Exact Moments and Percentage Points of the Order Statistics and the Distribution of the Range from the Logistic Distribution

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Mimeograph Series Number 5

May 1963

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1. Introduction and Summary

The logistic curve $y = k/(1 + \alpha e^{-\beta t})$ has been used in studies pertaining to population growth by Verhulst [17] and by Pearl and Reed [14] and by several later authors. The logistic function $P = 1/(1+e^{-(\alpha+\beta x)})$ has been very widely used by Berkson [1], Berkson and Hodges [2] as a model for analyzing bioassay and other experiments involving quantal response. Birnbaum [3] has mentioned it as a model in some problems pertaining to tests of mental ability. Gumbel [8] has shown that the asymptotic distribution of the midrange of the exponential type initial distributions is logistic. In connection with problems involving censored data, Plackett [15] [16] has considered the use of the logistic distribution.

In this paper a random variable X is said to follow a ~~standard~~ logistic distribution (denoted by $L(\mu, \sigma^2)$) if its cumulative distribution function (c.d.f.) is

$$F(y;\mu,\sigma) = \frac{1}{\left[1 + e^{-\left(\frac{y-\mu}{\sigma}\right)\cdot\frac{\pi}{\sqrt{3}}}\right]} \tag{1.1}$$

The probability density function (p.d.f.) corresponding to (1.1) is

(1.2) $$f(y;\mu,\sigma) = \frac{\pi}{\sigma\sqrt{3}} \frac{e^{-\pi(y-\mu)/\sqrt{3}\,\sigma}}{\left[1 + e^{-\frac{\pi}{\sqrt{3}}(y-\mu)/\sigma}\right]^2},$$

where $-\infty < y < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$.

It should be noted that the distribution (1.2) is symmetrical with mean μ and variance σ^2. The moment generating function of $X = \frac{Y-\mu}{\sigma}$ is easy to derive (See for example, Gumbel [8]) and is

(1.3) $$M_X(t) = \Gamma(1+t/g)/\Gamma(1-t/g), \; g = \pi/3^{1/2}$$

In this paper order statistics from the standard logistic distribution L(0, 1) are studied. If $X_1, X_2, \ldots, X_n$ are n independent and identically distributed logistic random variables with density function,

(1.4) $$f(x) = \frac{\pi}{3^{1/2}} \frac{e^{-x\pi/3^{1/2}}}{\left(1+e^{-x\pi/3^{1/2}}\right)^2}, \; -\infty < x < \infty$$

then we are concerned with the moments, the distribution and some estimation problems using the ordered random variables $X_{(1)}, X_{(2)} \cdots X_{(n)}$ where

(1.5) $$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(k)} \leq \cdots \leq X_{(n)}.$$

In the sequel, we shall call $X_{(k)}$', the kth order statistic in a sample of size n from the logistic distribution L(0, 1).

In this paper the exact expressions for the moments of $X_{(k)}$ have been derived. The values of the first four exact moments for all sample sizes n from 1 to 10 have been tabulated (Table 1). More generally, the moments of $X_{(k)}$ have been expressed in terms of expressions involving Bernoulli and Stirling numbers of 1st kind. These derivations are obtained from the moment generating function which has been derived. The cumulants of $X_{(k)}$ are expressed in terms of polygamma functions (a result also earlier mentioned by Plackett [15]) and using tabulated values of these functions [4] [5], we have given numerical values of the first four cumulants of all order statistics for all sample sizes n from 1 to 10 (Table 2). Table 3 gives the percentage points of $X_{(k)}$ (i) for all k $(k \leq n)$ and all n from 1 to 10 (ii) for $k = 1$, n and $\frac{n}{2}$ and $\frac{n+2}{2}$ (n even) or $\frac{n+1}{2}$(n odd) for n = 11 (1) 25. In Section 3, we obtain series expansions for the joint moment generations function and covariance of the two order statistics. In Section 5, the use of one and two order statistics for estimating μ and σ in $L(\mu, \sigma^2)$ is shown. In Section 6, expressions (closed form) are derived for the cumulative distribution function and the density function of the sample range. Section 6 also gives a short table of the c.d.f. of the sample range for the logistic for n = 2 and 3. Section 7 gives a description of the tables in this paper.

2. Moments and Cumulants of the kth Order Statistic

The moment generating function $M(t)$ of $X_{(k)}$ is

$$(2.1)\qquad M(t) = E(e^{tX_{(k)}}) = \frac{g}{B(k,n-k+1)} \int_{-\infty}^{\infty} \frac{e^{xt} e^{-xg(n-k+1)}}{(1+e^{-xg})^{n+1}}\, dx \quad .$$

$$(2.2)\qquad M(t) = \frac{B(k+t/g, n-k+1-t/g)}{B(k, n-k+1)}$$

where $B(p,q)$ and $\Gamma(x)$ are the usual beta and gamma functions.

After some algebraic simplification (2.2) can be written as

$$(2.3)\qquad M(t) = \frac{(-1)^{n-k} \pi \operatorname{cosec}(\pi t/g)(k-1+t/g)_n}{(k-1)!\,(n-k)!}$$

where $(x)_n = x(x-1) \ldots (x-n+1)$.

By expanding $\operatorname{cosec}(\pi t/g)$ in powers of t and by writing $(x)_n$ in terms of Stirling numbers of first kind, we obtain

$$(2.4)\qquad M(t) = \frac{(-1)^{n-k}\pi}{(k-1)!\,(n-k)!} \left[\frac{g}{\pi t} + \frac{\pi t}{6g} + \frac{7}{360}\frac{\pi^3 t^3}{g^3} + \right.$$

$$\left. + \ldots + \frac{2(2^{2\ell-1}-1)}{(2\ell)!} B_\ell \left(\frac{\pi t}{g}\right)^{2\ell-1} \right] \left[\sum_{i=1}^{n} (k-1+t/g)^i s(i,n) \right]$$

where B_n and $s(i,n)$ denote the Bernoulli numbers and Stirling numbers of first kind, respectively.

Now (2.4) can be expressed as

(2.5) $$M(t) = \frac{(-1)^{n-k}}{(k-1)!\,(n-k)!}\Big[\sum_{i=1}^{n}\sum_{j=0}^{i} s(i,n)\ \binom{i}{j}\left(\frac{t}{g}\right)^{j-1}(k-1)^{i-j}$$

$$+ 2\sum_{\ell=1}^{\infty}\sum_{i=1}^{n}\sum_{j=0}^{i}\frac{(2^{2\ell-1}-1)}{(2\ell)!}\,\pi^{2\ell}\,\binom{i}{j}(k-1)^{i-j}B_{\ell}\,s(i,n)\left(\frac{t}{g}\right)^{j+2\ell-1}\Big].$$

From (2.5) collecting the coefficient of t^{2r-1} and t^{2r} we obtain

(2.6) $$\mu'_{2r-1}(k,n) = \frac{1}{g^{2r-1}}\,\frac{(-1)^{n-k}(2r-1)!}{(k-1)!\,(n-k)!}\Big[\sum_{i=2r}^{n} b_{i(2r)}s(i,n) + 2\sum_{j=1}^{r}\sum_{i=2(r-j)}^{n} a_j b_{i,2(r-j)}s(i,n)\Big]$$

(2.7) $$\mu'_{2r}(k,n) = \frac{1}{g^{2r}}\,\frac{(-1)^{n-k}(2r)!}{(k-1)!\,(n-k)!}\Big[\sum_{i=2r+1}^{n} b_{i(2r+1)}\ s(i,n) + 2\sum_{j=1}^{r}\sum_{i=2(r-j)+1}^{n} a_j b_{i,2r-2j+1}\ s(i,n)\Big]$$

where $a_\ell = \frac{2^{2\ell-1}-1}{(2\ell)!}\ \beta_\ell\ \pi^{2\ell}$ and $b_{ij} = \binom{i}{j}(k-1)^{i-j},\ 0 \le j \le i.$

Since $\kappa'_1 = \mu'_1 = \Psi(k-1) - \Psi(n-k)$, we obtain the identity

(2.8) $$\psi(k-1) - \psi(n-k) = \frac{(-1)^{n-k}}{(k-1)!\,(n-k)!}\left[\sum_{i=2}^{n} \binom{i}{2}(k-1)^{i-2}s(i,n) + \frac{\pi^2}{18}\sum_{i=0}^{n}(k-1)^{i}s(i,n)\right].$$

Similar identities can be obtained by equating,

$$\kappa_r = \frac{1}{g^r}\left[\psi^{(r-1)}(k-1) - \psi^{(r-1)}(n-k)\right]$$ to the expression available for κ_r in terms of μ'_{2r-1} and μ'_{2r} and using (2.6) and (2.7).

It should be pointed out that for small n, the computation of exact moments is simpler if we collect the coefficient of the appropriate power of t in the right hand side of (2.3). This procedure was followed to obtain the exact values of moments that are given in Table I.

Cumulants of $X_{(k)}$

We rewrite (2.2) as

(2.9) $$M(t) = \frac{\Gamma(k+t/g)\,\Gamma(n-k+1-t/g)}{\Gamma(k)\,\Gamma(n-k+1)}$$

From (2.9) we obtain the rth cumulant $\kappa_r(k,n)$ as

(2.10) $$\mathcal{K}_r(k,n) = \frac{d^r}{dt^r} \log_e \Gamma(k+t/g) \Big|_{t=0} + \frac{d^r}{dt^r} \log_e \Gamma(n-k+1-t/g) \Big|_{t=0}$$

(2.11) $$\mathcal{K}_r(k,n) = \frac{1}{g^r}\left[\psi^{(r-1)}(k-1) + (-1)^r \psi^{(r-1)}(n-k)\right]$$

where $\psi^{(r-1)}(x) = \frac{d^r}{dx^r} \log_e \Gamma(1+x) = \frac{d^{r-1}}{dx^{r-1}} \frac{\Gamma'(1+x)}{\Gamma(1+x)}$

and $\psi^{(0)}(x) = \psi(x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)}$.

Using formula (2.11) and tables of polygamma functions $\psi^{(r)}(x)$ given in [4] [5], we have computed $\mathcal{K}_r(k,n)$ for r=1, 2, 3 and 4 for all order statistics in a sample of size n, n = 1(1)10. These are given in Table II.

It is clear from (2.11) that

(2.12) $$\mathcal{K}_{2r-1}(k,n) = -\mathcal{K}_{2r-1}(n-k+1,n)$$

(2.13) $$\mathcal{K}_{2r}(k,n) = \mathcal{K}_{2r}(n-k+1,n).$$

Using the series expansions for $\psi^{(r-1)}(x)$ and $\psi(x)$

(2.14) $$\psi^{(r-1)}(x) = (r-1)!(-1)^r \sum_{j=1}^{\infty} \frac{1}{(j+x)^r}, \quad r \geq 2$$

(2.15) $$\psi(x) = \sum_{\nu=1}^{\infty} \left[\frac{1}{\nu} - \frac{1}{\nu+x}\right]$$

(2.16) $$\kappa_r(k,n) = \frac{(r-1)!\,(-1)^r}{g^r}\left[\sum_{j=1}^{\infty}\frac{1}{(j+k-1)^r} + (-1)^r\sum_{j=1}^{\infty}\frac{1}{(j+n-k)^r}\right], \quad r \geq 2$$

(2.17) $$\kappa_1(k,n) = -\frac{1}{g}\left[\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n-k}\right] \text{ if } n-k > k-1 .$$

From (2.17), we obtain for the special cases

(2.18) $$\kappa_1(m,2m) = -\frac{1}{gm} \quad ,$$

(2.19) $$\kappa_1(1,n) = -\frac{1}{g}\left[1 + \frac{1}{2} + \ldots + \frac{1}{n-1}\right].$$

Formula (2.19) is given in [8,p.128]. It should be mentioned that Plackett [15] gave the $\kappa_r(k,n)$ $(r = 1, 2, 3, 4)$ in the form (2.16) and (2.17) for the case $n-k < k-1$.

3. Covariance Between the ℓth and mth Order Statistics $(m > \ell)$

The joint moment generating function of $X_{(\ell)}$ and $X_{(m)}$ $(m > \ell)$ is

(3.1) $$M(t_1,t_2) = E(e^{t_1x+t_2y}) = C\int_{-\infty}^{\infty}\int_{x}^{\infty} e^{t_1x+t_2y}F^{\ell-1}(x)\,[F(y)-F(x)]^{m-\ell-1}$$

$$\cdot[1-F(y)]^{n-m} f(x)f(y)dx\,dy \quad ,$$

where $C = \dfrac{n!}{(\ell-1)!\,(m-\ell-1)!\,(n-m)!}$ and where $f(x)$ and $F(x)$ are defined in Section 1.

After substituting $p_1 = 1/(1+\exp(-gx))$ and $p_2 = 1/(1+\exp(-gy))$ we obtain

$$(3.2)\quad M(t_1,t_2) = C\int_0^1 dp_2 \int_0^{p_2} \left(\frac{p_1}{1-p_1}\right)^{t_1/g} \left(\frac{p_2}{1-p_2}\right)^{t_2/g} p_1^{\ell-1}(p_2-p_1)^{m-\ell-1} (1-p_2)^{n-m} dp_1 .$$

$$(3.3)\quad = C \sum_{\alpha=0}^{m-\ell-1} \sum_{\nu=0}^{\infty} (-1)^{\alpha} \binom{m-\ell-1}{\alpha} \cdot \left[\frac{(\nu-1+t_1/g)_{\nu}\; B(m+\nu+(t_1+t_2)/g,\, n-m+1-t_2/g)}{\nu!\,(\nu+m-\alpha-1+t_1/g)}\right].$$

From the above expression for the joint m.g.f. of the ℓth and mth order statistics, one can obtain the bivariate moments as follows.

$$(3.4)\quad E(X_{(\ell)}^r X_{(m)}^s) = \mu'_{rs}(X_{(\ell)},X_{(m)}) = \frac{\partial^{r+s}}{\partial t_1^r\, \partial t_2^s} M(t_1,t_2)\Big|_{t_1=t_2=0}$$

The case $r = 1$ and $s = 1$ is important. In this case (details omitted) we obtain

$$(3.5)\quad \mu'_{11}(X_{(\ell)},X_{(m)}) = \text{Cov}\,(X_{(\ell)},X_{(m)}) + EX_{(\ell)}EX_{(m)}$$

$$(3.6) \quad = \frac{c}{g^2} \sum_{i=0}^{m-\ell-1} \sum_{c=0}^{n-m} (-1)^{i+1} \binom{m-\ell-i}{i} \binom{n-m}{c} \left[\frac{1}{(m-i-1)} \left\{ \frac{2}{(m+c)^3} - \sum_{r=1}^{\infty} \frac{1}{r(m+c+r)^2} \right\} + \frac{1}{(m-i-1)^2} \left\{ \frac{1}{(m+c)^2} - \frac{1}{m+c} \sum_{x=1}^{m+c} \frac{1}{x} \right\} - \sum_{r=1}^{\infty} \frac{1}{(m+r-i-1)} \left\{ \frac{1}{(m+r+c)^2} - \frac{1}{(m+r+c)} \sum_{x=1}^{m+r+c} \frac{1}{x} \right\} \right].$$

4. Percentage Points, Modes and Some Remarks on the Distribution of $X_{(k)}$

The density function $h_{k,n}(x)$ and the c.d.f. $H_{k,n}(x)$ of the kth order statistic in a sample of size n from $L(0, 1)$ are,

$$(4.1) \quad h_{k,n}(x) = \frac{g}{B(k,n-k+1)} \frac{e^{-xg(n-k+1)}}{(1+e^{-xg})^{n+1}}, \; -\infty < x < \infty .$$

By differentiating the above expression for $h_{k,n}(x)$ with respect to x, we find that the mode $\breve{x}(k,n)$ of the kth order statistic is

$$(4.2 \quad \breve{x}(k,n) = \frac{1}{g} \log_e \frac{k}{n-k+1}$$

Also the expression for the c.d.f. $H_{k,n}(x)$ is

$$(4.2) \quad H_{k,n}(x) = I_{1/(1+\exp(-xg))}(k,n-k+1)$$

where $I_x(p,q)$ is the incomplete beta function,

$$(4.3) \qquad H_{k,n}(x) = \frac{1}{B(k,n-k+1)} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \frac{1}{(j+k)(1+e^{-xg})^{j+k}}$$

$$(4.4) \qquad H_{k,n}(x) = \frac{1}{B(k,n-k+1)} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{1}{n-k+j+1} \left\{ \left(\frac{e^{-xg}}{1+e^{-xg}}\right)^{n-k+j+1} -1 \right\} .$$

From (4.2) we see that the $(100)\alpha$-percentage points $x_\alpha(k,n)$ of $X_{(k)}$ is the solution of

$$(4.5) \qquad x_\alpha(k,n) = \frac{1}{g} \log_e \frac{B_\alpha(k,n-k+1)}{1-B_\alpha(k,n-k+1)} .$$

Using values of $B_\alpha(k,n-k+1)$ from [7] [12], we solved for $x_\alpha(k,n)$. These are given in Table III. Note that for $k = 1$ and $k = n$, we see from (4.3), (4.4) or otherwise that

$$(4.6) \qquad -x_{1-\alpha}(1,n) = x_\alpha(n,n) = \frac{1}{g} \log_e \frac{\alpha^{1/n}}{1-\alpha^{1/n}} .$$

Note that relations (4.1)-(4.5) are true in general for any continuous distribution if $1/(1+e^{-xg})$ is replaced by the c.d.f. of the given distribution. This is in virtue of the well-known result that the c.d.f. of the kth order statistic is a beta variable. Further, from the symmetric relation satisfied by the incomplete beta functions, it follows that the 100α percentage of $X_{(k)}$ is $-100(1-\alpha)$ percentage point of $X_{(n-k+1)}$. This relation can be verified, mathematically, for the $X_{(k)}$ of the logistic distribution by using (4.5).

Remark

The distribution of the sum of two symmetrical order statistics $V = X_{(k)} + X_{(n-k+1)}$ is of interest in some problems. In this connection, the following remark is relevant. From (4.2), we see that

$$H_{k,n}(x) + H_{n-k+1,n}(-x) = 1 \tag{4.7}$$

$$h_{k,n}(x) = h_{n-k+1,n}(-x) \tag{4.8}$$

The above expressions are true for the order statistics from any continuous symmetric distributions.

5. Applications of the Distributions of $X_{(k)}$ in Obtaining Estimates of μ and σ Based on ~~Single~~ one and two Order Statistics

In some problems it may be desirable to obtain estimators of μ and σ using only a single order statistic. Such estimators are "inefficient." Mosteller [13] first described and studied such "inefficient" estimators. We are concerned with inefficient estimators where only a single order statistic is used.

(a) Point Estimators of μ Based on $Y_{(k)}$ (σ is Known)

If $y_{(k)}$ is the observed kth ordered statistic in a sample of size n from $L(\mu,\sigma^2)$, then an unbiased linear estimate of μ is

$$\mu = a + by_{(k)} \tag{5.1}$$

where $a=a(k,n)$ and $b=b(k,n)$ are so chosen that μ is unbiased estimate of μ.

It follows from (5.1) that

$$\mu = E(\hat{\mu}) = a + b\left[\mu + \sigma\mu_1'(k,n)\right] \tag{5.2}$$

and hence $b = 1$ and $a = -\sigma\mu_1'(k,n)$.

Hence

$$\hat{\mu} = y_{(k)} - \sigma\mu_1'(k,n) . \tag{5.2a}$$

$$\text{Var}\,\hat{\mu} = \sigma^2 \kappa_2(k,n) = \sigma^2\mu_2(k,n). \tag{5.3}$$

From (5.3) and from the expression for $\kappa_2(k,n)$, it is easily seen that Var $\hat{\mu}$ decreases(increases) if $k < n-k+1$ ($k > n-k+1$). Hence the unbiased minimum variance linear estimator in the class of inefficient single order statistics is

$$\hat{\mu} = -\sigma\,\mu_1'([\tfrac{n+1}{2}],n) + y_{([\frac{n+1}{2}])} \tag{5.4}$$

where $[x]$ denotes the largest integer $\leq x$.

$$\text{Var}\,(\hat{\mu}) = \sigma^2\kappa_2([\tfrac{n+1}{2}],n) = \sigma^2\mu_2\,([\tfrac{n+1}{n}],\,n)\ . \tag{5.5}$$

Now the Cramér-Rao lower bound for the variance of $\delta = \delta(X_1, X_2, \ldots, X_n)$ of an unbiased estimator μ in the distribution $L(\mu,\sigma^2)$ can be shown to be

$$\text{Var}\,\delta \geq \frac{3\sigma^2}{g^2 n}\ . \tag{5.6}$$

Hence the efficiency of $\hat{\mu}$ (as measured by the ratio of the above lower bound and the variance of $\hat{\mu}$ as given in (5.5)) is

$$(5.7) \qquad e(\hat{\mu}) = \frac{3}{n\left[\frac{\pi^2}{3} - \sum_{x=1}^{\left[\frac{n+1}{2}\right]-1} \frac{1}{x^2} - \sum_{x=1}^{n-\left[\frac{n+1}{2}\right]} \frac{+1}{x^2}\right]} .$$

The following brief table gives the values of $e(\hat{\mu})$ for selected values of n(odd)

n	3	5	7	9
$e(\hat{\mu})$	.78	.76	.75	.75

<u>Confidence limits for μ</u>

If σ is known, then the 100(1-α) percent confidence limits for μ based on a single order statistic are

$$(5.8) \qquad y_{(k)} - \sigma C_1 \text{ and } y_{(k)} - \sigma C_2 \quad ,$$

where C_1 and C_2 are the percentage points of $X_{(k)}$ such that

$$(5.9) \qquad \int_{C_1}^{C_2} f(X_{(k)})\, dX_{(k)} = 1 - \alpha .$$

The expected values of these limits are

$$(5.10)\qquad \mu + \sigma\mathcal{K}_1(k,n) - \sigma C_1 \text{ and } \mu + \sigma\mathcal{K}_1(k,n) - \sigma C_2 .$$

The limits get closer to each other and hence the length of the interval gets shorter as k approaches $\frac{n+1}{2}$ (n odd), $\frac{n}{2}$ or $\frac{n+2}{2}$(n even). Hence for $n = 2m + 1$, the two-sided symmetrical confidence limits for μ with confidence coefficient $100(1-\alpha)$ percent are

$$(5.11)\qquad y_{(m+1)} - \sigma x_{1-\frac{\alpha}{2}}(m+1,\ 2m+1),\quad y_{(m+1)} + \sigma x_{1-\frac{\alpha}{2}}(m+1,\ 2m+1).$$

The limits yield the shortest $100(1-\alpha)$ percent confidence interval estimator for μ with length $2\sigma x_{1-\frac{\alpha}{2}}(m+1,\ 2m+1)$.

Estimator of σ based on range and quasi-range

An unbiased estimator $\hat{\sigma}$ of σ based on a quasi-range can be written as follows

$$(5.12)\qquad \hat{\sigma} = \frac{y_{(n-k+1)} - y_{(k)}}{2\mathcal{K}_1(n-k+1,n)},\ k = 1,\ 2,\ \ldots,\ [\tfrac{n}{2}] .$$

This type of estimator for σ was studied first by Mosteller [13] who showed that for large samples the variance of $\hat{\sigma}$ is minimized if $\frac{k}{n} = \lambda \doteq .0694$. In another paper we intend to study a similar result for the logistic distribution. Here we just propose the range or the quasi-range as estimators of σ. The factor $\frac{1}{2\mathcal{K}_1(n-k+1,n)}$ which goes with the quasi-range and makes it unbiased can be obtained from the Table II of this paper.

If we use the estimator based on the range, which is,

(5.13) $$\hat{\sigma} = \frac{y_{(n)} - y_{(1)}}{2\,k_1(n,n)}$$

then we are naturally interested in the distribution of the range. We study this in the next Section.

6. Distribution of the Range

Let us define the sample range W_n by

(6.1) $$W_n = (Y_{(n)} - Y_{(1)})/\sigma = X_{(n)} - X_{(1)}$$

where $Y_{(k)}$ and $X_{(k)}$ denote the kth order statistics from the logistic distributions $L(\mu,\sigma^2)$ and $L(0,1)$, respectively.

Now, the joint density function of $X_{(1)}$ and $X_{(n)}$ is

(6.2) $$h(X_{(1)}, X_{(n)}) = n(n-1)\left[F(X_{(n)}) - F(X_{(1)})\right]^{n-2} f(X_{(1)})\, f(X_{(n)})$$

from which we obtain the density function $p(w)$ of the range W_n as

(6.3) $$p(w) = n(n-1) \int_{-\infty}^{\infty} \left[F(x+w) - F(x)\right]^{n-2} f(x)\, f(x+w)dx\ .$$

The cumulative distribution function, $P(W_n \leq w)$, of the range is

$$(6.4)\qquad P(W_n \leq w) = n \int_{-\infty}^{\infty} \left[F(x+w) - F(x)\right]^{n-1} f(x)dx .$$

or

$$(6.5)\qquad P(W_n \leq w) = n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \int_{-\infty}^{\infty} [F(x+w)]^{n-1-j} [F(x)]^j f(x)dx.$$

It should be pointed out that formulae (6.2)-(6.5) apply to the distribution of range from any continuous distributions. Also if the distribution is symmetric about $x = 0$, this fact can be utilized to express (6.2)-(6.5) slightly differently.

Now we shall derive the c.d.f. of the range W_n from $L(\mu, \sigma^2)$. We start with (6.5) and obtain

$$(6.6)\qquad P(W_n \leq w) = n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \int_{-\infty}^{\infty} \frac{g\, e^{-gx} dx}{(1+a\, e^{-gx})^{n-1-j}\, (1+e^{-gx})^{2+j}}$$

where $a = \exp(-gw)$.

Form (6.6) by substituting $t = 1/[1+a \exp(-g\, x)]$, we obtain

$$(6.7)\qquad P(W_n \leq w) = n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j a^{j+1} A(j,n)$$

where

$$(6.8) \qquad A(j,n) = \int_0^1 t^{n-1} (1+ct)^{-j-2}\, dt, \quad c = a-1$$

$$= \frac{1}{(-c)^{n-1}} \sum_{\alpha=0}^{n-1} \binom{n-1}{\alpha} (-1)^{\alpha} \int_0^1 (1+ct)^{\alpha-j-2}\, dt$$

or

$$(6.9) \quad A(j,n) = \frac{-1}{(1-a)^{n}} \left[\binom{n-1}{j+1} (-1)^{j+1} \log a + \sum_{\substack{\alpha=0 \\ \alpha \neq j+1}}^{n-1} \binom{n-1}{\alpha} (-1)^{\alpha} \frac{a^{\alpha-j-1}-1}{\alpha-j-1} \right] .$$

Hence, from (6.7) and (6.9), we obtain the c.d.f. of W_n as

$$(6.10) \qquad P(W_n \leq w) = \frac{n}{(1-a)^n} \sum_{j=0}^{n-1} \binom{n-1}{j} (-a)^{j+1}$$

$$\cdot \left\{ \binom{n-1}{j+1} (-1)^{j} \log \left(\frac{1}{a}\right) + \sum_{\substack{\alpha=0 \\ \alpha \neq j+1}}^{n-1} \binom{n-1}{\alpha} (-1)^{\alpha} \frac{a^{\alpha-j-1}-1}{\alpha-j-1} \right\} ,$$

where in the first term inside the braces $\binom{n-1}{j+1}$ is to be put equal to zero for $j > n-2$.

By differentiating (6.7) with respect to w one can obtain the density function $p(w)$ as follows

$$(6.11) \quad p(w) = n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{j}\, g\, a^{j+1} \left[(j+2)\, a\, A(j+1, n+1) - (j+1) A(j,n) \right] .$$

For $n = 2$ and 3 we obtain from (6.10)

(6.12) $$P(W_2 \leq w) = \frac{1-a^2-2(gwa)}{(1-a)^2}$$

(6.13) $$P(W_3 \leq w) = \frac{1+9a-9a^2-a^3-6(gwa)(1+a)}{(1-a)^3} .$$

Using (6.12) and (6.13), the probability integral of the range has been computed for $n = 2$ and 3. These values are given below in Table A along with the values of the probability integral of the sample range of the normal distribution $N(\mu,1)$ which have been taken from [12].

Table A

Probability Integrals of the Sample Range from $L(\mu,1)$(top) and from $N(\mu,1)$(bottom).

n \ w	.20	.40	.60	.80	1.00	1.50	2.00	2.50	3.00	3.50	4.00
2	.12039	.23768	.34902	.45212	.54213	.73047	.85109	.92224	.96113	.98121	.99115
	.1125	.2227	.3286	.4284	.5205	.7112	.8427	.9229	.9661	.9867	.9953
3	.01306	.05138	.11084	.18655	.28725	.50027	.69272	.82676	.93300	.95340	.97765
	.0110	.0431	.0944	.1616	.2407	.4614	.6665	.8195	.9145	.9870	.9988

It is interesting to note that for $-\infty < w < \infty$ or $0 < a < 1$, we obtain from (6.12) and (6.13)

(6.14) $$(1-a)^2+2(gwa) > 1 - a^2 > 2(gwa) ,$$

$$(1-a)^3+6(gwa)(1+a) > 1+9a(1-a)-a^3 > 6(gwa)(1+a),$$

where $a=\exp(-gw)$ and $g=\pi/3^{1/2}$.

In general, for $0 < a < 1$, we have

$$(6.15)\quad \frac{(1-a)^n}{n} > \sum_{j=0}^{n-1} \binom{n-1}{j}(-a)^{j+1} \left\{ \binom{n-1}{j+1}(-1)^j \log \frac{1}{a} + \sum_{\substack{\alpha=0 \\ \alpha \neq j+1}}^{n-1} \binom{n-1}{\alpha}(-1)^{\alpha} \frac{a^{\alpha-j-1}-1}{\alpha-j-1} \right\} > 0$$

and

$$(6.16)\quad \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j g\, a^{j+1}\Big[(j+2)a\, A(j+1,n+1)-(j+1)A(j,n)\Big] > 0$$

where $A(j,n)$ is given by (6.9) .

7. Description of Tables and Comparison with Normal Order Statistics

Birnbaum [3] has tables and graphs comparing normal and logistic order statistics. A . comparison of this type is omitted in this paper. It may be recalled that Plackett [16] observed that the standard logistic and standard normal are similar in shape between the range of logistic probability levels .05 and .95. A comparison with the tables of Owen [11, p.254] and Pearson and Hartley's tables [12, p.104] shows that the two c.d.f.'s agree to within 2 units in the second decimal place. The density function curve of the logistic crosses the density curve of the normal between 0.68 and 0.69. The inflection points of the standard logistic are $\pm$ 0.53 (approx.) whereas the inflection points of standard normal are $\pm$ 1.00 .

Table 1 of this paper gives the exact expressions for the moments about the origin of the kth order statistic in a sample of size n from the L(0,1).

Since $\mu_r'(k,n) = (-1)^r \mu_r'(n-k+1,n)$, we give only the values of $\mu_r'(k,n)$ for $k = 1, 2, \ldots, n/2$ (n even), $(n+1)/2$ (n odd). The range of values of n is $n = 1(1)10$.

Table 2 gives the numerical values of $\kappa_r(k,n)$, the cumulants of the kth order statistic in a sample of size n from the L(0,1) distribution. The values are correct to six decimal places as given in the table. The values of n range from 1 to 10 by jumps of 1. Only the lower half of n order statistics are tabulated for each n. The remaining values can be obtained from the symmetry relation $\kappa_r(k,n) = (-1)^r \kappa_r(n-k+1,n)$. In computing the table 2, we have used the tables of the polygamma functions [4], [5].

Table 3 gives the percentage points of the kth order statistic in a sample of size n from the L(0,1) distribution. These computations are similar to the computations described in [6], [9], [10]. The percentiles of the distributions were obtained from [12], [7]. The values are given to four decimal places. Independent checks have revealed no errors. However, the fourth decimal place may be off by one unit. The table contains values for k=1(1)n, n=1(1)10 and $k=1,\frac{n}{2}$(n even), $\frac{n+1}{2}$ (n odd) and n, for n=11(1)25. The 100α percentage points are listed for α = .50, .75, .90, .95, .975 and .99. The $100(1-\alpha)$ percentage point of kth order statistic is the negative of the 100α percentage point of the (n-k+1)th order statistic.

TABLE I

Exact Moments of the kth Order Statistic in a
Sample of size n from a Standard Logistic Distribution

n	k	μ_1'	μ_2'	μ_3'	μ_4'
1	1	0	1	0	21/5
2	1	-a	1	-3a	21/5
3	1	-3a/2	$b(1+\pi^2/3)$	-9a/2	$d(2\pi^2+7\pi^4/15)$
	2	0	$b(-2+\pi^2/3)$	0	$d(-4\pi^2+7\pi^4/15)$
4	1	11a/6	$b(2+\pi^2/3)$	$c(3+11\pi^2/2)$	$d(4\pi^2+7\pi^4/15)$
	2	-a/2	$b(-2+\pi^2/3)$	$c(9-3\pi^2/2)$	$d(-4\pi^2+7\pi^4/15)$
5	1	-25a/12	$b(35/12+\pi^2/3)$	$-c(15/2+25\pi^2/4)$	$d(1+35\pi^2/6+7\pi^4/15)$
	2	-5a/6	$b(-5/3+\pi^2/3)$	$c(15-5\pi^2/2)$	$d(-4-10\pi^2/3+7\pi^4/15)$
	3	0	$b(-5/2+\pi^2/3)$	0	$d(6-5\pi^2+7\pi^4/15)$
6	1	-137a/10	$b(35/4+\pi^2/3)$	$-c(51/4+137\pi^2/20)$	$d(3+15\pi^2/2+7\pi^4/15)$
	2	-13a/12	$b(-5/4+\pi^2/3)$	$c(75/4-13\pi^2/4)$	$d(-9-5\pi^2/2+7\pi^4/15)$
	3	-a/3	$b(-5/2+\pi^2/3)$	$c(15/2-\pi^2)$	$d(6-5\pi^2+7\pi^4/15)$
7	1	-147a/60	$b(203/45+\pi^2/3)$	$c(441/24+147\pi^2/20)$	$d(35/6+406\pi^2/45+7\pi^4/15)$
	2	-77a/60	$b(-49/60+\pi^2/3)$	$c(21-77\pi^2/20)$	$d(-14-49\pi^2/30+7\pi^4/15)$
	3	-7a/12	$b(-7/3+\pi^2/3)$	$c(105/8-7\pi^2/4)$	$d(7/2-14\pi^2/3+7\pi^4/15)$
	4	0	$b(-49/12+\pi^2/3)$	0	$d(28/3-$[illegible]$/9+7\pi^4/15)$

TABLE I (CONT'D)

n	k	μ_1'	μ_2'	μ_3'	μ_4'
	1	$-363a/140$	$b(469/90+\pi^2/3)$	$-c(967/40+1089\pi^2/140)$	$d(28/3+469\pi^2/45+7\pi^4/15)$
	2	$-29a/20$	$b(-7/18+\pi^2/3)$	$c(889/40-87\pi^2/20)$	$d(-56/3-7\pi^2/9+7\pi^4/15)$
8	3	$-47a/60$	$b(-21/10+\pi^2/3)$	$c(693/40-47\pi^2/20)$	$d(-21\pi^2/5+7\pi^4/15)$
	4	$-a/4$	$b(-49/18+\pi^2/3)$	$c(49/8-3\pi^2/4)$	$d(28/3-49\pi^2/9+7\pi^4/15)$
	1	$-761a/280$	$b(29531/5040+\pi^2/3)$	$c(2403/80-2283\pi^2/280)$	$d(1069/80+29531\pi^2/2520+7\pi^4/15)$
	2	$-223a$	$b(81315+\pi^2/3)$	$c(909/40-669\pi^2/140)$	$d(-229/10+16\pi^2/315+7\pi^4/15)$
9	3	$-19a/20$	$b(-331/180+\pi^2/3)$	$c(819/40-57\pi^2/20)$	$d(-77/20-331\pi^2/90+7\pi^4/15)$
	4	$-9a/20$	$b(-118/45+\pi^2/3)$	$c(441/40-27\pi^2/20)$	$d(181/30-236\pi^2/45+7\pi^4/15)$
	5	0	$b(-205/72+\pi^2/3)$	0	$d(273/24-205\pi^2/36+7\pi^4/15)$
	1	$-7129a/2520$	$b(6515/1008+\pi^2/3)$	$c(4523/126-7129\pi^2/840)$	$d(285/16+6515\pi^2/1008+7\pi^4/15)$
	2	$-481a/280$	$b(61/144+\pi^2/3)$	$c(1271/56-1443\pi^2/280)$	$d(-427/16+61\pi^2/72+7\pi^4/15)$
10	3	$-1377a/1260$	$b(85/84+\pi^2/3)$	$c(1271/56-1377\pi^2/420)$	$d(-31/4+85\pi^2/42+7\pi^4/15)$
	4	$-37a/60$	$b(-89/36+\pi^2/3)$	$c(359/24-37\pi^2/20)$	$d(21/4-89\pi^2/18+7/15\pi^4)$
	5	$-a/5$	$b(-205/72+\pi^2/3)$	$c(41/8-3\pi^2/5)$	$d(91/8-205\pi^2/36+7/15\pi^4)$

Here $a = \sqrt{3}/\pi = 0.5513,2889$

$b = 3/\pi^2 = 0.3039,6355$

$c = \sqrt{3}/\pi^3 = 0.0558,6130$

$d = 9/\pi^4 = 0.0923,9384$

23.

TABLE II

Cumulants of the kth Order Statistic in a Sample of Size n from a Standard Logistic Distribution

n	k	k_1	k_2	k_3	k_4
1	1	0.000000	1.000000	0.000000	1.200000
2	1	-0.551329	0.696036	-0.335168	0.645636
3	1	-0.826993	0.620046	-0.377064	0.610989
	2	0.000000	0.392073	0.000000	0.091274
4	1	-1.010770	0.580272	-0.389477	0.604145
	2	-0.275664	0.316082	-0.041896	0.056626
5	1	-1.148502	0.567274	-0.394714	0.601980
	2	-0.459441	0.282308	-0.054310	0.049782
	3	0.000000	0.240091	0.000000	0.021949
6	1	-1.258868	0.555116	-0.397396	0.601093
	2	-0.597273	0.263331	-0.059547	0.047617
	3	-0.183776	0.206317	-0.012414	0.015135
7	1	-1.350756	0.546672	-0.398947	0.600665
	2	-0.707539	0.251152	-0.062228	0.046730
	3	-0.321609	0.187320	-0.017651	0.013006
	4	0.000000	0.172544	0.000000	0.008291
8	1	-1.429517	0.540469	-0.399925	0.600434
	2	-0.799432	0.242709	-0.063780	0.046302
	3	-0.4318~~69~~74	0.175161	-0.020332	0.012082
	4	-0.137832	0.153546	-0.005237	0.006125
9	1	-1.498433	0.535719	-0.400579	0.600299
	2	-0.878188	0.236505	-0.064757	0.046071
	3	-0.523763	0.166718	-0.021884	0.011654
	4	-0.248098	0.141387	-0.007918	0.005238
	5	0.000000	0.134548	0.000000	0.003960
10	1	-1.559692	0.531967	-0.401039	0.602143
	2	-0.947104	0.231756	-0.065411	0.045936
	3	-0.602524	0.160514	-0.022861	0.011423
	4	-0.339986	0.132944	-0.009470	0.004810
	5	-0.110266	0.122390	-0.002681	0.003073

TABLE III

Percentage Points of the kth Order Statistic in a Sample of Size n from a Standard Logistic Distribution

n	k \ α	0.500	0.750	0.900	0.950	0.975	0.990
1	1	0.0000	0.6057	1.2114	1.6234	2.0198	2.5334
2	1	-0.4859	0.0000	0.4251	0.6863	0.9220	1.2114
	2	0.4859	1.0289	1.6083	2.0126	2.4054	2.9170
3	1	-0.7428	-0.2933	0.0792	0.2972	0.4872	0.7126
	2	0.0000	0.3996	0.7789	1.0224	1.2472	1.5278
	3	0.7428	1.2659	1.8367	2.2385	2.6305	3.1410
4	1	-0.9179	-0.4859	-0.1382	0.0598	0.2290	0.4251
	2	-0.2565	0.0966	0.4144	0.6098	0.7848	0.9968
	3	0.2565	0.6277	0.9892	1.2262	1.4469	1.7241
	4	0.9179	1.4312	1.9977	2.3983	2.7893	3.2999
5	1	-1.0507	-0.6290	-0.2957	-0.1090	0.0482	0.2279
	2	-0.4313	-0.1013	0.1868	0.3593	0.5109	0.6912
	3	0.0000	0.3186	0.6156	0.8021	0.9710	1.1777
	4	0.4313	0.7861	1.1402	1.3738	1.5922	1.8677
	5	1.0507	1.5583	2.1222	2.5221	2.9128	3.4230
6	1	-1.1578	-0.7428	-0.4188	-0.2396	-0.0900	0.0792
	2	-0.5640	-0.2478	0.0227	0.1820	0.3202	0.4822
	3	-0.1748	0.1178	0.3825	0.5446	0.6889	0.8623
	4	0.1748	0.4753	0.7612	0.9428	1.1084	1.3120
	5	0.5640	0.9094	1.2583	1.4897	1.7070	1.9813
	6	1.1578	1.6615	2.2237	2.6230	3.0136	3.5236
7	1	-1.2474	-0.8373	-0.5199	-0.3457	-0.2015	-0.0396
	2	-0.6709	-0.3639	-0.1049	0.0456	0.1751	0.3253
	3	-0.3074	-0.0307	0.2148	0.3626	0.4925	0.6497
	4	0.0000	0.2726	0.5245	0.6808	0.8210	0.9905
	5	0.3074	0.5967	0.8756	1.0541	1.2176	1.4193
	6	0.6709	1.0102	1.3554	1.5854	1.8017	2.0752
	7	1.2474	1.7485	2.3095	2.7084	3.0987	3.6087
8	1	-1.3245	-0.9179	-0.6054	-0.4351	-0.2948	-0.1382
	2	-0.7604	-0.4601	-0.2092	-0.0647	0.0585	0.2005
	3	-0.4143	-0.1482	0.0844	0.2228	0.3434	0.4850
	4	-0.1326	0.1229	0.3544	0.4857	0.6209	0.7705
	5	0.1326	0.3927	0.6365	0.7891	0.9366	1.0935
	6	0.4143	0.6959	0.9701	1.1455	1.3085	1.5088
	7	0.7604	1.0953	1.4378	1.6668	1.8824	2.1555
	8	1.3245	1.8236	2.3835	2.7824	3.1723	3.6825

n	k \ α	0.500	0.750	0.900	0.950	0.975	0.990
9	1	-1.3921	-0.9883	-0.6795	-0.5122	-0.3748	-0.2223
	2	-0.8374	-0.5420	-0.2973	-0.1573	-0.0386	0.0973
	3	-0.5037	-0.2453	-0.0219	0.1098	0.2238	0.3566
	4	-0.2395	0.0044	0.2222	0.3536	0.4690	0.6056
	5	0.0000	0.2421	0.4646	0.6017	0.7238	0.8704
	6	0.2395	0.4910	0.7291	0.8791	1.0147	1.1799
	7	0.5037	0.7799	1.0506	1.2254	1.3863	1.5857
	8	0.8374	1.1689	1.5094	1.7377	2.0961	2.2255
	9	1.3921	1.8899	2.4493	2.8478	3.2373	3.7478
10	1	-1.4523	-1.0507	-0.7450	-0.7069	-0.4450	-0.2957
	2	-0.9050	-0.6135	-0.3734	-0.2370	-0.1217	0.0096
	3	-0.5808	-0.3282	-0.1117	0.0152	0.1243	0.2506
	4	-0.3290	-0.0935	0.1145	0.2388	0.3472	0.4747
	5	-0.1069	0.1228	0.3311	0.4579	0.5699	0.7031
	6	0.1069	0.3266	0.5559	0.6900	0.8099	0.9544
	7	0.3290	0.5742	0.8082	0.9562	1.0904	1.2543
	8	0.5808	0.8528	1.1206	1.2943	1.4543	1.6531
	9	0.9050	1.2340	1.5729	1.8007	2.0151	2.2876
	10	1.4523	1.9484	2.5073	2.9056	3.2953	3.8051
11	1	-1.5066	-1.1068	-0.8035	-0.6403	-0.5073	-0.3606
	6	0.0000	0.2199	0.4214	0.5449	0.6544	0.7853
	11	1.5066	2.0017	2.5601	2.9583	3.3482	3.8580
12	1	-1.5561	-1.1578	-0.8564	-0.6948	-0.5634	-0.4189
	6	-0.0895	0.1210	0.3118	0.4277	0.5298	0.6511
	7	0.0895	0.3022	0.4985	0.6196	0.7273	0.8564
	12	1.5561	2.0503	2.6082	3.0064	3.3962	3.9060
13	1	-1.6014	-1.2044	-0.9047	-0.7445	-0.6144	-0.4716
	7	0.0000	0.2029	0.3883	0.5016	0.6017	0.7210
	13	1.6014	2.1216	2.6526	3.0506	3.4404	3.9501
14	1	-1.6434	-1.2474	-0.9492	-0.7900	-0.6611	-0.5198
	7	-0.0770	0.1185	0.2955	0.4029	0.4974	0.6092
	8	0.0770	0.2740	0.4552	0.5664	0.6651	0.7827
	14	1.6434	2.1362	2.6936	3.0915	3.4813	3.9910
15	1	-1.6823	-1.2873	-0.9904	-0.8322	-0.7042	-0.5642
	8	0.0000	0.1893	0.3620	0.4672	0.5606	0.6701
	15	1.6823	2.1746	2.7318	3.1296	3.5194	4.0290
16	1	-1.7187	-1.3245	-1.0286	-0.8713	-0.7442	-0.6054
	8	-0.0675	0.1156	0.2815	0.3820	0.4703	0.5747
	9	0.0675	0.2519	0.4211	0.5245	0.6160	0 7248
	16	1.7187	2.2105	2.7675	3.1653	3.5550	4.0646

TABLE III (CONT'D)

n	k \ α	0.500	0.750	0.900	0.950	0.975	0.990
17	1	-1.7528	-1.3593	-1.0644	-0.9078	-0.7815	-0.6437
	9	0.0000	0.1781	0.3403	0.4390	0.5259	0.6287
	17	1.7528	2.2443	2.8010	3.1987	3.5884	4.0981
18	1	-1.7850	-1.3921	-1.0981	-0.9421	-0.8164	-0.6795
	9	-0.0602	0.1128	0.2694	0.3642	0.4474	0.5455
	10	0.0602	0.2341	0.3933	0.4905	0.5761	0.6778
	18	1.7850	2.2760	2.8326	3.2303	3.6200	4.1296
19	1	-1.8153	-1.4230	-1.1298	-0.9744	-0.8493	-0.7132
	10	0.0000	0.1687	0.3222	0.4153	0.4973	0.5941
	19	1.8153	2.3061	2.8625	3.2602	3.6498	4.1594
20	1	-1.8441	-1.4523	-1.1598	-1.0049	-0.8803	-0.7450
	10	-0.0542	0.1101	0.2588	0.3487	0.4276	0.5203
	11	0.0542	0.2194	0.3702	0.4620	0.5429	0.6387
	20	1.8441	2.3346	2.8909	3.2885	3.6781	4.1877
21	1	-1.8715	-1.4802	-1.1882	-1.0338	-0.9097	-0.7750
	11	0.0000	0.1606	0.3066	0.3951	0.4729	0.5647
	21	1.8715	2.3617	2.9179	3.3154	3.7050	4.2146
22	1	-1.8975	-1.5066	-1.2152	-1.0613	-0.9377	-0.8035
	11	-0.0494	0.1075	0.2494	0.3351	0.4103	0.4987
	12	0.0494	0.2069	0.3506	0.4379	0.5147	0.6055
	22	1.8975	2.3875	2.9436	3.3411	3.7307	4.2402
23	1	-1.9224	-1.5319	-1.2411	-1.0875	-0.9642	-0.8306
	12	0.0000	0.1536	0.2931	0.3776	0.4518	0.5392
	23	1.9224	2.4121	2.9680	3.3656	3.7552	4.2648
24	1	-1.9462	-1.5561	-1.2657	-1.1125	-0.9896	-0.8564
	12	-0.0453	0.1050	0.2409	0.3231	0.3949	0.4794
	13	0.0453	0.1962	0.3336	0.4171	0.4903	0.5768
	24	1.9462	2.4357	2.9916	3.3891	3.7787	4.2882
25	1	-1.9691	-1.5792	-1.2892	-1.1364	-1.0138	-0.8811
	13	0.0000	0.1475	0.2813	0.3623	0.4333	0.5169
	25	1.9691	2.4584	0.3016	3.4116	3.8012	4.3107

For given n, k and α, the above table gives the values of y for which

$$\frac{1}{B(k,n-k+1)} \int_0^{F(y)} x^{k-1}(1-x)^{n-k}dx = \alpha$$

where $F(y)$ denotes the c.d.f. of a standard logistic random variable.

References

[1] Berkson, J. (1944). Application of the logistic function to Bio-assay. J. Amer. Statist. Assoc. 39 357-365.

[2] Berkson, J. and Hodges, Jr. J. L. (1960). A minimax estimator for the logistic function. Proceedings of the fourth Berkeley Symposium on Mathematical Statistics and Probability Vol. 4. University of California Press 1961, Berkeley and Los Angeles 77-86.

[3] Birnbaum, A. (1958). On logistic order statistics. Ann. Math. Statist. 29. 1285 (Abstract).

[4] British Association Mathematical Tables, Vol. 1 (1931)

[5] Davis, H. T. (1935). Tables of the higher mathematical functions. The Principia Press, Inc.

[6] Eisenhart, Churchill, Deming, Lola S. and Martin, Celia S. (1948). On the arithmetic mean and the median in small samples from the normal and certain non-normal populations. Ann. Math. Statist. 19. 599.

[7] Govindrajulu, Zakkula and Hubacker, N. W. (1962). Percentile points of order statistics in samples from beta, normal, chi (1 d.f.) populations. Report 101. Statistical Laboratory. Case Institute of Technology, Cleveland.

[8] Gumbel, E. J. (1958). Statistics of Extremes. Columbia University Press.

[9] Gupta, Shanti S. (1960). Order statistics from the gamma distribution. Technometrics 2 243-262.

[10] Gupta, Shanti S. (1961). Percentage points and modes of order statistics from the normal distribution. Ann. Math. Statist. 32 888-893.

[11] Owen, D. B. (1962). Handbook of Statistical Tables. Addison-Wesley Publishing Co., Inc.

[12] Pearson, E. S. and Hartley, H. O. (1958). Biometrika Tables for Statisticians. Cambridge University Press.

[13] Mosteller, F. (1946). On some useful 'inefficient' statistics. Ann. Math. Statist. 17 377-408.

[14] Pearl, R. and Reed, L. J. (1920). On the rate of growth of the population of the United States since 1790 and its mathematical representation. Proc. Nat. Acad. Sc. 6 275-288.

[15] Plackett, R. L. (1958). Linear estimation from censored data. Ann. Math. Statist. 29 131-142

[16] Plackett, R. L. (1959). The analysis of life test data. Technometrics 1 9-19.

[17] Verhulst, P F. (1845). Recherches mathématiques sur la loi d'accroisement de la population. Nouveaux mémoires de l'ac. de Bruxelles. 18.