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On the Non-Central Multivariate Beta Distribution and the Moments of Traces of Some Matrices

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1. Introduction and summary. Let A_1 and A_2 be two positive definite symmetric matrices of order p, A_1 having a Wishart distribution [3, 15] with f_1 degrees of freedom and A_2 having an independent non-central Wishart distribution with f_2 degrees of freedom, corresponding to the linear case [1,2]. Now let

$$A_1 = C\ L\ C'$$

where C is a lower triangular matrix such that

$$A_1 + A_2 = C\ C' \quad .$$

It has been shown [6] that the density function of L is given by

$$(1.1)\quad f(L) = Ke^{-\lambda^2/2}\ {}_1F_1\{\tfrac{1}{2}(f_1+f_2), \tfrac{1}{2}f_2, \tfrac{1}{2}\lambda^2(1-\ell_{11})\}|L|^{(f_1-p-1)/2}|I-L|^{(f_2-p-1)/2},$$

where

$$K = \pi^{-p(p-1)/4} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(f_1+f_2+1-i)]/\{\Gamma[\tfrac{1}{2}(f_1+1-i)]\Gamma[\tfrac{1}{2}(f_2+1-i)]\} ,$$

λ^2 is the single non-centrality parameter in the linear case, ℓ_{11} is the element in the top left corner of the L matrix, and ${}_1F_1$ denotes the confluent hypergeometric function.

In this paper, the density function of L given by (1.1) has been observed to be a product of density functions of $p(p+1)/2$ independent beta variables, explicit expressions for these variables being given for $p = 2, 3, 4$ and 5. In view of the independence of the beta variables, it has been shown how the moments of the trace of L (say $W^{(p)}$) and of I-L (say $V^{(p)}$, which is actually Pillai's $V^{(s)}$ criterion with $s = p$ [8]) can be computed from those of the beta variables. Again, if we denote the characteristic roots of I-L by θ_i $(i = 1, 2, \ldots, p)$, a method has been given for computing the moments of $U^{(2)} = \sum_{i=1}^{2} [\theta_i/(1-\theta_i)] = \sum_{i=1}^{2} \lambda_i$ (a constant times Hotelling's T_o^2, $s = 2$), [8], also from those of the independent beta variables. The case of $p = 2$ has been considered in detail, deriving the first four moments of $W^{(2)}$, $V^{(2)}$ and $U^{(2)}$ and suggesting approximate distributions for them.

In addition, for tests of the hypothesis: H_o: $\lambda = o$ against H_1: $\lambda > o$ based on the three criteria, $V^{(2)}$, $U^{(2)}$ and Wilks' criterion, $\Lambda = \prod_{i=1}^{2} (1-\theta_i)$, [16] comparison of power functions has been carried out for different values of f_1 and f_2 using the moments of these criteria. Further, such comparison has been extended to include also Roy's largest root criterion in testing the hypothesis $H_o: \rho = o$ against $H_1: \rho > o$ where ρ is the single non-null population canonical correlation coefficient.

2. Independent beta variables. Let

$$L = T\,T'$$

where T is a lower triangular matrix $[t_{ij}]$. It has been shown [4] that then the diagonal elements t_{ii} are independently distributed and that $t_{ii}^2 (i = 2, 3, \ldots, p)$ follows the distribution

$$(2.1) \qquad f_i(t_{ii}^2) = (t_{ii}^2)^{\frac{1}{2}(f_1+1-i)-1}(1-t_{ii}^2)^{\frac{1}{2}f_2-1}/\beta\{\tfrac{1}{2}(f_1+1-i),\tfrac{1}{2}f_2\}$$

$$(0 \leq t_{ii}^2 \leq 1),$$

while t_{11}^2 is distributed as

$$(2.2) \qquad f_1(t_{11}^2) = \frac{e^{-\lambda^2/2}(t_{11}^2)^{\frac{f_1}{2}-1}(1-t_{11}^2)^{\frac{f_2}{2}-1}\,{}_1F_1\{\tfrac{1}{2}(f_1+f_2),\tfrac{1}{2}f_2,\tfrac{1}{2}\lambda^2(1-t_{11}^2)\}}{\beta(\tfrac{1}{2}f_1,\ \tfrac{1}{2}f_2)}$$

$$(0 \leq t_{11}^2 \leq 1) \quad .$$

(i) <u>$p = 2$</u>. Now, if $p = 2$, it can be shown that

$$(2.3) \qquad f(\ell_{11},\ell_{22},\ell_{21}) = f_1(u_{11})f_2(u_{22})f_{21}(u_{21})$$

where

$$(2.4) \qquad u_{11} = t_{11}^2,\ u_{22} = t_{22}^2 \text{ and } u_{21} = t_{21}^2/\{(1-t_{11}^2)(1-t_{22}^2)\}\ ,$$

$f_1(u_{11})$ is given by (2.2), $f_2(u_{22})$ by (2.1) with $i = 2$, and

$$(2.5)\qquad f_{21}(u_{21}) = u_{21}^{\frac{1}{2}-1}(1-u_{21})^{\frac{1}{2}(f_2-1)-1}/\beta\{\tfrac{1}{2}, \tfrac{1}{2}(f_2-1)\},\ (0 \le u_{21} \le 1) \quad .$$

Thus, from (2.3) it may be seen that u_{11}, u_{22} and u_{21} are independently distributed.

(ii) <u>p = 3</u>. When p = 3, it can be shown that

$$(2.6)\quad f(\ell_{11},\ell_{22},\ell_{33},\ell_{21},\ell_{32},\ell_{31}) = f_1(u_{11})f_2(u_{22})f_3(u_{33})f_{21}(u_{21})f_{21}(u_{32})f_{31}(v_{31})$$

where u's are defined in a similar manner as in (2.4), v_{31} is defined by

$$(2.7)\qquad v_{31} = (\sqrt{u_{31}} + \sqrt{u_{21}u_{32}u_{22}})^2/[(1-u_{21})(1-u_{32})] \quad ,$$

$f_1(u_{11})$ follows (2.2), $f_i(u_{ii})(i = 2, 3)$ is given in (2.1), $f_{21}(u_{21})$ and $f_{21}(u_{32})$ both follow the form as in (2.5) and $f_{31}(v_{31})$ is given by

$$(2.8)\qquad f_{31}(v_{31}) = v_{31}^{\frac{1}{2}-1}(1-v_{31})^{\frac{1}{2}(f_2-2)-1}/\beta\{\tfrac{1}{2}, \tfrac{1}{2}(f_2-2)\},\ (0 \le v_{31} \le 1) \quad .$$

(iii) <u>p = 4</u>. Now, if p = 4,

$$f(L) = \left[\prod_{i=1}^{4} f_i(u_{ii})\right] f_{21}(u_{21})f_{21}(u_{32})f_{21}(u_{43})f_{31}(v_{31})f_{31}(v_{42})f_{41}(w_{41}),$$

where u's are similarly defined as before, v_{31} is given in (2.7), v_{42} is given by

(2.9) $$v_{42} = (\sqrt{u_{42}} + \sqrt{u_{43}u_{32}u_{33}})^2 / [(1-u_{32})(1-u_{43})] \quad ,$$

(2.10) $$w_{41} = (\sqrt{v_{41}} + \sqrt{v_{31}\ z_{42}})^2 / [(1-v_{31})(1-v_{42})] \quad ,$$

where

(2.11) $$v_{41} = (\sqrt{u_{41}} + \sqrt{u_{42}u_{21}u_{22}})^2 / [(1-u_{21})(1-u_{43})] \quad ,$$

(2.12) $$z_{42} = (\sqrt{u_{42}u_{32}} + \sqrt{u_{43}u_{33}})^2 / [(1-u_{32})(1-u_{43})] \quad ,$$

and where $f_1(u_{11})$ as before is given by (2.2), $f_i(u_{ii})(i = 2, 3, 4)$ by (2.1), $f_{21}(u_{21})$, $f_{21}(u_{32})$ and $f_{21}(u_{43})$ follow the form (2.5), $f_{31}(v_{31})$ and $f_{31}(v_{42})$ follow the form (2.8) and

(2.13) $$f_{41}(w_{41}) = w_{41}^{\frac{1}{2}-1}(1-w_{41})^{\frac{1}{2}(f_2-3)-1} / \beta\left(\tfrac{1}{2}, \tfrac{1}{2}(f_2-3)\right), \ (0 \le w_{41} \le 1) \quad .$$

(iv) <u>p = 5</u>. When $p = 5$,

(2.14) $$f(L) = \left[\prod_{i=1}^{5} f_i(u_{ii})\right]\left[\prod_{i=1}^{4} f_{21}(u_{i+1,i})\right]\left[\prod_{i=1}^{3} f_{31}(v_{i+2,i})\right] \times$$

$$\left[\prod_{i=1}^{2} f_{41}(w_{i+3,i})\right] f_{51}(x_{51})$$

where u's are defined as before, v_{31} and v_{42} are given by (2.7) and (2.9) respectively v_{53} is given by

$$v_{53} = (\sqrt{u_{53}} + \sqrt{u_{43}u_{54}u_{44}})^2/[(1-u_{43})(1-u_{54})] , \tag{2.15}$$

w_{41} is defined in (2.10), w_{52} is given by

$$w_{52} = (\sqrt{v_{52}} + \sqrt{v_{42}\, z_{53}})^2/(1-v_{42})(1-v_{53}) , \tag{2.16}$$

where

$$v_{52} = (\sqrt{u_{52}} + \sqrt{u_{32}u_{53}u_{33}})^2/[(1-u_{32})(1-u_{54})] \tag{2.17}$$

$$z_{53} = (\sqrt{u_{53}u_{43}} + \sqrt{u_{54}u_{44}})^2/[(1-u_{43})(1-u_{54})] , \tag{2.18}$$

and where x_{51} is given by

$$x_{51} = \frac{[(\sqrt{v_{51}} + \sqrt{v_{31}z_{52}})(1-v_{42}) + (\sqrt{v_{41}} + \sqrt{v_{31}z_{42}})(\sqrt{v_{42}v_{52}} + \sqrt{z_{53}})]^2}{(1-v_{31})(1-v_{42})^2(1-v_{53})(1-w_{41})(1-w_{52})} \tag{2.19}$$

and where

$$v_{51} = (\sqrt{u_{51}} + \sqrt{u_{21}u_{52}u_{22}})^2/[(1-u_{21})(1-u_{54})] , \tag{2.20}$$

(2.21) and

$$z_{52} = (\sqrt{u_{52}u_{32}} + \sqrt{u_{53}u_{33}})^2/[(1-u_{32})(1-u_{54})] .$$

Here again $f_1(u_{11})$ is given by (2.2), $f_i(u_{ii})(i = 2, 3, 4, 5)$ is given by (2.1), $f_{21}(u_{21})$, $f_{21}(u_{32})$, $f_{21}(u_{43})$ and $f_{21}(u_{54})$ follow the form (2.5), $f_{31}(v_{31})$, $f_{31}(v_{42})$ and $f_{31}(v_{53})$ follow the form (2.8), $f_{41}(w_{41})$ and $f_{41}(w_{52})$ follow the form (2.13) and $f_{51}(x_{51})$ is given by

(2.22) $$f_{51}(x_{51}) = x_{51}^{\frac{1}{2}-1}(1-x_{51})^{\frac{1}{2}(f_2-4)-1}/\beta(\tfrac{1}{2},\tfrac{1}{2}(f_2-4)),\ 0 \le x_{51} \le 1.$$

(v) General case (p). In this subsection, for convenience, let us relabel $u_{i+1,i}$ as $u^{(2)}_{i+1,i}$, $i = 1, 2, \dots p-1$; $v_{i+2,i}$ as $u^{(3)}_{i+2,i}$ $i = 1, \dots, p-2$; $w_{i+3,i}$ as $u^{(4)}_{i+3,i}$, $i = 1, \dots, p-3$; $x_{i+4,i}$, as $u^{(5)}_{i+4,i}$, $i = 1, \dots, p-4$; etc. Now from (2.4),

(2.23) $$1 - u^{(2)}_{21} = |I - L|(p=2)/[(1-u_{11})(1-u_{22})] ,$$

where $L = T\,T'$. Further, $u^{(2)}_{32}$ is obtained from $u^{(2)}_{21}$ by adding simultaneously unity to both suffixes of each of the t's involved in $u^{(2)}_{21}$, which is reflected in the notation 32 which replaces 21. Similarly $u^{(2)}_{43}$ is obtained from $u^{(2)}_{32}$, $u^{(2)}_{54}$ from $u^{(2)}_{43}$ etc. Again,

(2.24) $$1 - u^{(3)}_{31} = \frac{|I - L|(p = 3)}{(1-u_{11})(1-u_{22})(1-u_{33})(1-u^{(2)}_{21})(1-u^{(2)}_{32})} .$$

Further, $u_{42}^{(3)}$ is obtained from $u_{31}^{(3)}$ by increasing as before both suffixes in each of the t's in $u_{31}^{(3)}$ by unity. Similarly $u_{53}^{(3)}$ is obtained from $u_{42}^{(3)}$ etc. Following this pattern, it is easy to see that

$$(2.25) \qquad 1 - u_{p1}^{(p)} = \frac{|I - L|}{\prod_{i=1}^{p}(1-u_{ii}) \prod_{i=1}^{p-1} (1-u_{i+1,i}^{(2)}) \prod_{i=1}^{p-2} (1-u_{i+2,i}^{(3)}) \cdots \prod_{i=1}^{2}(1-u_{i+p-2,i}^{(p-1)})} .$$

Hence it may be seen that in the case of p variables *

$$(2.26) \qquad f(L) = \left[\prod_{i=1}^{p} f_i(u_{ii})\right]\left[\prod_{i=1}^{p-1} f_{21}(u_{i+1,i}^{(2)})\right] \times$$

$$\left[\prod_{i=1}^{p-2} f_{31}(u_{i+2,i}^{(3)})\right] \cdots \left[\prod_{i=1}^{2} f_{p-1,1}(u_{i+p-2,i}^{(p-1)})\right] f_{p1}(u_{p1}^{(p)}) ,$$

where

$$(2.27) \qquad f_{j1}(u_{i+j-1,i}^{(j)}) = (u_{i+j-1,i}^{(j)})^{\frac{1}{2}-1} (1-u_{i+j-1,i}^{(j)})^{\frac{1}{2}(f_2-j+1)-1} / \beta(\tfrac{1}{2}, \tfrac{1}{2}(f_2-j+1)) ,$$

$$0 \leq u_{i+j-1,i}^{(j)} \leq 1, \; j = 2, 3, \ldots, p.$$

Further, it may be noted that K in (1.1) equals

$$(2.28) \qquad \prod_{i=1}^{p} \beta(\tfrac{1}{2}(f_1+1-i), \tfrac{1}{2}(f_2))[\beta(\tfrac{1}{2}, \tfrac{1}{2}(f_2-i))]^{p-i} .$$

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*Since this paper was written, a theorem was proved to establish this. (See Khatri, C.G. and Pillai, K.C.S. (1965) "Some Results on the Non-Central Multivariate Beta Distribution and Moments of Traces of Two Matrices", Ann. Math. Statist., 36, October Issue.)

3. Traces of some matrices as functions of independent beta variables.

First, consider the trace of L when $p = 2$. Noting that

$$\ell_{11} + \ell_{22} = t_{11}^2 + t_{22}^2 + t_{21}^2 \tag{3.1}$$

and using (2.4) we get

$$W^{(2)} = \ell_{11} + \ell_{22} = u_{11} + u_{22} + u_{21}(1-u_{11})(1-u_{22}) \ . \tag{3.2}$$

Similarly

$$V^{(2)} = 2 - W^{(2)} = (1-u_{11}) + (1-u_{22}) - u_{21}(1-u_{11})(1-u_{22}) \ . \tag{3.3}$$

When $p = 3$,

$$\begin{aligned} W^{(3)} = {} & u_{11} + u_{22} + u_{33} + u_{21}(1-u_{11})(1-u_{22}) + u_{32}(1-u_{22})(1-u_{33}) \\ & + (1-u_{11})(1-u_{33})\Big[v_{31}(1-u_{21})(1-u_{32}) + u_{21}u_{22}u_{32} \\ & -2\sqrt{v_{31}(1-u_{21})(1-u_{32})u_{21}u_{22}u_{32}}\Big] \end{aligned} \tag{3.4}$$

and $V^{(3)} = 3 - W^{(3)}$.

Similarly, $W^{(4)}$, $V^{(4)}$, and $W^{(5)}$ and $V^{(5)}$ can be expressed explicitly as functions of independent beta variables.

Now consider $U^{(2)} = \sum_{i=1}^{2} \lambda_i$. It may be seen that

(3.5) $$U^{(2)} = \sum_{i=1}^{2} [\theta_i/(1-\theta_i)] = \{[(1-\theta_1) + (1-\theta_2)]/[(1-\theta_1)(1-\theta_2)]\} -2 .$$

Noting that $(1-\theta_1) + (1-\theta_2) = W^{(2)}$ and $(1-\theta_1)(1-\theta_2) = u_{11}u_{22}$ we get

(3.6) $$U^{(2)} = \frac{1-u_{11}}{u_{11}} + \frac{1-u_{22}}{u_{22}} + \frac{u_{21}(1-u_{11})(1-u_{22})}{u_{11}\, u_{22}} .$$

4. Moments of $W^{(2)}$, $V^{(2)}$ and $U^{(2)}$. The first four moments of $W^{(2)}$ will be given by

(4.1) $$\mu_1'(W^{(2)}) = \left\{2f_1 e^{-\lambda^2/2}/(\nu-1)\right\} \sum_{i=o}^{\infty} a_i \, (\tfrac{1}{2}\lambda^2)^i/i!$$

where

(4.2) $$a_i = (\nu+i-1)/g_i$$

$$\nu = (f_1+f_2) \quad \text{and} \quad g_i = \nu + 2i .$$

(4.3) $$\mu_2'(W^{(2)}) = \{4f_1 e^{-\lambda^2/2}/(\nu^2-1)\} \sum_{i=o}^{\infty} b_i (\tfrac{1}{2}\lambda^2)^i/i!$$

where

(4.4) $b_i = \{f_1\nu^2+2(i+1)f_1^2+(i^2+3i-1)f_1+(2i+3)f_1f_2+f_2^2+(2i-1)f_2+2(i^2-1)\}/e_0$,

where $e_0 = g_i(g_i + 2)$.

(4.5) $\mu_3'(W^{(2)}) = [8f_1e^{-\lambda^2/2}/\{(\nu^2-1)(\nu+3)\}] \sum_{i=o}^{\infty} c_i(\frac{1}{2}\lambda^2)^i/i!$,

where

(4.6) $$c_i = e_1/e_2 ,$$

and where

$$\begin{aligned} e_1 = {} & f_1^2\nu^3+(3i+9)f_1^4+(6i+21)f_1^3f_2+(3i+15)f_1^2f_2^2+(3i^2+21i+25)f_1^3 \\ & + (3i^2+30i+41)f_1^2f_2+(i^3+18i^2+44i+15)f_1^2+(9i+18)f_1f_2^2 \\ & + 3f_1f_2^3+2f_2^3+(12i^2+39i+9)f_1f_2+6if_2^2 \\ & +(6i^3+30i^2+18i-26)f_1+(12i^2+6i-26)f_2+8i^3+12i^2-20i-24 , \end{aligned}$$

and

$$e_2 = g_i(g_i+2)(g_i+4) .$$

(4.7) $$\mu_4'(W^{(2)}) = [f_1 e^{-\lambda^2/2}/\{(\nu^2-1)(\nu+3)(\nu+5)\}] \sum_{i=0}^{\infty} d_i(\tfrac{1}{2}\lambda^2)^i/i!\ ,$$

where

(4.8) $$d_i = e_3/e_4\ ,$$

and where

$$e_3 = (\nu+5)[(f_1+2)(f_1+4)(\nu+1)(\nu+3)(\nu f_1+4f_1g_i+23f_1+6\nu-4g_i-30)$$

$$+ 4(f_1+2)(\nu+3)h_i(f_1\nu+3f_1g_i+19f_1+4\nu-3g_i-14)$$

$$+ 2(f_1^2-1)(g_i+4)(g_i+6)(2f_1g_i+3f_1\nu+13f_1+6\nu+6g_i+30)$$

$$+12(f_1-1)(g_i+6)h_i(f_1g_i+4f_1+g_i+3h_i+10)$$

$$+ 6h_i(h_i+2)(3f_1\nu+9f_1+6\nu+10h_i+58)]$$

$$+ (g_i+5)[(f_1-1)(g_i+6)\{(f_1(g_i+5)+h_i+4)(f_1(g_i+5)+12h_i+6)+45h_i(h_i+2)\}$$

$$+ 105h_i(h_i+2)(h_i+4)]\ ,$$

and

$$e_4 = g_i(g_i+2)(g_i+4)(g_i+6)\ ,$$

and where

$$h_i = f_2 + 2i\ .$$

It may be observed that the moments of $V^{(2)}$ can be obtained from those of $W^{(2)}$ using the relation $V^{(2)} = 2-W^{(2)}$, which is given in terms of the u's in (3.3).

Now consider the moments of $U^{(2)}$. From (3.6)

$$U^{(2)} = z_1 + z_2 + z_1 z_2 u_{21} \tag{4.9}$$

where $z_1 = (1-u_{11})/u_{11}$ and $z_2 = (1-u_{22})/u_{22}$.

From (2.2) we get

$$f(z_1) = e^{-\lambda^2/2} z_1^{\frac{1}{2}f_2-1} {}_1F_1\{\tfrac{1}{2}\nu, \tfrac{1}{2}f_2, (\tfrac{1}{2}\lambda^2 z_1/(1+z_1))\}/\{(1+z_1)^{\frac{\nu}{2}}\beta(\tfrac{1}{2}f_2, \tfrac{1}{2}f_1)\} . \tag{4.10}$$

Similarly from (2.1) ,

$$f(z_2) = z_2^{\frac{1}{2}f_2-1}/\{(1+z_2)^{\frac{1}{2}(\nu-1)}\beta(\tfrac{1}{2}f_2, \tfrac{1}{2}(f_1-1))\} . \tag{4.11}$$

Now using (4.10), (4.11) and (2.5) we obtain the first four moments of $U^{(2)}$ as follows:

$$\mu_1'(U^{(2)}) = \{2\, e^{-\lambda^2/2}/(f_1-3)\} \sum_{i=0}^{\infty} (f_2+i)(\tfrac{1}{2}\lambda^2)^i/i! , \tag{4.12}$$

$$= (2f_2+\lambda^2)/(f_1-3) .$$

Similarly

(4.13) $\mu_2'(U^{(2)}) = [\lambda^4(f_1-2)+4(\lambda^2+f_2)(f_1+f_2(f_1-3)-1)]/[(f_1-2)(f_1-3)(f_1-5)]$,

(4.14) $\mu_3'(U^{(2)}) = [\lambda^6(f_1-2)+3\lambda^4\{(f_1-2)(f_2+4)+f_2(f_1-6)+4\}$

$+(3\lambda^2+2f_2)(f_2+2)\{(f_1-2)(f_2+4)$

$+3(f_2(f_1-6)+4)\}]/[(f_1-2)(f_1-3)(f_1-5)(f_1-7)]$,

and

(4.15) $\mu_4'(U^{(2)})=[\lambda^8 b+\lambda^6\{(12+s_1)b+4B\}+\lambda^4\{(28+6s_1+s_2)b+12(f_2+4)B+6A\}$

$+\lambda^2\{(8+4s_1+2s_2+s_3)b+16(f_2+2)(f_2+4)B+12(f_2+2)A\}$

$+2f_2(f_2+2)\{(f_2+4)(f_2+6)b$

$+4(f_2+4)B+3A\}]/[(f_1-2)(f_1-3)(f_1-4)(f_1-5)(f_1-7)(f_1-9)]$,

where s_i is the ith $(i=1,2,3)$ elementary symmetric function in the arguments f_2, f_2+2, f_2+4 and f_2+6,

$A = f_2^2(f_1-6)(f_1-8)+2f_2(f_1-4)(f_1-6)+16f_1-72$,

$B = (f_1-4)(f_2(f_1-8)+6)$ and $b = (f_1-2)(f_1-4)$.

It may be observed that when $\lambda = 0$, the moments given in this section reduce to those obtained by Pillai [8], [9], [10], [11] .

5. Approximations to the distributions of $W^{(2)}$, $V^{(2)}$ and $U^{(2)}$.

On the basis of the moments presented in the preceding section, the following approximation to the distribution of $W^{(2)}$ is suggested for small values of λ:

$$(5.1) \quad g_1(W^{(2)}) = (W^{(2)})^{p_1-1}(1-W^{(2)}/2)^{q_1-1}/[2^{p_1}\beta(p_1,q_1)], 0 < W^{(2)} < 2 ,$$

where

$$p_1 = [(2K_1-K_2)K_1] \ / \ [2(K_2-K_1^2)] ,$$

$$q_1 = [(2-K_1)(2K_1-K_2)] \ / \ [2(K_2-K_1^2)] ,$$

where

$$K_1 = 2f_1\{1-(\lambda^2/2)/(\nu+2)\} \ / \ \nu$$

and

$$K_2 = 4f_1(f_1\nu+\nu-2)\{1-\lambda^2/(\nu+4)\}/\{(\nu-1)\nu(\nu+2)\} .$$

A comparison of the lower order moments from (5.1) with the respective exact ones may be made from Table 1.

Since $V^{(2)} = 2-W^{(2)}$, an approximation to the distribution of $V^{(2)}$ can be obtained from (5.1) in the following form:

$$(5.2) \quad g_2(V^{(2)})=(V^{(2)})^{q_1-1}(1-V^{(2)}/2)^{p_1-1}/[2^{q_1}\beta(q_1, p_1)], \ 0 < V^{(2)} < 2 .$$

Table 1

Moments (central) of $W^{(2)}$ from the exact and approximate distributions for different values of f_1 and f_2 and $\lambda = 2$.

Moments	$f_1 = 10$ $f_2 = 5$			$f_1 = 100$ $f_2 = 5$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	1.2134	1.1765	.9696	1.8708	1.8692	.9991
μ_2	0.0506	0.0578	1.1404	$0.0^{2}269$	$0.0^{2}284$	1.0560
μ_3	$-0.0^{2}151$	$-0.0^{2}229$	1.5230	$-0.0^{4}919$	$-0.0^{3}113$	1.2319
μ_4	$0.0^{2}731$	$0.0^{2}907$	1.2405	$0.0^{4}255$	$0.0^{4}303$	1.1900
$\sqrt{\mu_2}$	0.2250	0.2403	1.0679	0.0518	0.0533	1.0276
β_1	0.0175	0.0273	1.5639	0.4350	0.5605	1.2886
β_2	2.8510	2.7192	0.9538	3.5265	3.7632	1.0671
Moments	$f_1 = 20$ $f_2 = 20$			$f_1 = 100$ $f_2 = 80$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	0.9575	0.9524	.9947	1.0992	1.0989	0.9997
μ_2	0.0229	0.0232	1.0103	$0.0^{2}5419$	$0.0^{2}5425$	1.0012
μ_3	$0.0^{3}112$	$0.0^{3}100$	0.8899	$-0.0^{4}109$	$-0.0^{4}117$	1.0722
μ_4	$0.0^{2}151$	$0.0^{2}154$	1.0148	$0.0^{4}872$	$0.0^{4}874$	1.0020
$\sqrt{\mu_2}$	0.1514	0.1522	1.0052	0.0736	0.0737	1.0006
β_1	$0.0^{2}105$	$0.0^{3}806$	0.7678	$0.0^{3}748$	$0.0^{3}856$	1.1456
β_2	2.8849	2.8681	0.9942	2.9695	2.9688	0.9997
Moments	$f_1 = 5$ $f_2 = 20$			$f_1 = 5$ $f_2 = 100$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	0.3751	0.3704	0.9875	0.0935	0.0935	0.9991
μ_2	0.0208	0.0203	0.9782	$0.0^{2}162$	$0.0^{2}162$	0.9976
μ_3	$0.0^{2}160$	$0.0^{2}167$	1.0437	$0.0^{4}522$	$0.0^{4}530$	1.0136
μ_4	$0.0^{2}139$	$0.0^{2}136$	0.9806	$0.0^{4}102$	$0.0^{4}103$	1.0075
$\sqrt{\mu_2}$	0.1442	0.1426	0.9890	0.0403	0.0403	0.9988
β_1	0.2848	0.3314	1.1637	0.6368	0.6591	1.0350
β_2	3.2123	3.2920	1.0248	3.8780	3.9262	1.0124

Again, consider $U^{(2)}$. An approximation to the distribution of $U^{(2)}$ for $f_1 > f_2$ and which is good even for very small values of f_2 is given below:

$$(5.3) \quad g_3(U^{(2)}) = (U^{(2)})^{p_2-1} / \{(1+U^{(2)}/K_3)^{p_2+q_2+1} K_3^{p_2} \beta(p_2, q_2+1)\},$$

$$0 < U^{(2)} < \infty ,$$

where

$$p_2 = 2q_2 \,/\, \{q_2(h-1)-2h\}$$

$$q_2 = 2\{c^2(f_1-5)h-(c+d)^2(f_1-3)\}/\{c^2(f_1-5)(h+1)-2(c+d)^2(f_1-3)\} ,$$

$$K_3 = c\{q_2(h-1)-2h\}/\{2(f_1-3)\} ,$$

$$h = (c+1.99d)^3(f_1-3)/\{(c+d)^2(f_1-7)c\} ,$$

$$c = 2f_2+\lambda^2 \quad \text{and} \quad d = (f_1-f_2-1)/(f_1-2) .$$

A comparison of the moments from (5.3) with the respective exact ones may be made from Table 2.

6. Power functions of tests of hypothesis: $\lambda = 0$ against $\lambda > 0$ based on $V^{(2)}$, $U^{(2)}$ and Λ. Using the results on the moments of $W^{(2)}$ in section 4, and the relation $V^{(2)} = 2-W^{(2)}$, the central moments, μ_2, μ_3, and μ_4, and the moment quotients, β_1 and β_2, were computed for various values of f_1, f_2, and λ. Similar computations were made for $U^{(2)}$ and Wilks' criterion, using the expressions in section 4 for the

Table 2

Moments (central) of $U^{(2)}$ from the exact and approximate distributions for different values of $f_1 > f_2$ and $\lambda = 1,3,$ and 5

Moments	$f_1 = 10,\ f_2 = 2,\ \lambda = 1$ Exact	Approximate	Ratio (A/E)	$f_1 = 15,\ f_2 = 5,\ \lambda = 5$ Exact	Approximate	Ratio (A/E)
μ_1'	0.7143	0.7143	1.0000	2.9167	2.9167	1.0000
μ_2	0.5041	0.4760	0.9442	2.3937	2.1092	0.8812
μ_3	1.5792	1.5333	0.9709	8.0457	6.8781	0.8549
μ_4	25.7893	27.0736	1.0498	82.9146	67.0244	0.8084
$\sqrt{\mu_2}$	0.7100	0.6899	0.9717	1.5472	1.4523	0.9387
β_1	19.4703	21.8049	1.1199	4.7198	5.0415	1.0682
β_2	101.4935	119.5121	1.1775	14.4708	15.0656	1.0411
Moments	$f_1 = 50,\ f_2 = 10,\ \lambda = 1$ Exact	Approximate	Ratio (A/E)	$f_1 = 100,\ f_2 = 10,\ \lambda = 1$ Exact	Approximate	Ratio (A/E)
μ_1'	0.4468	0.4468	1.0000	0.2165	0.2165	1.0000
μ_2	0.0258	0.0253	0.9823	0.0^2532	0.0^2522	0.9798
μ_3	0.0^2373	0.0^2407	1.0911	0.0^3295	0.0^3309	1.0467
μ_4	0.0^2296	0.0^2314	1.0616	0.0^3112	0.0^3114	1.0114
$\sqrt{\mu_2}$	0.1605	0.1591	0.9911	0.0730	0.0722	0.9898
β_1	0.8126	1.0207	1.2561	0.5772	0.6724	1.1648
β_2	4.4566	4.9032	1.1002	3.9610	4.1731	1.0535
Moments	$f_1 = 100,\ f_2 = 20,\ \lambda = 3$ Exact	Approximate	Ratio (A/E)	$f_1 = 100,\ f_2 = 20,\ \lambda = 5$ Exact	Approximate	Ratio (A/E)
μ_1'	0.5052	0.5052	1.0000	0.6701	0.6701	1.0000
μ_2	0.0155	0.0140	0.9031	0.0252	0.0209	0.8292
μ_3	0.0^2116	0.0^2106	0.9199	0.0^2236	0.0^2186	0.7886
μ_4	0.0^3874	0.0^3736	0.8417	0.0^2231	0.0^2161	0.6995
$\sqrt{\mu_2}$	0.1246	0.1184	0.9503	0.1587	0.1446	0.9106
β_1	0.3578	0.4111	1.1488	0.3467	0.3782	1.0907
β_2	3.6295	3.7458	1.0320	0.3631	0.3694	1.0173

moments of the former, and deriving the expressions for the moments of the latter as the product of the respective moments of u_{11} and u_{22}.

For a given size α, using the β_1 and β_2 values computed for fixed f_1 and f_2 for $\lambda = 0$, the critical region was determined for each criterion referring to tables of "Percentage points of Pearson curves for β_1 and β_2 expressed in standardized measure" [7]. Further, for the same values of f_1 and f_2 and a value of $\lambda > 0$, the computed values of β_1 and β_2 were used to determine from the same table by interpolation the power of the test based on the critical region determined previously. The following table presents the results of these computations.

Table 3

Powers of tests of hypothesis: $\lambda = 0$ against $\lambda > 0$ based on $V^{(2)}$, $U^{(2)}$ and Λ.

f_1	f_2	α	λ	Power $V^{(2)}$	$U^{(2)}$	Λ
50	10	.005	1	.0076	.0076	.0113
50	10	.005	2	.0215	.0217	.0470
100	10	.01	1	.0156	.0156	.0217
100	20	.025	1	.0306	.0306	.0345
100	30	.025	1	.0300	.0300	.0303
100	50	.025	1	.0278	.0277	.0280
100	100	.025	1	.0270	.0270	.0270
50	50	.005	1	.0057	.0056	.0056
50	50	.005	2	.0085	.0083	.0085
50	50	.005	3	.0151	.0151	.0153

Table 3 shows that a) there is practically very little difference between the powers of tests based on $V^{(2)}$ and $U^{(2)}$ and b) for small values of f_2 Wilks' criterion seems to have marked power compared to both $V^{(2)}$ and $U^{(2)}$. This point needs further investigation.

7. Power functions for tests of hypothesis: $\rho = 0$ against $\rho > 0$ based on $V^{(2)}$, $U^{(2)}$, Λ and the largest root. In the case of relation between a p-set of variates, $x' = (x_1, \ldots, x_p)$, and a q-set, $y' = (y_1, \ldots, y_q)$, from a (p+q)-variate normal population, where there is only one non-null population canonical correlation coefficient, ρ, and $p \leqq q$, $(p+q) < n'$ where n' is the sample size,

$$(7.1) \qquad \lambda^2 = \rho^2 \sum_{t=1}^{\nu} y_{1t}^2 \,/\, (1-\rho^2)$$

where $y_{1t}(t=1,\ldots,\nu)$ are related to the sample observations of y_1, and y, here, is considered fixed [6]. Further, $f_2 = q$ and $f_1 = n'-q-1$ such that $\nu = f_1+f_2$. If, however, y is not fixed, then $\sum_{t=1}^{\nu} y_{1t}^2$ in λ^2 of (7.1) is a chi-square with ν degrees of freedom and, therefore, for obtaining the moments of $W^{(2)}$ in this case the following changes may be made in the moments of $W^{(2)}$ given in section 4:

$$(7.2) \qquad e^{-\frac{1}{2}\lambda^2} \longrightarrow (1-\rho^2)^{\nu/2}, \quad (\lambda^2)^i \longrightarrow (\rho^2)^i \quad \text{and}$$

$$(a_i, b_i, c_i, d_i) \longrightarrow [\nu(\nu+2) \ldots (\nu+2(i-1))]\,(a_i, b_i, c_i, d_i) \;.$$

Similar changes apply for Wilks' criterion. But for $U^{(2)}$, $(\lambda^2)^i$ is replaced by $(2\rho^2/(1-\rho^2))^i\Gamma(\frac{1}{2}\nu+1)/\Gamma(\frac{\nu}{2})$. Now for the test of the hypothesis: $\rho = 0$ against $\rho > 0$ using $V^{(2)}$, $U^{(2)}$ and Λ, powers were evaluated for $\rho = .05$ and $\rho = .1$ for certain values of f_1 and f_2 using the method discussed in the foregone section. For the largest root, the power was computed using Constantine's form of the distribution of the canonical correlation coefficients [4], [5] in the following manner:

First the joint distribution for $p = 2$ and a single nonzero ρ was obtained as a series of determinants using a lemma by Pillai [12] . Further taking into account the first seven terms of the series and integrating out the smallest root by employing Pillai's method [8, 10] , the following expression was obtained for the cdf of the largest canonical correlation coefficient, r_2^2 .

$$\Pr\{r_2^2 \le x\} = K_4 \Big\{-I_o(m+1,n+1)\Big[I(x;m,n)\{\sum_{j=o}^{6}(B_j x^{6-j}/(m+n+8-j))\}$$

$$-xI(x;m+1,n)\{\sum_{j=o}^{4}(C_j x^{4-j}/(m+n+7-j))\}$$

$$-x^2I(x;m+2,n)\{\sum_{j=o}^{2}(D_j x^{2-j}/(m+n+6-j))\}$$

$$-x^3I(x;m+3,n)E_o/(m+n+5)\Big]$$

$$+2I(x;2m+7,2n+1)\Big[\{B_o/(m+n+8)\}-\{C_o/(m+n+7)\}-\{D_o/(m+n+6)\}-\{E_o/(m+n+5)\}\Big]$$

$$+2I(x;2m+6,2n+1)\Big[\{B_1/(m+n+7)\}-\{C_1/(m+n+6)\}-\{D_1/(m+n+5)\}\Big]$$

$$+2I(x;2m+5,2n+1)\Big[\{B_2/(m+n+6)\}-\{C_2/(m+n+5)\}-\{D_2/(m+n+4)\}\Big]$$

$$+2I(x;2m+4,2n+1)\Big[\{B_3/(m+n+5)\}-\{C_3/(m+n+4)\}\Big]$$

$$+2I(x;2m+3,2n+1)\Big[\{B_4/(m+n+4)\}-\{C_4/(m+n+3)\}\Big]$$

$$+2I(x;2m+2,2n+1)\{B_5/(m+n+3)\}+2I(x;2m+1,2n+1)\{B_6/(m+n+2)\}\Big\}$$

where

$$f_1 = 2n+3,\ f_2 = 2m+3,$$

$$K_4 = (1-\rho^2)^{\nu/2}\ C(2,\ m,\ n)\ ,$$

$$C(2,m,n) = \Gamma(2m+2n+5)/\{4\Gamma(2m+2)\Gamma(2n+2)\}\ ,$$

$$I_o(m+1,n+1) = x^{m+1}(1-x)^{n+1}\ ,\ I(x;c',d') = \int_o^x \theta^{c'}(1-\theta)^{d'}d\theta\ ,$$

$$B_o = 231A_6, B_1 = 63A_5+(m+7)B_o/(m+n+8), B_2 = 35A_4+(m+6)B_1/(m+n+7)\ ,$$

$$B_3 = 5A_3+(m+5)B_2/(m+n+6),\ B_4 = 3A_2+(m+4)B_3/(m+n+5)\ ,$$

$$B_5 = A_1+(m+3)B_4/(m+n+4)\ ,\ B_6 = 1+(m+2)B_5/(m+n+3)\ ,$$

$$C_o = 105A_6,\ C_1 = 28A_5+(m+6)C_o/(m+n+7),\ C_2 = 15A_4+(m+5)C_1/(m+n+6)\ ,$$

$$C_3 = 2A_3+(m+4)C_2/(m+n+5)\ ,\ C_4 = A_2+(m+3)C_3/(m+n+4)$$

$$D_o = 21A_6,\ D_1 = 5A_5+(m+5)D_o/(m+n+6), D_2 = 2A_4+(m+4)D_1/(m+n+5)\ ,$$

$$E_o = 5A_6,\ A_1 = \nu^2\rho^2/2^2 f_2 ,$$
$$A_2 = [\nu(\nu+2)]^2\rho^4/[f_2(f_2+2)2^6] ,$$
$$A_3 = A_2(\nu+4)^2\rho^2/[(f_2+4)2.3!] ,$$
$$A_4 = A_3(\nu+6)^2\rho^2/\{2^6(f_2+6)\} ,$$
$$A_5 = A_4(\nu+8)^2\rho^2/\{2^2.5(f_2+8)\} , \text{ and}$$
$$A_6 = A_5(\nu+10)^2\rho^2/\{2^3.6(f_2+10)\} .$$

For $\rho = 0$, upper 1% points of the largest root were taken from Pillai's tables [11] for values of $m = 2$ and 5 and $n = 10$, 15,20,25, 30,40 and 60. Using these $x_{.99}$ values to determine the critical region, the powers of the largest root test were computed for $\rho = .05$ and $\rho = .1$ for values of m and n given above. These are shown in Table 4.

Table 4

Powers of the largest root test for testing $\rho = 0$ against $\rho = .05$ and $\rho = .1$ and $\alpha = .01$

	Power			
	$\rho = .05$		$\rho = .1$	
n	m = 2	m = 5	m = 2	m = 5
10	.010321	.010247	.011254	.010898
15	.010366	.010311	.011798	.011294
20	.010621	.010394	.012612	.011728
25	.010744	.010518	.013310	.012226
30	.010779	.010611	.013927	.012705
40	.011437	.010812	.016011	.013735
60	.011625	.011248	.019072	.016063

Now, a comparison of the powers of the test of hypothesis: $\rho = 0$ against $\rho > 0$ based on $V^{(2)}$, $U^{(2)}$, Λ and the largest root may be made from Table 5.

Table 5

Powers of the test $\rho = 0$ against $\rho > 0$ based on $V^{(2)}$, $U^{(2)}$, Λ and largest root for $\rho = .05$ and $\rho = .1$, and $\alpha = .01$

	Power							
	$\rho = .05$							
f_1	$f_2 = 7$				$f_2 = 13$			
	$V^{(2)}$	$U^{(2)}$	Λ	largest root	$V^{(2)}$	$U^{(2)}$	Λ	largest root
53	.0108	.0107	.0118	.0107	.0106	.0106	.0110	.0105
83	.0115	.0115	.0140	.0114	.0112	.0109	.0122	.0108
123	.0123	.0120	.0155	.0116	.0115	.0115	.0130	.0112
	$\rho = .1$							
53	.0135	.0135	.0190	.0133	.0125	.0125	.0135	.0122
83	.0165	.0165	.0280	.0160	.0144	.0142	.0180	.0137
123	.0202	.0200	.0440	.0190	.0170	.0170	.0240	.0161

Table 5 shows that a) the largest root has comparatively less power than the other test criteria b) $V^{(2)}$ and $U^{(2)}$ practically have equal power and c) Wilks' criterion as in the previous case seems to have greater power for the (small) values of f_2 considered here. Further investigation is being made to clear this point.

The author wishes to thank Mrs. Louise Mao Lui, Statistical Laboratory, Purdue University, for the excellent programming of the material for the computations in this paper carried out on the IBM 7094 Computer, Purdue University's Computer Science's Center.

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