A Simple Model for Studying Active Defense Against Ballistic Missiles

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Abstract: In the design of an active defense system it is necessary to consider the reliabilities of the intercept system. In this paper some interrelationships between these reliabilities and a defensive policy are examined.

Introduction. In defending a given target, two forms of rather independent, radically different types of defense are available. One, the passive type of defense, is essentially concerned with fallout and blast shelters for protecting a population and with hardness of sites for protecting targets of military importance. The other, the active defense, is concerned with the actual interception of attacking enemy warheads. In this paper only the active defense will be considered.

The objective of this paper is to study the role and the interactions of the various major components in an active defense system. By such a study it may be possible to attain an optimal active defense posture for each city or target.

1. Intercept reliability.

Let us denote the i-th target, whether defended or not by the index i. We will use the term "target" exclusively to denote a target to the enemy, such as a city or a missile site. Let us assume that an interceptor at target i can kill an incoming object, such as a warhead or a decoy, with a probability

*This paper was written almost entirely when the author was a consultant for the Office of Aerospace Research, Holloman Air Force Base, New Mexico denoted by p_i. This probability is assumed to be dependent only on radar accuracy, guidance reliability and lethal radius of the interceptor warhead and thus is not necessarily a function of the target. However, it is possible to increase the of p_i by, for example, improving the radar and computer equipment. This variation presumably would be available for every intercept installation and thus could be considered to be constant among the targets. On the other hand the value of p_i can possibly be altered by a change of the ratio of interceptors to radars. Thus p_i can probably be varied considerably from target to target by changes in both the quality and quantity of ground support equipment of interceptors.

On the other hand, for a given p_i it is also possible to vary the probability of killing an incoming warhead by a redundancy of interceptors. Suppose at target i k_i interceptors are committed to the interception of each object identified as a warhead. Then the probability of killing each object with one or more of the k_i interceptors is

$$1 - (1 - p_i)^{k_i} = 1 - p_i^*$$

where $p_i^* = (1 - p_i)^{k_i}$ is the probability that all interceptors miss; i.e., that the object slips through the defense. That is based, of course, on the assumptions that the interceptors act independently.

2. Discrimination reliability (q_i and r_i)

Generally, an interceptor is launched towards an object if and only if the object is identified as a warhead. It is possible that the identification is incorrect; that is, for a warhead to be classified as a decoy conversely. The classification of an object as a warhead or a decoy is dependent on the signature of the object, and the probability of the two types of classification error are related.

For example, suppose B is a random variable representing the signature of an object. Then there is a value, b_i^*, for each target, $i = 1, 2, \ldots,$ such that if $B \leq b_i^*$, the object is called a decoy and if $B \geq b_i^*$, the object is called a warhead. If F_w is the distribution function of the signature of a warhead, then we can define the quantity q_i as follows

$$q_i = \int_{b > b_i^*} dF_w = P \left[B > b_i^* / \text{object is a warhead} \right] ;$$

and if F_d is the distribution function of the signature of a decoy we define

$$r_i = \int_{b \leq b_i^*} dF_d = P \left[B \leq b_i^* / \text{object is a decoy} \right] .$$

The quantities of q_i and r_i are the conditional probabilities of correctly classifying a warhead and a decoy, respectively, and thus represent the reliability of the discrimination system.

It should be noted that changing b^* changes q_i and r_i. For example, if $b_i^* = +\infty$, then $q_i = 0$ and $r_i = 1$. This corresponds to the case of identifying everything as a decoy. On the other hand if $b_i^* = -\infty$, then $q_i = 1$ and $r_i = 0$. This corresponds to the case in which all objects are identified as warheads. The determination of b_i^* can have a profound effect on the active defense posture.

3. Detection reliability (s)

It should also be apparent that the ability of the system to detect an incoming warhead will be important in determining an active defense. We will let s be the probability of detecting an object. This will largely be dependent on early warning nets, such as the DEW line, and is probably constant from target to target.

4. Enemy policy (n_i or $p(n_i)$)

In order that a defense can be constructed, the extent of the threat must be known. Let us assume that n_i is the number of warheads the enemy will direct towards the target i. The specification of the value of each n_i is of course quite difficult. One would expect it to be related to the importance of the target, its geographical area, the size of the warhead used by the enemy and the total number of warheads available to him. The set of values of n_i, $i = 1, 2, \ldots$, finally determined will be known as the enemy policy.

It may be impossible to determine the value of n_i. In this case at least the a priori probabilities, $p_i(n)$, of n warheads at target i must be known.

5. Other parameters

In addition to the above the following parameters will be used.

$P(A)$: the probability an enemy missile aborts; that is, it does not reach its target.

$P(\bar{A}) = 1 - P(A)$: the probability an enemy missile does not abort.

$P(1)$: the probability a single enemy warhead destroys its target if it penetrates the defense.

α: the number of decoys per warhead

V_i: value of i-th target to the enemy. This value can be something simple like population, although this criterion tends to overlook such measures as political importance or the role of military targets. Even though the determination of this value is important this paper will overlook this problem and assume that the set of values, V_i, is known.

6. Defense objective: Equal unattractiveness

In devising a countering active defense policy some objective must be used as a basis for establishing a defense posture. One objective is based on the criterion of "equal unattractiveness." That is, assuming a given enemy policy for undefended targets an active defense posture is to be constructed so as to make a given set of targets no more attractive than a given value. This could be the value of the most attractive undefended target. That is, if $V_1 \geq V_2 \geq V_3 \geq \cdots$ and if the first c targets are to be defended but the (c+1)st will not, then we wish to build an active defense stature which will yield new target values V_i', $i = 1, 2, \ldots, c$ in such a way that $V_i' \leq V_{c+1}$, $i = 1, 2, \ldots,$ or more generally to make $V_i' \leq K$ where K is some given constant.

Since the capabilities of interception, detection and identification are not perfect, then there is a certain probability that a missile will evade the defense and hit the target. This means that a defensive posture will yield expected values $\overline{V}_i$, $i = 1, 2, \ldots,$. And the objective will be to determine a defense which will make $\overline{V}_i \leq V_{c+1}$, $i = 1, 2, \ldots$.

On the other hand the criterion of equal unattractiveness can also imply that the objective of the defense is to define a posture which minimizes the absolute values

$$| \bar{V}_i - V_{c+1} |$$

for $i = 1, 2, \ldots, c$.

7. Case of $P(1) = 1$, $P(\bar{A}) = 1$

The simplest case occurs when we can assume that no enemy missiles will abort, $P(\bar{A}) = 1$, and the target is destroyed if just one missile evades the defense, $P(1) = 1$. The assumption of non abortion of the enemy missile implies that if n_i missiles are aimed at target; then all n_i missiles will successfully arrive at the target, and a defensive system must be constructed to cope with n_i missiles.

The assumption of $P(1) = 1$ implies that the target will be destroyed if one or more warheads reaches the target. One or more warheads will reach the target either because the interceptors fail to stop a warhead, or because a warhead is incorrectly identified as a decoy, or because a warhead is not detected.

Then the probability of destroying the target under the above conditions becomes

$$P(k_i) = s^{n_i} q_i^{n_i} \left[1 - \left(1 - (1-p_i)^{k_i}\right)^{n_i} \right] + s^{n_i}(1-q_i^{n_i}) + (1 - s^{n_i})$$

where the first term represents the joint occurrence of all n warheads being detected and identified correctly but at least one reaches the target; the second term is the joint occurrence of all n_i warheads detected but at least one is incorrectly identified as a decoy; and the last term is the probability of at least one warhead being undetected. This expression can be simplified to:

$$P(k_i) = 1 - \left[s\ q_i \left(1 - (1 - p_i)^{k_i}\right) \right]^{n_i} \tag{7.1}$$

This makes the expected value of each target to be

$$\bar{V}_i(n_i,k_i) = V_i \left\{ 1 - \left[sq_i\ (1 - (1 - p_i{}^{k})\right]^{n_i} \right\} \tag{7.2}$$

Thus, under the objective of equal unattractiveness we need only to compute k_i such that

$$\bar{V}_i(n_i,k_i) \leq V_{c+1} \quad i = 1, 2, \ldots, c \tag{7.3}$$

or, solving for k_i we want the smallest integer value of k_i such that

$$k_i \geq \log \left\{ 1 - \left[(V_i - V_{c+1})/V_i q_i{}^{n_i}\ s^{n_i} \right]^{\frac{1}{n_i}} \right\} / \log\,(1-p_i) \tag{7.4}$$

Thus, to achieve the stated defense objective, k_i interceptors will be assigned to intercept each object that is classified as a warhead where

k_i satisfies (7.4). If k_i interceptors are assigned to each object classified as a warhead, then the expected number of interceptors required at the i-th target is

$$k_i \left(n_i q_i + n_i \alpha (1 - r_i) \right)$$

In the expression for the defense posture, (7.4), it is interesting to note that k_i will be undefined if

$$(V_i - V_{c+1}) / V_i q_i^{n_i} s^{n_i} \geq 1 \; ;$$

That is, if

$$q_i s \leq \left[\frac{V_i - V_{c+1}}{V_i} \right]^{\frac{1}{n_i}}$$

Thus, to define k_i as in (7.4) we must have

$$q_i s > \left[\frac{V_i - V_{c+1}}{V_i} \right]^{\frac{1}{n_i}} \qquad (7.5)$$

If this relationship does not hold, then no matter how large k_i might be, the expected value of the i-th target can never be reduced to the value of the given constant or the value of the (c+1) st city.

The relationships of the various reliabilities is even more evident in graphs numbered 1, 2, 3, 4.. If

$$\Lambda_i = \left[(V_i - V_{c+1}) / V_i \right]^{\frac{1}{n_i}}$$

is small, then the reliability of the interception system, p_i, is of very little importance for most values of the product $q_i s$. Only when Λ_i is quite large will the interceptor reliability be of any importance.

More important is the role of the detection and classification probabilities. If the product of these two is less than Λ, then it is impossible to achieve the objective of equal unattractiveness. For example, suppose $V_i = 1,000,000$ and $V_{c+1} = 640,000$ and $n_i = 2$, then $\Lambda = .8$. If both q_i and s were below .89, it would be impossible to defend the target to the desired degree. This would be true even if the rest of the system is perfect, and if an infinite number of interceptors are committed to each object called a warhead.

Of the three types of reliabilities, that of detection is likely to be the most difficult to estimate and control. Detection depends on the early warning system, the alertness of early warning crews, the quality of radars both at early warning sites and at interceptor locations. As can be seen from Graph 4, if the probability of detecting is as low as .8, then for a larger city it is impossible to meet the criterion of equal unattractiveness, even if all other systems are operating perfectly.

A similar role is played by q_i, the reliability of the identification system. Just as for detection, if q_i is sufficiently low, it will be impossible to defend the city so that the equal unattractiveness criterion is met. However, unlike detection, identification reliability can be controlled. In fact, since q_i is found from

$$q_i = \int_{b > b_i^*} dF_W(b)$$

it is possible to make q_i as large as desired by decreasing the critical value of the signature, b_i^*. This in turn, of course, increases r_i which would increase the expected number of decoys intercepted and thus increase the inventory of interceptors required at each interceptor site. On the other hand, better engineering designs can change the character of both F_d and F_w which in turn can vary q_i and r_i.

As has been seen an alternative objective is to find the number of interceptors such that the expected value of each defended target would be as close to a given constant as is possible. That is, for a given enemy policy, n_i, we wish to find k_i which minimizes

$$\left| \bar{V}_i (k_i) - V_{c+1} \right|, \quad i = 1, 2, \ldots, c \tag{7.6}$$

Substituting for $\bar{V}_i(k_i)$ its value from (7.2) we find we want to minimize

$$\left| [V_i - V_{c+1}] - V_i \left[s\, q_i \left(1 - (1-p_i)^{k_i}\right)\right]^{n_i} \right| \qquad i = 1, 2, \ldots, c \tag{7.7}$$

Since the first term is a non-negative constant for a given i, and since the second term is also non-negative, this minimum can be found from setting these two equal to each other. That is, we want to find k_i such that

$$V_i - V_{c+1} = V_i \left[s\, q_i \left(1 - (1 - p_i)^{k_i}\right)\right]^{n_i} \tag{7.8}$$

When $k_i = 0$, the right-hand side is zero. As k_i increases, the right-hand side increases. Unfortunately, there may be no finite real number for k_i, for which this expression holds; that is, the right-hand side might be always smaller than $V_i - V_{c+1}$ regardless of the magnitude of k_i.

For example, only if

$$s\, q_i > \frac{(V_i - V_{c+1})^{\frac{1}{n_i}}}{V_i} \tag{7.9}$$

will it be possible to determine k_i such that (7.8) holds. If this does not hold, even an infinite value for k_i would not be sufficient for satisfying this criterion. This requirement is identical to that of (7.5) given before. Thus, the detection and identification probabilities have identical importance in the two models.

If (7.9) holds, then (7.8) can be solved for k_i

$$k_i = \log \left\{ 1 - \left[(V_i - V_{c+1}) \,/\, V_i q_i^{\,n_i} s^{n_i} \right]^{\frac{1}{n_i}} \right\} / \log (1 - p_i)$$

which gives the number of interceptors to commit to the interception of each object identified as a warhead. Unfortunately, k_i is most likely not an integer. The usual round off procedure may be reasonable, but a better and an easily calculable rule can be found from the following analysis.

It is easy to see that for increasing k_i, $\overline{V}(k_i)$ is monotonically decreasing. For k_i as found above, $V(k_i) = V_{c+1}$. Let $\underline{k}_i$ be the greatest integer less than k_i and $\overline{k}_i$ the smallest integer larger than k_i. Then

$$\underline{\delta}_i = V_i(\underline{k}_i) - V_i(k_i) \geq 0$$

$$\bar{\delta}_i = V_i(k_i) - V_i(k_i) \geq 0$$

If $\underline{\delta}_i > \bar{\delta}_i$, we round up to $\bar{k}_i$; if $\underline{\delta}_i < \bar{\delta}_i$, we round down to $\underline{k}_i$; if $\underline{\delta}_i = \bar{\delta}_i$, round either way, probably up since this would tend to give greater coverage of the target by the interceptors. Now, it is easy to see that $\underline{\delta}_i > \bar{\delta}_i$ whenever

$$(1-p)^{\underline{k}_i} - (1-p)^{k_i} > (1-p)^{k_i} - (1-p)^{\bar{k}_i}$$

or when

$$\Delta = (1-p_i)^{\underline{k}_i} + (1-p_i)^{\bar{k}_i} - 2(1-p_i)^{k_i} > 0 \qquad (7.10)$$

Thus, the rounding rule can be stated as follows:

$$\text{If } \Delta \geq 0, \text{ then round up to } \bar{k}_i$$
$$\text{If } \Delta < 0, \text{ then round down to } \underline{k}_i \qquad (7.11)$$

where

$$\Delta = (1-p_i)^{\underline{k}_i} + (1-p_i)^{\bar{k}_i} - 2(1-p_i)^{k_i} \qquad (7.12)$$

8. The Case of $P(1) = 1$, $P(\bar{A}) < 1$

In the preceding sections it was assumed that each missile aimed at a particular target would arrive at the target. Thus, if n_i warheads are allocated to target i by the enemy, then all n_i warheads are assumed to arrive within the range of the interceptors. This may not be true; that is, the enemy may experience a certain probability of abort. In other words, $P(\bar{A}) < 1$. Then the number of missiles actually to arrive at target i is a random variable described by

$$P \left\{ \begin{matrix} m \text{ missiles arrive} \\ \text{at target } i \end{matrix} \right\} = \binom{n_i}{m} P(\bar{A})^m \left(1 - P(\bar{A})\right)^{n_i - m}$$

Thus, the expected value of each target would be

$$\bar{V}(n_i, k_i) = V_i \sum_{m=0}^{n_i} \left\{ 1 - \left[s\, q_i \left(1 - (1-p_i)^{k_i}\right) \right]^m \right\} \times \left\{ \binom{n_i}{m} P(\bar{A})^m \left(1 - P(\bar{A})\right)^{n_i - m} \right\} \tag{8.1}$$

Using this expected value we can defend each defended target such that $\bar{V}(n_i, k_i) \leq V_{c+1}$, where V_{c+1} is either the value of the first undefended city, or a given constant. Substituting for $\bar{V}(n_i, k_i)$ its value from (8.1) and solving for k_i we find we want the smallest integer value of k_i such that

$$k_i \geq \frac{\log \left\{ 1 - \frac{1}{s\, q_i} \left[\frac{1}{P(\bar{A})} \left(\left(\frac{V_i - V_{c+1}}{V_i} \right)^{\frac{1}{n_i}} - 1 \right) + 1 \right] \right\}}{\log (1 - p_i)} \tag{8.2}$$

as the number of interceptors to commit to each incoming warhead. If we now define

$$\Lambda_i = \frac{1}{P(\bar{A})} \left[\left(\frac{V_i - V_{c+1}}{V_i} \right)^{\frac{1}{n_i}} - 1 \right] + 1 \tag{8.3}$$

we again have Graphs 1, 2, 3, 4 to be representative of the relationship between the product $s\ q_i$ and k_i for varying values of p_i.

Again, it is evident that k_i will be undefined if

$$s\ q_i > \frac{1}{P(\overline{A})}\left[\left(\frac{V_i - V_{c+1}}{V_i}\right)^{\frac{1}{n_i}} - 1\right] + 1 = \Delta_i \qquad (8.4)$$

When $P(\overline{A}) = 1$, this is merely (7.5). As $P(\overline{A})$ decreases, the value of Λ_i decreases. In fact, the relationship is as given in Graph 5, where it is interesting to note that admissible values of $s\ q_i$ is highly related not only to the value of the target but more important to the probability of non-abort of the missile. Thus, if $P(\overline{A})$ is sufficiently small, each target can always be defended to the desired level, regardless of the reliability of the system, provided a sufficient (but finite) number of interceptors are committed to each incoming object called a warhead.

Using the criterion of minimizing $\left|\overline{V}(n_i, k_i) - V_{c+1}\right|$ we want the nearest integer value to

$$k^* = \frac{\log\left\{1 - \left(\left[(V_i - V_{c+1} / V_i)^{\frac{1}{n_i}} - 1\right] / P(\overline{A}) + 1\right) / s\ q_i\right\}}{\log(1 - p_i)}$$

Round-off rules similar to (7.11) and (7.12) can be devised and relationships similar to that of Graphs 1, 2, 3, 4 and 5 can be stated.

9. <u>The case of $P(1) < 1$, $P(\overline{A}) < 1$</u>

If a single hit does not destroy a target, but instead destroys a target with a given probability, $P(1)$, where $0 < P(1) < 1$, then if $y \geq 1$ missiles reached the target, the probability of destroying the target is $1 - \left[1 - P(1)\right]^y$, on the assumption that the effects of the y warheads are independent. Now, each of the y warheads can reach the target either by not being detected, or by being detected but incorrectly identified as decoys or by being detected, correctly identified but evading the active defense.

Let

y be the number of warheads successfully evading active defense

$p_i^* = (1-p_i)^k$ be the probability an object is not effectively intercepted

z be the number of warheads called decoys

ℓ be the number of detected warheads

m_i be the number of warheads arriving at the target i

n_i be the number of warheads aimed at target i

Then the conditional probability of the target not being destroyed by the y warheads evading active defense is

$$[1 - P(1)]^y$$

The conditional probability of y identified warheads evading active defense is

$$\binom{\ell - z}{y} (p_i^*)^y (1 - p_i^*)^{\ell - z - y}$$

when ℓ warheads are detected and z are called decoys. Thus, the probability of y warheads evading active defense and the target surviving is

$$\left[1 - P(1)\right]^y \binom{\ell - z}{y} (p_1^*)^y (1-p_1^*)^{\ell - z - y}$$

Now y can be 0 or 1 or 2 or, ..., or $(\ell - z)$. Thus if ℓ warheads are detected and z are called decoys, the probability of the target not being destroyed by warheads evading active defense is

$$\sum_{y=0}^{\ell - z} \left[1 - P(1)\right]^y \binom{\ell - z}{y} (p_1^*)^y (1-p_1^*)^{\ell - z - y}$$

If, out of a total of ℓ detected warheads, exactly z are called decoys, then the conditional probability of the target not being destroyed by one or more of the z warheads is

$$\left[1 - P(1)\right]^z$$

The joint occurrence of the target surviving z warheads called decoys and y evading active defense given ℓ warheads detected is

$$\left[1 - P(1)\right]^z \sum_{y=0}^{\ell - z} \left[1 - P(1)\right]^y \binom{\ell - z}{y} (p_1^*)^y (1 - p_1^*)^{\ell - z - y}$$

The probability that z warheads out of a total of ℓ detected warheads are called decoys is

$$\binom{\ell}{z} q_1^{\ell - z} (1 - q_1)^z$$

Thus, the probability of the target surviving an attack of ℓ detected warheads is

$$\sum_{z=0}^{\ell} \binom{\ell}{z} q_1^{\ell-z}(1-q_1)^z \left[1-P(1)\right]^z \times \sum_{y=0}^{\ell-z}\left[1-P(1)\right]^y \binom{\ell-z}{y} (p_1*)^y (1-p_1*)^{\ell-z-y}$$

If m warheads arrive and only ℓ are detected, then $m-\ell$ will not be detected. The target will survive the attack by the $m-\ell$ undetected warheads with probability

$$\left[1 - P(1)\right]^{m-\ell}$$

A total of ℓ warheads will be detected with probability

$$\binom{m}{\ell} s^{\ell} (1 - s)^{m-\ell}$$

Thus, the probability of the target surviving if m warheads reach the target is

$$\sum_{\ell=0}^{m} \binom{m}{\ell} s^{\ell} (1-s)^{m-\ell}\left[1-P(1)\right]^{m-\ell} \times \sum_{z=0}^{\ell} \binom{\ell}{z} q_1^{\ell-z}(1-q_1)^z \left[1-P(1)\right] \times \sum_{y=0}^{\ell-z} \left[1-P(1)\right]^y \binom{\ell-z}{y} (p_1*)^y (1-p_1*)^{\ell-z-y}$$

Finally, if the enemy assigns n_i weapons to target i, it is possible that some abort. Thus, given n_i warheads directed at target i and a probability of non-abort $P(\bar{A}) < 1$, the probability of destroying the target with one or more of the n_i warheads when k_i interceptors are assigned to each object called a warhead is

$$P(n_i,k_i) = 1 - \sum_{m=0}^{n_i} \binom{n_i}{m} P(\bar{A})^m \left[1-P(\bar{A})\right]^{n_i-m} \times$$
$$\sum_{\ell=0}^{m} \binom{m}{\ell} s^{\ell} (1-s)^{m-\ell} \left[1-P(1)\right]^{m-\ell} \times \tag{9.1}$$
$$\sum_{z=0}^{\ell} \binom{\ell}{z} q_i^{\ell-z}(1-q_i)^z \left[1-P(1)\right]^z \times$$
$$\sum_{y=0}^{\ell-z} \left[1-P(1)\right]^y \binom{\ell-z}{y}(1-p_i)^y \left[1-(1-p_i)^{k_i}\right]^{\ell-z-y}$$

This can be simplified rather readily to

$$P(n_i,k_i) = 1 - \left[1-P(\bar{A})\left\{1 - \left[(1-sq_ip_i)\left[1-P(1)\right] + sq_i\left(1-(1-p_i)^{k_i}\right)\right]\right\}\right]^{n_i} \tag{9.2}$$

from which the expected value of the i-th target is

$$\bar{V}(n_i,k_i) = V_i - V_i\left[1-P(\bar{A})\left\{1 - \left[(1-sq_ip_i)\left[1-P(1)\right] + sq_i\left(1 - (1-p_i)^{k_i}\right)\right]\right\}\right]^{n_i} \tag{9.3}$$

Using the criterion of $\bar{V}(n_i,k_i) \leq V_{c+1}$ for $i = 1, 2, \ldots, c$ we have that k_i is the smallest integer value such that

$$k \geq \frac{\log\left\{1 - \frac{1}{sq}\left[\frac{1}{P(\bar{A})}\left(\left(\frac{V_i - V_{c+1}}{V_i}\right)^{\frac{1}{n_i}} - 1\right) + 1 - (1-sq_i p_i)(1-P(1))\right]\right\}}{\log(1-p)} \qquad (9.4)$$

as the number of interceptors to use to intercept each object classified a warhead.

It is interesting to note the similarity between this expression and that of (8.2). Thus a Λ_i can be defined in a similar fashion and Graphs 1, 2, 3, 4 can be used.

10. Unknown Policy

In the preceding it was assumed that the enemy targeting policy, n_i, $i = 1, 2, \ldots,$ was known, and on the basis of this known policy a defensive policy, k_i, for the given enemy policy and given reliabilities was determined. Unfortunately, only a person with unusual perception could come close to indicating the exact enemy policy. On the other hand, it may be possible that the a priori probabilities $p_i(n)$, of n warheads at target i can be estimated with a fair degree of accuracy. If so then the probability of kill at the i-th target is

$$P(k_i) = \sum_{n=0}^{\infty} P(n,k_i)p_i(n) \qquad (10.1)$$

where $P(n,k_i)$ is computed from one of the preceding sections. This will give an average value to the enemy of target i to be

$$\overline{V}_i(k_i) = V_i P(k_i) \tag{10.2}$$

As an example $p_i(n)$ might be considered to be Poisson probability function. That is

$$p_i(n) = \frac{e^{-\lambda_i} \lambda_i^{\,n}}{n!}$$

where λ_i is the average value of the random variable of the number of missiles sent to target i. This assumption is actually quite a reasonable assumption since the number of missiles is an integer and is randomly distributed in space.

Using the simple model given by (7.1) where

$$P(n_i, k_i) = 1 - \left[s\ q_i\ (1 - (1 - p_i)^{k_i}) \right]^{n_i}$$

we have

$$P(k_i) = 1 - \sum_{n=0}^{\infty} \left[s\ q_i\ (1 - (1 - p_i)^{k_i}) \right]^{n_i} \cdot \left[\frac{e^{-\lambda_i} \lambda_i^{\,n}}{n!} \right]$$

Then

$$\overline{V}_i(k_i) = V_i - V_i \sum_{n=0}^{\infty} \left[s\ q_i\ (1 - (1 - p_i)^{k_i}) \right]^{n_i} \cdot \frac{e^{-\lambda_i} \lambda_i^{\,n}}{n!}$$

Using the criterion of $\overline{V}_i(k_i) \leq V_{c+1}$ for $i = 1, 2, \ldots, n$ we have

$$V_i - V_i \sum_{n=0}^{\infty} \left[s\ q_i\ (1 - (1 - p_i)^{k_i}) \right]^{n_i} \cdot \left[\frac{e^{-\lambda_i} \lambda_i^{\,n}}{n!} \right] \leq V_{c+1};$$

or solving for k_i we find we want the smallest integer value of k_i such that

$$k_i \geq \log \left\{ 1 - \frac{1 + \log_e \left(\frac{V_i - V_{c+1}}{V_i} \right)^{\frac{1}{\lambda_i}}}{s\ q_i} \right\} / \log (1 - p_i) \qquad (10.3)$$

This will be defined only if

$$\frac{1 + \log_e \left(\frac{V_i - V_{c+1}}{V_i} \right)^{\frac{1}{\lambda_i}}}{s\ q_i} \leq 1$$

or if

$$1 + \log_e \left[(V_i - V_{c+1}) / V_i \right]^{\frac{1}{\lambda_i}} \leq s\ q_i \qquad (10.4)$$

It is interesting to compare this with (7.4) and (7.5), the analogous case without uncertainty, as shown in Graph 6. For the same target value and the same average number of attacking warheads there is a greater range of admissible values for $s\ q_i$ in the uncertainty case than there is for the deterministic case.

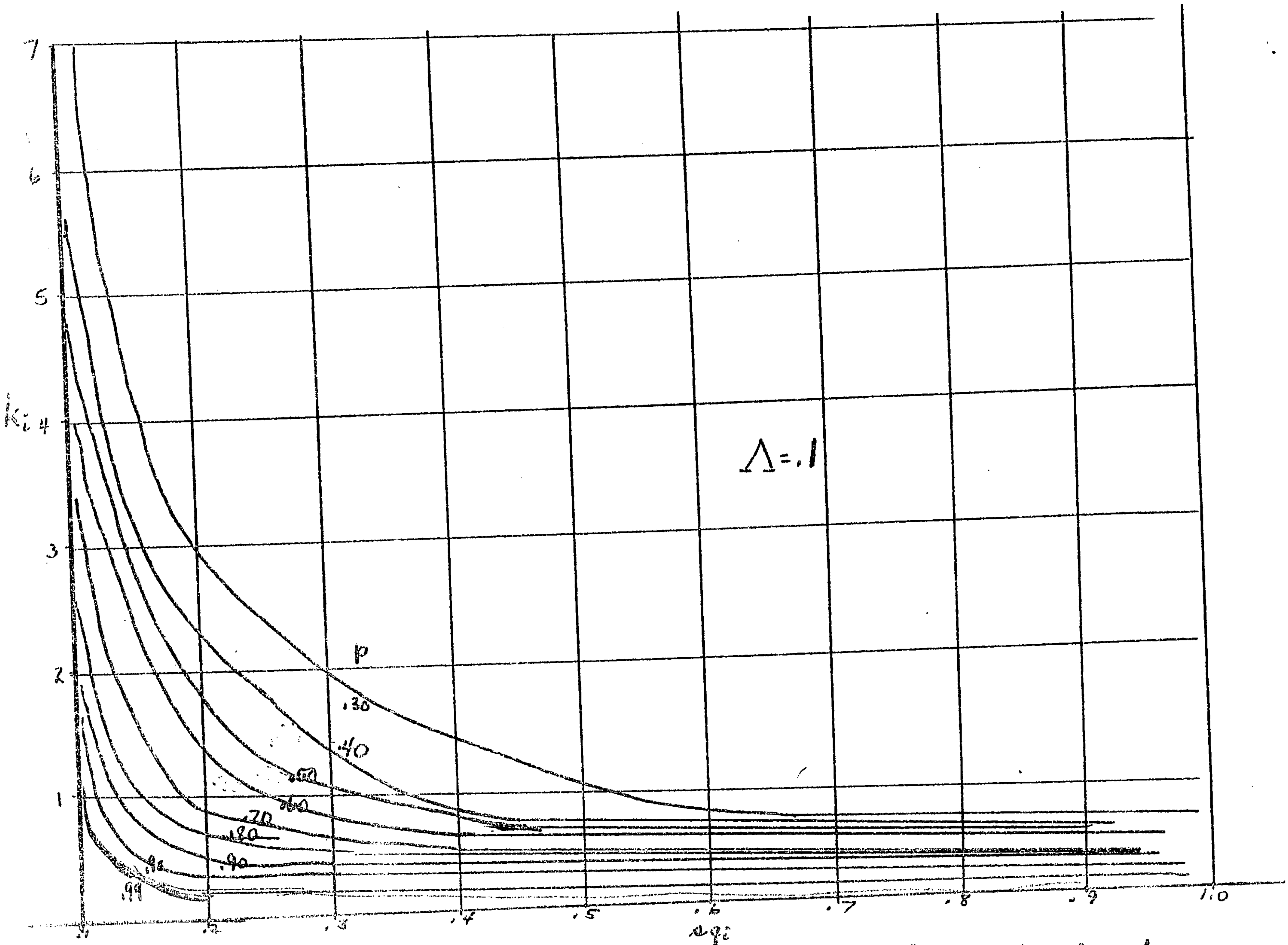
Graphs similar to Graphs 1, 2, 3 and 4 can be constructed with conversion factors as indicated by Table I. where $\Lambda_i = \left[(V_i - V_{c+1}) / V_i \right]^{\frac{1}{n_i}}$ for the deterministic case and $\Lambda_i = \left[(V_i - V_{c+1}) / V_i \right]^{\frac{1}{\lambda}}$ for the uncertainty model.

Table I. Conversion Factors Between
Deterministic and Uncertainty Models for Graphs 1, 2, 3, 4

Graph Number	Deterministic Λ_i	Uncertainty Λ_i
1	.1	.40
2	.2	.45
3	.4	.55
4	.8	.82

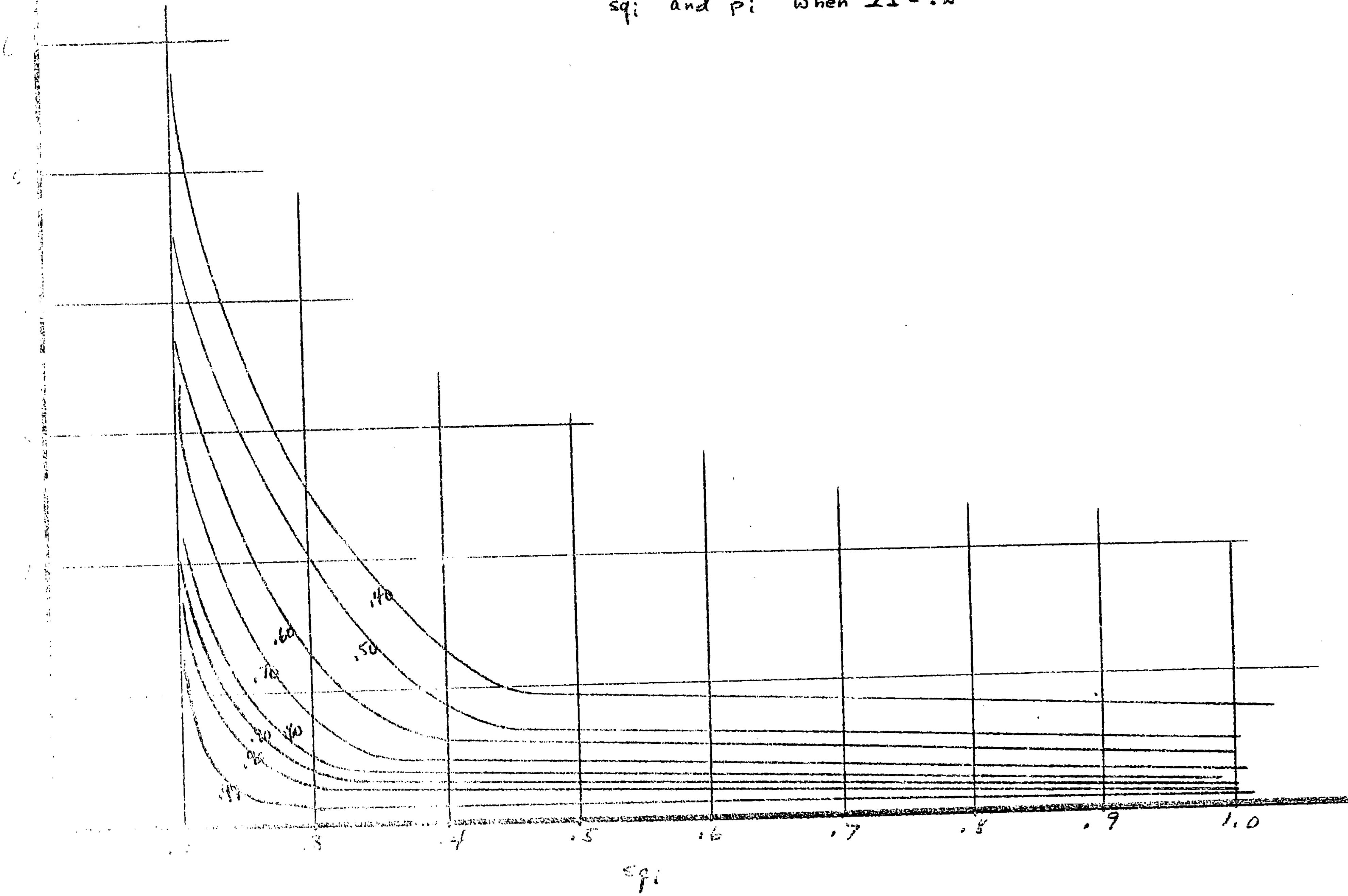
11. Conclusions

In this paper the effectiveness requirements of an interceptor system for point targets were presented. It was interesting to note that the reliability of detection and identification were very important, especially for relatively valuable targets.

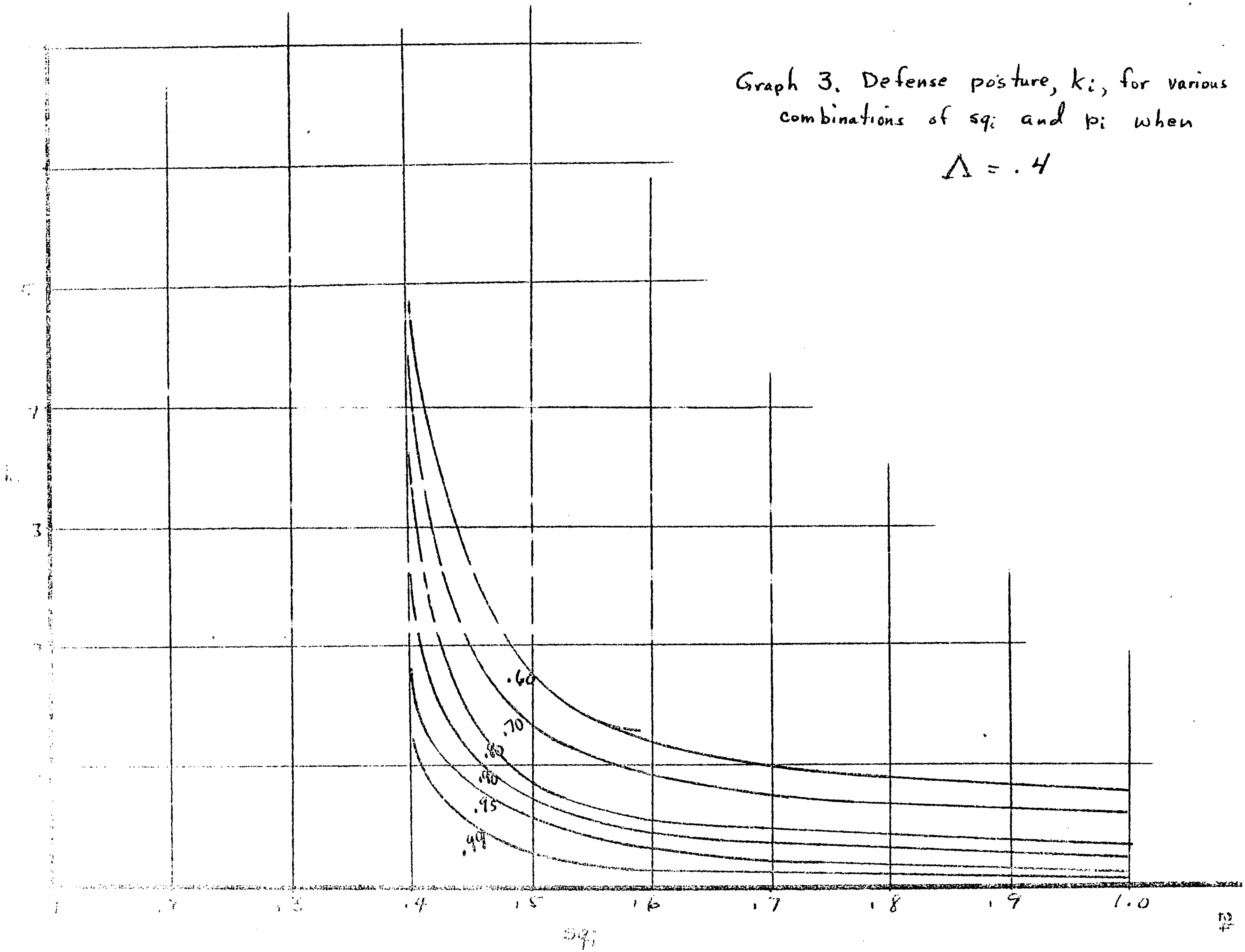


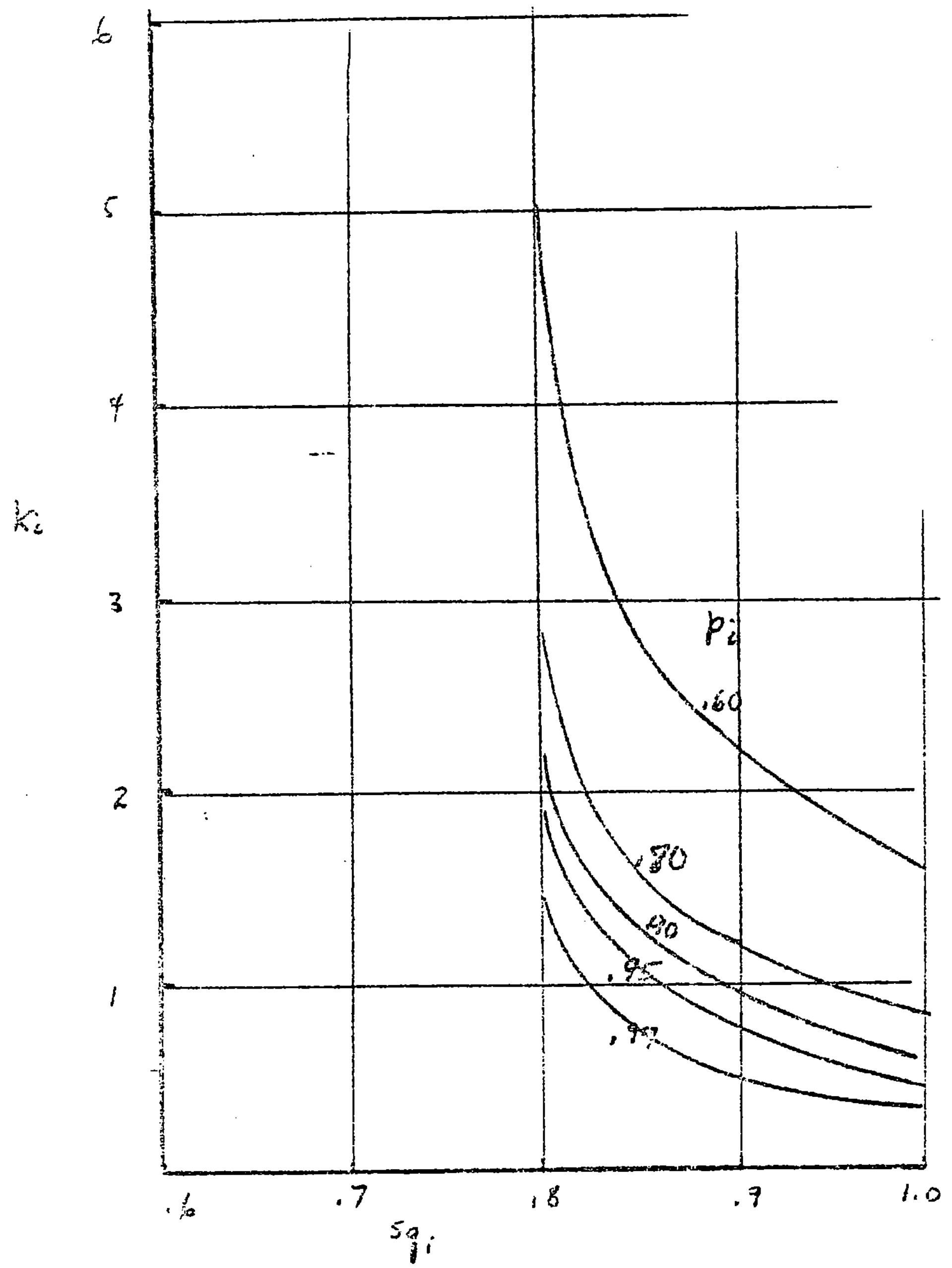
Graph 1. Determination of Defence, k_i, for various values of sq_i and p_i when $\Lambda = .1$

Graph 2 Defense posture, k_i, for various combinations of sq_i and p_i when $\Delta = .2$

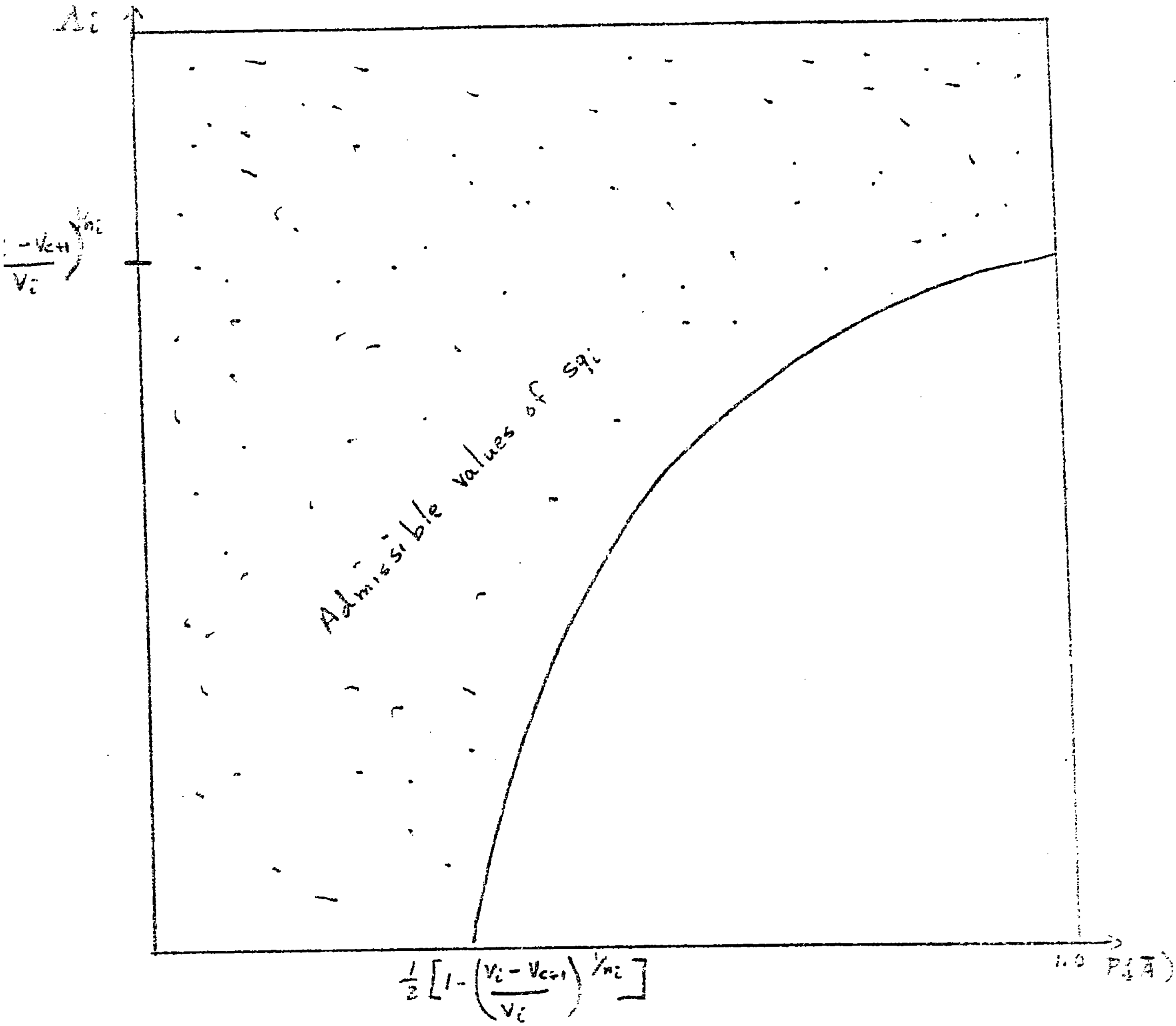


Graph 3. Defense posture, k_i, for various combinations of sq_i and p_i when $\Lambda = .4$

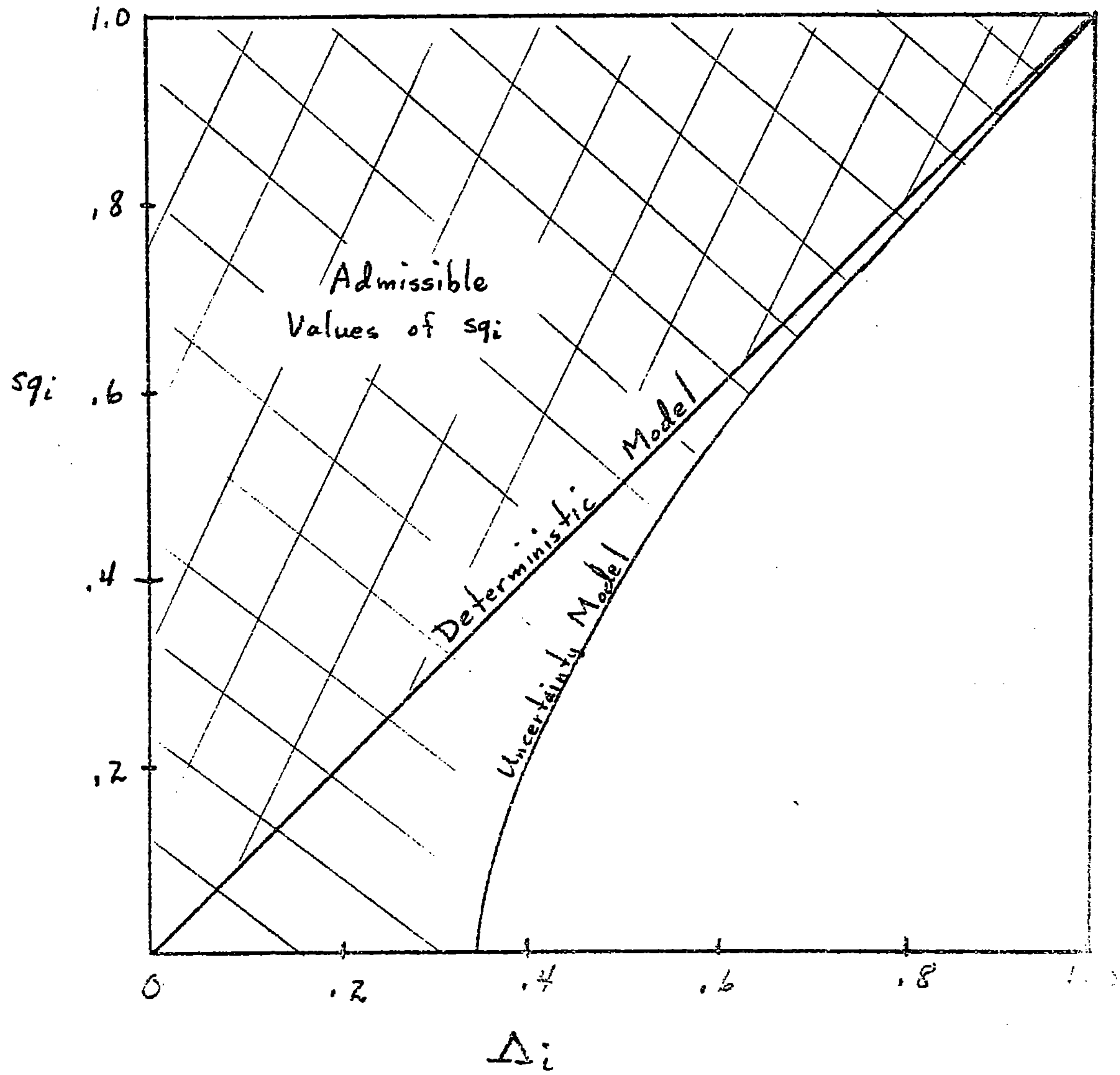




Graph 4. Defense posture k_i for various combinations of sq_i and p_i when $\Delta = .8$



Graph 5. The set of points $(\Delta_i, P(\bar{A}))$ which yield admissible values of the product sq_i



Graph 6 Set of values of sqi and Δ yielding admissible values of sqi

$$\Delta_i = \left(\frac{V_i - V_{[illegible]}}{V_i}\right)^{\frac{1}{n_i}} \text{ or } \Delta_i = [illegible]$$