cell I, a partition η of I is said to be p-bordering $(p \geq 1)$, if for each cell $J \in \eta^u$ and each complex $\mathcal{K} \subset \eta^u J$ with $A(\mathcal{K}) \neq J$, there exists a bordering complex $\mathcal{K}^b$ of $\mathcal{K}$ with $\mathcal{K}^b \subset \eta^u J$ and $P(\mathcal{K}^b) \leq p\, P(\mathcal{K})$. $\mathcal{S}$ will be said to have the p-bordering property, if to every cell I and every complex $\mathcal{K}$ of subcells of I, there corresponds a $\mathcal{K}$-fine p-bordering partition of I.

Assume that the family Λ of all partitions λ of Ω forms a directed set with respect to the order $>$ (refinement). For each $\lambda \in \Lambda$, let $\mathcal{F}_\lambda$ be the σ-algebra generated by the λ-cells.

Theorem 3. Let $(x_\lambda, \mathcal{F}_\lambda, \Lambda)$ be a martingale and $\mathcal{J}$ have the p-bordering property with $1 < p < \infty$. Let B be an $\mathcal{F}_{\lambda_0}$-cell $V = [\sup_{\lambda > \lambda_0} |x_\lambda| < b]$ for $0 < b < \infty$, and $c = 2pb$. For any given $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots, \lambda_m$ in Λ $(\lambda > \lambda_1)$ and $\mathcal{F}_{\lambda_i}$-sets $L_i \subset BV^*_{\lambda_i}$, there exists $\eta > \lambda_i$, $i = 1, 2, \ldots, m$ in Λ such that

$$(3.1) \quad \int_{L_i} x(\lambda_i) \leq cP[L_i(x(\eta) \geq b)] + \int_{L_i[x(\eta) < b]} x^{(i)}(\eta), \; i=1,\ldots,n,$$

$$(3.2) \quad \int_{L_j} x(\lambda_j) \geq -cP[L_j(x(\eta) \leq -b)] + \int_{L_j[x(\eta) > -b]} x^{(j)}(\eta), \; j=n+1,\ldots,m,$$

where

$$(3.3) \quad x^{(i)}_\eta(\omega) = x_\eta(\omega) = x^{(j)}_\eta(\omega) \text{ if } \omega \in I \in \eta, \quad IV \neq \emptyset,$$
$$i = 1, \ldots, n; \; j = n+1, \ldots, m,$$

$$(3.4) \quad x^{(i)}_\eta(\omega) = c = -x^{(j)}_\eta(\omega) \text{ if } \omega \in I \in \eta, \; IV = \emptyset, \; i=1,\ldots,n;$$
$$j = n+1, \ldots, m.$$

Proof. We can and will assume that each L_i is an $\mathcal{F}_{\lambda_i}$-cell. Let η' be a partition of Ω such that $\eta' > \lambda_i$, $i = 1, \ldots, n, \ldots, m$. Let $\mathcal{G} = \eta' B$. Then $\mathcal{G}$ is a complex and $L_i \in \mathcal{G}^u$ for each $i = 1, \ldots, m$. By the p-bordering property of $\mathcal{J}$, there exists a $\mathcal{G}$-fine, p-bordering partition δ of B. Put $\eta = \eta'(\Omega - B) \cup \delta$. Then, $\eta \in \Lambda$ and $\eta > \eta' > \lambda_i$, $i = 1, \ldots, m$. For each $i = 1, \ldots, n$, let $\mathcal{K}_i = [I | I \in \eta L_i, IV = \emptyset]$. If $\mathcal{K}_i = \emptyset$, then

$$\int_{L_i} x(\lambda_i) = \int_{L_i} x(\eta) = \int_{L_i[|x(\eta)| < b]} x(\eta) = \int_{L_i[|x(\eta)| < b]} x^{(i)}(\eta)$$

$$[illegible] \int_{L_i[\ldots < b]} x^{(i)}(\eta).$$

If $\mathcal{K}_i = L_i$, then since $L_i \subset V^*_{\lambda_i}$,

$$\int_{L_i} x(\lambda_i) \leq c\,P(L_i) \ldots [illegible] \ldots L_i(x(\eta) \geq b)] + c\,P\,[L_i(x(\eta) \leq -b)]$$

$$\leq c\,P[L_i(x(\eta) \geq b)] + \int_{L_i[x(\eta) < b]} x^{(i)}(\eta) = cP[L_i(x(\eta) \geq b)] +$$

$$+ \int_{L_i[x(\eta) < b]} x^{(i)}(\eta).$$

Now assume that [illegible] $\neq 0$ and $\mathcal{K}_i \neq L_i$. Since $A(\mathcal{K}_i) \neq L_i \in \delta^u$ and $\mathcal{K}_i \subset \delta^u L_i$, by the ρ-bordering property of δ, there exists a complex $\mathcal{K}^b_i \subset \delta^u L_i$ such that every $\mathcal{K}_i$-cell is contained in some $\mathcal{K}^b_i$-cell, $IV \neq 0$ for every $\mathcal{K}^b_i$-cell I, and that $P(\mathcal{K}^b_i) \leq pP(\mathcal{K}_i)$. Hence

$$\int_{A(\mathcal{K}_i)} x(\eta) = \int_{A(\mathcal{K}^b_i)} x(\eta) - \int_{A(\mathcal{K}^b_i) - A(\mathcal{K}_i)} x(\eta)$$

$$\leq bP(\mathcal{K}^b_i) + b[P(\mathcal{K}^b_i) - P(\mathcal{K}_i)] \leq 2bP(\mathcal{K}^b_i) \leq cP(\mathcal{K}_i).$$

Therefore

$$\int_{L_i} x(\lambda_i) = \int_{L_i} x(\eta) = \int_{A(\mathcal{K}_i)} x(\eta) + \int_{L_i - A(\mathcal{K}_i)} x(\eta)$$

$$\leq cP(\mathcal{K}_i) + \int_{L_i - A(\mathcal{K}_i)} x(\eta)$$

$$\leq cP\,[A(\mathcal{K}_i)(|x(\eta)| \geq b)] + cP[A(\mathcal{K}_i)(|x(\eta)| < b)] +$$

$$\int_{[L_i - A(\mathcal{K}_i)][|x(\eta)| < b]} x(\eta) \leq cP[L_i(|x(\eta)| < b)] +$$

$$+ \int L_i[|x(\eta)| < b] x^{(i)}(\eta).$$

Since by (3.4)

$$\int L_i[x(\eta) \leq - b] \; x^{(i)}(\eta) = cP[L_i(x(\eta) \leq - b)],$$

$$\int_{L_i} x(\lambda_i) \leq cP[L_i(x(\eta) \geq b)] + \int L_i[x(\eta) < b] \; x^{(i)}(\eta).$$

Similarly we can prove (3.2).

Theorem 4. Let $(x_\lambda, \mathcal{F}_\lambda, \Lambda)$ be a martingale satisfying the weak Vitali condition V_1 and $_\lambda\mathcal{L}$ have the p-bordering property with $1 < p < \infty$. Then x_λ converges a.e. where $\sup|x_\lambda| < \infty$.

Proof. Theorem 3 states that $(x_\lambda, \mathcal{F}_\lambda, \Lambda)$ satisfies the condition $(A, b)_\infty$ for every $b > 0$. Therefore, Theorem 4 follows from Theorem 1 immediately.

Theorem 3 includes Theorem II of Rutowitz' [7], which in turn (See [7; p.29]) includes a theorem of Ward [8; p.141].

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