Lecture 3. Experiments with a Single Factor: ANOVA

Montgomery Sections 3.1 through 3.5 and Section 15.1.1
# Tensile Strength Experiment

Investigate the tensile strength of a new synthetic fiber. The factor is the weight percent of cotton used in the blend of the materials for the fiber and it has five levels.

<table>
<thead>
<tr>
<th>percent of cotton</th>
<th>tensile strength</th>
<th>total</th>
<th>average</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>7</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>17</td>
<td>12</td>
</tr>
<tr>
<td>25</td>
<td>14</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>30</td>
<td>19</td>
<td>25</td>
<td>22</td>
</tr>
<tr>
<td>35</td>
<td>7</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>
## Data Layout for Single-Factor Experiments

<table>
<thead>
<tr>
<th>treatment</th>
<th>observations</th>
<th>totals</th>
<th>averages</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$y_{11}$</td>
<td>$y_{12}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>2</td>
<td>$y_{21}$</td>
<td>$y_{22}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>a</td>
<td>$y_{a1}$</td>
<td>$y_{a2}$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>
Analysis of Variance

- Statistical Model (Factor Effects Model):

\[ y_{ij} = \mu + \tau_i + \epsilon_{ij} \]

\[ \begin{cases} 
  i = 1, 2 \ldots, a \\
  j = 1, 2, \ldots, n_i 
\end{cases} \]

\( \mu \) - grand mean; \( \tau_i \) - \( i \)th treatment effect; \( \epsilon_{ij} \), iid \( \sim N(0, \sigma^2) \) - error

Constraint: \( \sum_{i=1}^{a} \tau_i = 0 \) (Conceptual Approach; SAS: \( \tau_a = 0 \)).

- Estimates for parameters:

\[ \hat{\mu} = \bar{y}.. \]

\[ \hat{\tau}_i = (\bar{y}_{i.} - \bar{y}..) \]

\[ \hat{\epsilon}_{ij} = y_{ij} - \bar{y}_i. \quad \text{(residual)} \]

- Basic Hypotheses:

\[ H_0 : \tau_1 = \tau_2 = \ldots = \tau_a = 0 \text{ vs } H_1 : \tau_i \neq 0 \text{ for at least one } i \]
## Analysis of Variance (ANOVA) Table

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>( F_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between</td>
<td>( SS_{\text{Treatment}} )</td>
<td>( a - 1 )</td>
<td>( MS_{\text{Treatment}} )</td>
<td>( F_0 )</td>
</tr>
<tr>
<td>Within</td>
<td>( SS_E )</td>
<td>( N - a )</td>
<td>( MSE )</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>( SS_T )</td>
<td>( N - 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **If balanced:** \( N = n \times a \)
  
  \[
  SS_T = \sum \sum y_{ij}^2 - y^2../N; \quad SS_{\text{Treatment}} = \frac{1}{n} \sum y_{i.}^2 - y^2../N \\
  SS_E = SS_T - SS_{\text{Treatment}}
  \]

- **If unbalanced:** \( N = \sum_{i=1}^{a} n_i \)
  
  \[
  SS_T = \sum \sum y_{ij}^2 - y^2../N; \quad SS_{\text{Treatment}} = \sum \frac{y_{i.}^2}{n_i} - y^2../N \\
  SS_E = SS_T - SS_{\text{Treatment}}
  \]

- \( SS_{\text{Treatments}} = \sum_{i=1}^{a} n_i \hat{\tau}_i^2 \) and \( SS_E = \sum_i \sum_j \hat{\epsilon}_{ij}^2. \)
• The Expected Mean Squares (EMS) are

\[ E(MS_E) = \sigma^2 \]

\[ E(MS_{\text{Treatment}}) = \sigma^2 + \sum n_i \tau_i^2 / (a - 1) \]

• Test Statistic

\[ F_0 = \frac{SS_{\text{Treatments}}/(a - 1)}{SS_E/(N - a)} = \frac{MS_{\text{Treatments}}}{MS_E} \]

• Under \( H_0 \):

\[ F_0 = \frac{SS_{\text{Treatment}}/\sigma^2(a - 1)}{SS_E/\sigma^2(N - a)} = \frac{\chi^2_{a-1}/(a - 1)}{\chi^2_{N-a}/(N - a)} \sim F_{a-1,N-a} \]

• Decision Rule: If \( F_0 > F_{\alpha,a-1,N-a} \) then reject \( H_0 \)

• When \( a = 2 \), the square of the t-test statistic \( t^2_0 = \frac{MS_{\text{Treatment}}}{MS_E} = F_0 \).
  
  \( F \)-test and two-sample two-sided test are equivalent.
Example

Twelve lambs are randomly assigned to three different diets. The weight gain (in two weeks) is recorded. Is there a difference between the diets?

<table>
<thead>
<tr>
<th>Diet 1</th>
<th>8</th>
<th>16</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diet 2</td>
<td>9</td>
<td>16</td>
<td>21</td>
</tr>
<tr>
<td>Diet 3</td>
<td>15</td>
<td>10</td>
<td>17</td>
</tr>
</tbody>
</table>

- \( N = 12, \sum \sum y_{ij} = 156 \) and \( \bar{y}_{..} = 156/12 = 13 \).
- \( n_1 = 3, y_1. = 33, \bar{y}_1. = 11; n_2 = 5, y_2. = 75, \bar{y}_2. = 15; n_3 = 4, y_3. = 48 \) and \( \bar{y}_3. = 12 \).
- \( \hat{\tau}_1 = \bar{y}_1. - \bar{y}_{..} = 11 - 13 = -2; \) Similarly, \( \hat{\tau}_2 = 15 - 13 = 2 \) and \( \hat{\tau}_3 = 12 - 13 = -1 \).
- \( \text{SS}_T = \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2 = 246 \).
- \( \text{SS}_{\text{Treatment}} = 3 \times (-2)^2 + 5 \times (2)^2 + 4 \times (-1)^2 = 36 \).
- \( \text{SS}_E = 246 - 36 = 210; \text{MS}_E = \hat{\sigma}^2 = 210/(12 - 3) = 23.33 \).
- \( F_0 = (36/2)/(210/9) = 0.77; \) P-value \( > 0.25 \).
- Fail to reject \( H_0 : \tau_1 = \tau_2 = \tau_3 = 0 \).
Using SAS (lambs.sas)

```sas
option nocenter ps=65 ls=80;
data lambs;
  input diet wtgain @@;
datalines;
1 8 1 16 1 9 2 9 2 16 2 21
2 11 2 18 3 15 3 10 3 17 3 6
;
symbol1 bwidth=5 i=box; axis1 offset=(5);
proc gplot; plot wtgain*diet / frame haxis=axis1; run; quit;
```

![Box plot showing weight gain by diet group]
proc glm;
   class diet;
   model wtgain=diet;
   output out=diag r=res p=pred; run; quit;

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Squares</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>2</td>
<td>36.00000000</td>
<td>18.00000000</td>
<td>0.77</td>
<td>0.4907</td>
</tr>
<tr>
<td>Error</td>
<td>9</td>
<td>210.00000000</td>
<td>23.33333333</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corrected Total</td>
<td>11</td>
<td>246.00000000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

R-Square: 0.146341
Coeff Var: 37.15738
Root MSE: 4.830459
wtgain Mean: 13.00000

Source         | DF    | Type I SS    | Mean Square | F Value | Pr > F |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>diet</td>
<td>2</td>
<td>36.000000000</td>
<td>18.000000000</td>
<td>0.77</td>
<td>0.4907</td>
</tr>
</tbody>
</table>

Source         | DF    | Type III SS  | Mean Square | F Value | Pr > F |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>diet</td>
<td>2</td>
<td>36.000000000</td>
<td>18.000000000</td>
<td>0.77</td>
<td>0.4907</td>
</tr>
</tbody>
</table>
proc gplot; plot res*diet /frame haxis=axis1;

proc sort; by pred;
symbol1 v=circle i=sm50;
proc gplot; plot res*pred / haxis=axis1;
run; quit;
Model Checking and Diagnostics

• Model Assumptions

1 Model is correct
2 Independent observations
3 Errors normally distributed
4 Constant variance

\[ y_{ij} = \left( \bar{y}_{..} + (\bar{y}_i - \bar{y}_{..}) \right) + \left( y_{ij} - \bar{y}_i \right) \]

\[ y_{ij} = \hat{y}_{ij} + \hat{\epsilon}_{ij} \]

observed = predicted + residual

• Note that the predicted response at treatment \( i \) is \( \hat{y}_{ij} = \bar{y}_i \).

• Diagnostics use predicted responses and residuals.
Diagnostics

• Normality
  – Histogram of residuals
  – Normal probability plot / QQ plot
  – Shapiro-Wilk Test

• Constant Variance
  – Plot $\hat{\epsilon}_{ij}$ vs $\hat{y}_{ij}$ (residual plot)
  – Bartlett’s or Modified Levene’s Test

• Independence
  – Plot $\hat{\epsilon}_{ij}$ vs time/space
  – Plot $\hat{\epsilon}_{ij}$ vs variable of interest

• Outliers
Diagnostics Example: Tensile Strength Experiment

options ls=80 ps=60 nocenter;
goptions device=win target=winprtm rotate=landscape ftext=swiss
   hsize=8.0in vsize=6.0in htext=1.5 htitle=1.5 hpos=60 vpos=60
   horigin=0.5in vorigin=0.5in;
data one;
   infile 'c:\saswork\data\tensile.dat';
   input percent strength time;

title1 'Tensile Strength Example';
proc print data=one; run;

<table>
<thead>
<tr>
<th>Obs</th>
<th>percent</th>
<th>strength</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>7</td>
<td>19</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>24</td>
<td>35</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>25</td>
<td>35</td>
<td>11</td>
<td>23</td>
</tr>
</tbody>
</table>
symbol v=circle i=none;
title1 'Plot of Strength vs Percent Blend';
proc gplot data=one; plot strength*percent/frame; run;

proc boxplot;
plot strength*percent/boxstyle=skeletal pct1def=4; run;
proc glm data=one;
  class percent; model strength=percent;
  means percent / hovtest=bartlett hovtest=levene;
  output out=diag p=pred r=res; run;

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Squares</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>4</td>
<td>475.7600000</td>
<td>118.9400000</td>
<td>14.76</td>
<td>&lt;.0001</td>
</tr>
<tr>
<td>Error</td>
<td>20</td>
<td>161.2000000</td>
<td>8.0600000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corrected Total</td>
<td>24</td>
<td>636.9600000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Levene’s Test for Homogeneity of strength Variance
ANOVA of Squared Deviations from Group Means

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Squares</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>percent</td>
<td>4</td>
<td>91.6224</td>
<td>22.9056</td>
<td>0.45</td>
<td>0.7704</td>
</tr>
<tr>
<td>Error</td>
<td>20</td>
<td>1015.4</td>
<td>50.7720</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Bartlett’s Test for Homogeneity of strength Variance

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Chi-Square</th>
<th>Pr &gt; ChiSq</th>
</tr>
</thead>
<tbody>
<tr>
<td>percent</td>
<td>4</td>
<td>0.9331</td>
<td>0.9198</td>
</tr>
</tbody>
</table>
proc sort; by pred;
symbol1 v=circle i=sm50; title1 'Residual Plot';
proc gplot; plot res*pred/frame; run;

Residual Plot
proc univariate data=diag normal;
  var res; qqplot res / normal (L=1 mu=est sigma=est);
  histogram res / normal; run;

Moments

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>25</td>
<td>Sum Weights</td>
<td>25</td>
</tr>
<tr>
<td>Mean</td>
<td>0</td>
<td>Sum Observations</td>
<td>0</td>
</tr>
<tr>
<td>Std Deviation</td>
<td>2.59165327</td>
<td>Variance</td>
<td>6.71666667</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.11239681</td>
<td>Kurtosis</td>
<td>-0.8683604</td>
</tr>
<tr>
<td>Uncorrected SS</td>
<td>161.2</td>
<td>Corrected SS</td>
<td>161.2</td>
</tr>
<tr>
<td>Coeff Variation</td>
<td>.</td>
<td>Std Error Mean</td>
<td>0.51833065</td>
</tr>
</tbody>
</table>

Tests for Normality

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shapiro-Wilk</td>
<td>W</td>
<td>Pr &lt; W 0.1818</td>
</tr>
<tr>
<td>Kolmogorov-Smirnov</td>
<td>D</td>
<td>Pr &gt; D 0.0885</td>
</tr>
<tr>
<td>Cramer-von Mises</td>
<td>W-Sq</td>
<td>Pr &gt; W-Sq 0.2026</td>
</tr>
<tr>
<td>Anderson-Darling</td>
<td>A-Sq</td>
<td>Pr &gt; A-Sq 0.1775</td>
</tr>
</tbody>
</table>
Histogram of Residuals & QQ Plot
/* Time Serial Plot */
symbol v=circle i=none;
title 'Plot of residuals vs time';
proc gplot; plot res*time / vref=0 vaxis=-6 to 6 by 1;
run;
Non-Constant Variance: Impact and Remedy

- Does not affect F-test dramatically when the experiment is balanced

- Why concern?
  - Comparison of treatments depends on MSE
  - Incorrect intervals and comparison results

- Variance-Stabilizing Transformations
  - Ideas for Finding Proper Transformations ($E[Y] = \mu, \text{Var}(Y) = \sigma^2$)

\[
f(Y) \approx f(\mu) + (Y - \mu)f'(\mu)
\]

\[
\implies \text{Var}(f(Y)) \approx [f'(\mu)]^2 \text{Var}(Y) = [f'(\mu)]^2 \sigma^2
\]

* Find $f$ such that $\text{Var}(f(Y))$ does not depend on $\mu$ anymore. So, $\tilde{Y} = f(Y)$ has constant variance for different $f(\mu)$.

- Common transformations
  \[
  \sqrt{x}, \log(x), 1/x, \arcsin(\sqrt{x}), \text{ and } 1/\sqrt{x}
  \]
Transformations

• Suppose $\sigma^2$ is a function of $\mu$, that is $\sigma^2 = g(\mu)$

• Want to find transformation $f$ such that $\tilde{Y} = f(Y)$ has constant variance: $\text{Var}(\tilde{Y})$ does not depend on $\mu$.

• Have shown $\text{Var}(\tilde{Y}) \approx [f'(\mu)]^2 \sigma^2 \approx [f'(\mu)]^2 g(\mu)$

• Want to choose $f$ such that $[f'(\mu)]^2 g(\mu) \approx c$

Examples

$g(\mu) = \mu$ \hspace{1cm} (Poisson) \hspace{1cm} $f(\mu) = \int \frac{1}{\sqrt{\mu}} d\mu \rightarrow f(X) = \sqrt{X}$

$g(\mu) = \mu(1 - \mu)$ \hspace{1cm} (Binomial) \hspace{1cm} $f(\mu) = \int \frac{1}{\sqrt{\mu(1-\mu)}} d\mu \rightarrow f(X) = \text{arcsin}(\sqrt{X})$

$g(\mu) = \mu^{2\beta}$ \hspace{1cm} (Box-Cox) \hspace{1cm} $f(\mu) = \int \mu^{-\beta} d\mu \rightarrow f(X) = X^{1-\beta}$

$g(\mu) = \mu^2$ \hspace{1cm} (Box-Cox) \hspace{1cm} $f(\mu) = \int \frac{1}{\mu} d\mu \rightarrow f(X) = \log X$
Identify Box-Cox Transformation Using Data: Approximate Method

- From the previous slide, if $\sigma = \theta \mu^\beta$, the transformation is

$$f(Y) = \begin{cases} 
Y^{1-\beta} & \beta \neq 1; \\
\log Y & \beta = 1
\end{cases}$$

So it is crucial to estimate $\beta$ based on data $y_{ij}$, $i = 1, \ldots, a$.

- We have $\log\sigma_i = \log\theta + \beta\log\mu_i$

- Let $s_i$ and $\bar{y}_i.$ be the sample standard deviations and means. Because $\hat{\sigma}_i = s_i$ and $\hat{\mu}_i = \bar{y}_i.$, for $i = 1, \ldots, a$,

$$\log s_i \approx \text{constant} + \beta\log\bar{y}_i.$$ $$\Rightarrow \log s_i = \text{constant} + \beta\log\bar{y}_i + \text{error}_i.$$  

- We can plot $\log s_i$ against $\log\bar{y}_i.$, fit a straight line and use the slope to estimate $\beta$. 

Identify Box-Cox Transformation: Formal Method

1. For a fixed $\lambda$, perform analysis of variance on

$$y_{ij}(\lambda) = \begin{cases} 
\frac{y_{ij}^\lambda - 1}{\hat{y}^\lambda - 1} & \lambda \neq 0 \\
\hat{y} \log y_{ij} & \lambda = 0 
\end{cases}$$

where $\hat{y} = \left( \prod_{i=1}^{a} \prod_{j=1}^{n_i} y_{ij} \right)^{1/N}$.

2. Step 1 generates a transformed data $y_{ij}(\lambda)$. Apply ANOVA to the new data and obtain $SS_E$. Because $SS_E$ depends on $\lambda$, it is denoted by $SS_E(\lambda)$.

- Repeat 1 and 2 for various $\lambda$ in an interval, e.g., $[-2,2]$, and record $SS_E(\lambda)$

3. Find $\lambda_0$ which minimizes $SS_E(\lambda)$ and pick up a meaningful $\lambda$ in the neighborhood of $\lambda_0$. Denote it again by $\lambda$. (Maximum Likelihood Principle)

4. The transformation is:

$$\tilde{y}_{ij} = y_{ij}^{\lambda_0} \text{ if } \lambda_0 \neq 0;$$
$$\tilde{y}_{ij} = \log y_{ij} \text{ if } \lambda_0 = 0.$$
Example: Approximate Method (trans.sas)

data one;
    infile 'c:\saswork\data\boxcox.dat'; input trt resp;
proc glm data=one; class trt;
    model resp=trt; output out=diag p=pred r=res; run;

title1 'Residual Plot'; symbol1 v=circle i=none;
proc gplot data=diag; plot res*p=pred /frame; run;
proc univariate data=one noprint;
    var resp; by trt; output out=two mean=mu std=sigma;
data three; set two; logmu = log(mu); logsig = log(sigma);
proc reg; model logsig = logmu;

title1 'Mean vs Std Dev'; symbol1 v=circle i=rl;
proc gplot; plot logsig*logmu / regeqn; run;
Example: Formal Method (trans1.sas)

data one;
  infile 'c:saswork\data\boxcox.dat';
  input trt resp;
  logresp = log(resp);

proc univariate data=one noprint;
  var logresp; output out=two mean=mlogresp;

data three;
  set one; if _n_ eq 1 then set two;
  ydot = exp(mlogresp);
  do l=-2.0 to 2.0 by .25;
    den = l*ydot**(l-1);   if abs(l) eq 0 then den = 1;
    yl=(resp**(l -1))/den; if abs(l) < 0.0001 then yl=ydot*log(resp);
    output;
  end;
  keep trt yl l;

proc sort data=three out=three; by l;
proc glm data=three noprint outstat=four;
   class trt; model yl=trt; by l;

data five; set four;
   if _SOURCE_ eq 'ERROR'; keep l SS;

proc print data=five; run;

<table>
<thead>
<tr>
<th>OBS</th>
<th>L</th>
<th>SS</th>
<th>OBS</th>
<th>L</th>
<th>SS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2.00</td>
<td>2150.06</td>
<td>10</td>
<td>0.25</td>
<td>112.37</td>
</tr>
<tr>
<td>2</td>
<td>-1.75</td>
<td>1134.83</td>
<td>11</td>
<td>0.50</td>
<td>154.23</td>
</tr>
<tr>
<td>3</td>
<td>-1.50</td>
<td>628.94</td>
<td>12</td>
<td>0.75</td>
<td>253.63</td>
</tr>
<tr>
<td>4</td>
<td>-1.25</td>
<td>369.35</td>
<td>13</td>
<td>1.00</td>
<td>490.36</td>
</tr>
<tr>
<td>5</td>
<td>-1.00</td>
<td>232.32</td>
<td>14</td>
<td>1.25</td>
<td>1081.29</td>
</tr>
<tr>
<td>6</td>
<td>-0.75</td>
<td>158.56</td>
<td>15</td>
<td>1.50</td>
<td>2636.06</td>
</tr>
<tr>
<td>7</td>
<td>-0.50</td>
<td>119.28</td>
<td>16</td>
<td>1.75</td>
<td>6924.95</td>
</tr>
<tr>
<td>8</td>
<td>-0.25</td>
<td>100.86</td>
<td>17</td>
<td>2.00</td>
<td>19233.39</td>
</tr>
<tr>
<td>9</td>
<td>0.00</td>
<td>98.09</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
symbol1 v=circle i=sm50;
proc gplot; plot SS*l; run;

Plot of $SS_E(\lambda)$ vs $\lambda$

Increasing Variance Example
Using PROC TRANSREG

```plaintext
proc transreg data=one;
    model boxcox(y/lambda=-2.0 to 2.0 by 0.1)=class(trt); run;
```

---

**Transformation Information**

for BoxCox(y)

<table>
<thead>
<tr>
<th>Lambda</th>
<th>R-Square</th>
<th>Log Like</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.0</td>
<td>0.10</td>
<td>-108.906</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.18</td>
<td>-22.154</td>
</tr>
<tr>
<td>-0.4</td>
<td>0.19</td>
<td>-19.683</td>
</tr>
<tr>
<td>-0.3</td>
<td>0.20</td>
<td>-17.814 *</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.20</td>
<td>-16.593 *</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.21</td>
<td>-16.067 &lt;</td>
</tr>
<tr>
<td>0.0</td>
<td>0.21</td>
<td>-16.284 *</td>
</tr>
<tr>
<td>0.1</td>
<td>0.22</td>
<td>-17.289 *</td>
</tr>
<tr>
<td>0.2</td>
<td>0.22</td>
<td>-19.124</td>
</tr>
<tr>
<td>0.3</td>
<td>0.22</td>
<td>-21.820</td>
</tr>
<tr>
<td>2.0</td>
<td>0.10</td>
<td>-174.641</td>
</tr>
</tbody>
</table>

*Confidence Interval

< Best Lambda

* Convenient Lambda
Kruskal-Wallis Test: a Nonparametric Alternative for Nonnormality

\(a\) treatments, \(H_0\): \(a\) treatments are not different.

- Rank the observations \(y_{ij}\) in ascending order
- Replace each observation by its rank \(R_{ij}\) (assign average for tied observations), and apply one-way ANOVA to \(R_{ij}\).
- Test statistic \((a = 2 \implies\) Wilcoxon rank-sum test\)
  \[
  H = \frac{1}{S^2} \left[ \sum_{i=1}^{a} \frac{R_{i.}^2}{n_i} - \frac{N(N+1)^2}{4} \right] \approx \chi^2_{a-1} \text{ under } H_0
  \]
  where \(S^2 = \frac{1}{N-1} \left[ \sum_{i=1}^{a} \sum_{j=1}^{n_i} R_{ij}^2 - \frac{N(N+1)^2}{4} \right] \)
- Decision Rule: reject \(H_0\) if \(H > \chi^2_{\alpha,a-1}\).
- Let \(F_0\) be the \(F\)-test statistic in ANOVA based on \(R_{ij}\). Then
  \[
  F_0 = \frac{H/(a - 1)}{(N - 1 - H)/(N - a)}
  \]
A Nonnormality Example

data new;
   input strain nitrogen @@;
cards;
   1 19.4 1 32.6 1 27.0 1 32.1 1 33.0
   2 17.7 2 24.8 2 27.9 2 25.2 2 24.3
   3 17.0 3 19.4 3 9.1 3 11.9 3 15.8
   4 20.7 4 21.0 4 20.5 4 18.8 4 18.6
   5 14.3 5 14.4 5 11.8 5 11.6 5 14.2
   6 17.3 6 19.4 6 19.1 6 16.9 6 20.8;
;
proc glm data=new;
   class strain; model nitrogen=strain;
   output out=newres r=res; run;

proc univariate data=newres normal;
   var res; qqplot res / normal (L=1 mu=est sigma=est);
   run; quit;
## Tests for Normality

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shapiro-Wilk</td>
<td>W</td>
<td>0.910027</td>
</tr>
<tr>
<td>Kolmogorov-Smirnov</td>
<td>D</td>
<td>0.174133</td>
</tr>
<tr>
<td>Cramer-von Mises</td>
<td>W-Sq</td>
<td>0.155870</td>
</tr>
<tr>
<td>Anderson-Darling</td>
<td>A-Sq</td>
<td>0.908188</td>
</tr>
</tbody>
</table>

![Q-Q Plot](image-url)
proc npar1way data=new; /* May need option: wilcoxon */
   class strain; var nitrogen; run;

Analysis of Variance for Variable nitrogen
   Classified by Variable strain

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Among</td>
<td>5</td>
<td>847.046667</td>
<td>169.409333</td>
<td>14.3705</td>
<td>&lt;.0001</td>
</tr>
<tr>
<td>Within</td>
<td>24</td>
<td>282.928000</td>
<td>11.788667</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Kruskal-Wallis Test
 Chi-Square    21.6593
 DF            5
 Pr > Chi-Square 0.0006

Median One-Way Analysis
 Chi-Square    13.5333
 DF            5
 Pr > Chi-Square 0.0189
Linear Combinations of Treatment Means

- ANOVA Model:

\[ y_{ij} = \mu + \tau_i + \epsilon_{ij} \quad (\tau_i: \text{treatment effect}) \]
\[ = \mu_i + \epsilon_{ij} \quad (\mu_i: \text{treatment mean}) \]

- Linear combination with given coefficients \(c_1, c_2, \ldots, c_a:\)

\[ L = c_1 \mu_1 + c_2 \mu_2 + \ldots + c_a \mu_a = \sum_{i=1}^{a} c_i \mu_i, \]

- Want to test: \(H_0 : L = \sum c_i \mu_i = L_0\)

- Examples:

1. Pairwise comparison: \(\mu_i - \mu_j = 0\) for all possible \(i\) and \(j\).

2. Compare treatment vs control: \(\mu_i - \mu_1 = 0\) when treatment 1 is a control and \(i = 2, \ldots, a\) are new treatments.

3. General cases such as \(\mu_1 - 2\mu_2 + \mu_3 = 0, \mu_1 + 3\mu_2 - 6\mu_3 = 0\), etc.
• Estimate of \( L \):

\[
\hat{L} = \sum c_i \hat{\mu}_i = \sum c_i \bar{y}_i.
\]

\[
\text{Var}(\hat{L}) = \sum c_i^2 \text{Var}(\bar{y}_i) = \sigma^2 \sum \frac{c_i^2}{n_i} \left( = \frac{\sigma^2}{n} \sum c_i^2 \right)
\]

• Standard Error of \( \hat{L} \)

\[
\text{S.E.}\{\hat{L}\} = \sqrt{\text{MSE} \sum \frac{c_i^2}{n_i}}
\]

• Test statistic

\[
t_0 = \frac{(\hat{L} - L_0)}{\text{S.E.}\{\hat{L}\}} \sim t(N - a) \text{ under } H_0
\]
Example: Lambs Diet Experiment

- Denote the treatment means of three diets by $\mu_1$, $\mu_2$ and $\mu_3$. Suppose one wants to test $H_0 : L = 60$ with
  
  $L = \mu_1 + 2\mu_2 + 3\mu_3 = 6\mu + \tau_1 + 2\tau_2 + 3\tau_3$.

  ```
  proc glm data=lambs;
  class diet;
  model wtgain=diet;
  means diet;
  estimate 'l1' intercept 6 diet 1 2 3;
  run;
  ```

  | Standard Parameter Estimate | Error   | t Value | Pr > |t| |
  |-----------------------------|---------|---------|------|---|
  | l1                          | 77.0000000 | 8.88506862 | 8.67 | <.0001 |

  - $t_0 = (77.0 - 60)/8.89 = 1.91$

  $$P \text{ - value} = P(t \leq -1.91 \text{ or } t \geq 1.91|t(12 - 3)) = .088$$

  - Fail to reject $H_0 : \mu_1 + 2\mu_2 + 3\mu_3 = 60$ at $\alpha = 5\%$. 
Contrasts

- $\Gamma = \sum_{i=1}^{a} c_i \mu_i$ is a contrast if $\sum_{i=1}^{a} c_i = 0$.

  Equivalently, $\Gamma = \sum_{i=1}^{a} c_i \tau_i$.

- Examples

  1. $\Gamma_1 = \mu_1 - \mu_2 = \mu_1 - \mu_2 + 0\mu_3 + 0\mu_4$,
     $c_1 = 1, c_2 = -1, c_3 = 0, c_4 = 0$
     Comparing $\mu_1$ and $\mu_2$.

  2. $\Gamma_2 = \mu_1 - 0.5\mu_2 - 0.5\mu_3 = \mu_1 - 0.5\mu_2 - 0.5\mu_3 + 0\mu_4$
     $c_1 = 1, c_2 = -0.5, c_3 = -0.5, c_4 = 0$
     Comparing $\mu_1$ and the average of $\mu_2$ and $\mu_3$.

- Estimate of $\Gamma$:

  $$C = \sum_{i=1}^{a} c_i \bar{y}_i.$$
• Contrast Sum of Squares

\[ SS_C = \left( \sum c_i \bar{y}_i. \right)^2 / \sum \left( \frac{c_i^2}{n_i} \right) \]

\( SS_C \) represents the amount of variation attributable \( \Gamma \).

• Test: \( H_0 : \Gamma = 0 \)

\[ t_0 = \frac{C}{\text{S.E.}_C} \sim H_0 t(N - a) \]

\[ t_0^2 = \frac{\left( \sum c_i \bar{y}_i. \right)^2}{\text{MSE} \sum \frac{c_i^2}{n_i}} = \frac{SS_C / 1}{\text{MSE}} \sim F_{1, N-a}^H \]
### Tensile Strength Example (cont.sas)

```sas
proc glm data=one;
  class percent;
  model strength=percent;
  contrast 'C1' percent 0 0 0 1 -1;
  contrast 'C2' percent 1 0 1 -1 -1;
  contrast 'C3' percent 1 0 -1 0 0;
  contrast 'C4' percent 1 -4 1 1 1;
```

---

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Squares</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>4</td>
<td>475.76000</td>
<td>14.76</td>
<td>0.0001</td>
</tr>
<tr>
<td>Error</td>
<td>20</td>
<td>161.20000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corrected Total</td>
<td>24</td>
<td>636.96000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Type I SS</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>PERCENT</td>
<td>4</td>
<td>475.76000</td>
<td>14.76</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Contrast</th>
<th>DF</th>
<th>Contrast SS</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>1</td>
<td>291.60000</td>
<td>36.18</td>
<td>0.0001</td>
</tr>
<tr>
<td>C2</td>
<td>1</td>
<td>31.25000</td>
<td>3.88</td>
<td>0.0630</td>
</tr>
<tr>
<td>C3</td>
<td>1</td>
<td>152.10000</td>
<td>18.87</td>
<td>0.0003</td>
</tr>
<tr>
<td>C4</td>
<td>1</td>
<td>0.81000</td>
<td>0.10</td>
<td>0.7545</td>
</tr>
</tbody>
</table>
Orthogonal Contrasts

- Two contrasts \( \{ c_i \} \) and \( \{ d_i \} \) are **Orthogonal** if
  \[
  \sum_{i=1}^{a} \frac{c_i d_i}{n_i} = 0
  \quad (\sum_{i=1}^{a} c_i d_i = 0 \text{ for balanced experiments})
  \]

- Example (Balanced Experiment)

  \( \Gamma_1 = \mu_1 + \mu_2 - \mu_3 - \mu_4 \), So \( c_1 = 1, c_2 = 1, c_3 = -1, c_4 = -1 \).
  \( \Gamma_2 = \mu_1 - \mu_2 + \mu_3 - \mu_4 \). So \( d_1 = 1, d_2 = -1, d_3 = 1, d_4 = -1 \).

  It is easy to verify that both \( \Gamma_1 \) and \( \Gamma_2 \) are contrasts. Furthermore,

  \[
  c_1 d_1 + c_2 d_2 + c_3 d_3 + c_4 d_4 =
  1 \times 1 + 1 \times (-1) + (-1) \times 1 + (-1) \times (-1) = 0.
  \]

  Hence, \( \Gamma_1 \) and \( \Gamma_2 \) are orthogonal to each other.

- A **complete set** of orthogonal contrasts \( C = \{ \Gamma_1, \Gamma_2, \ldots, \Gamma_{a-1} \} \) if contrasts are mutually orthogonal and there does not exist a contrast orthogonal outside of \( C \) to all the contrasts in \( C \).
If there are \( a \) treatments, \( C \) must contain \( a - 1 \) contrasts.

Complete set is not unique. For example, in the tensile strength example

\[
\begin{align*}
\Gamma_1 &= (0, 0, 0, 1, -1) \\
\Gamma_2 &= (1, 0, 1, -1, -1) \\
\Gamma_3 &= (1, 0, -1, 0, 0) \\
\Gamma_4 &= (1, -4, 1, 1, 1)
\end{align*}
\]

\( C_1 \): includes:

\[
\begin{align*}
\Gamma_1' &= (-2, -1, 0, 1, 2) \\
\Gamma_2' &= (2, -1, -2, -1, 2) \\
\Gamma_3' &= (-1, 2, 0, -2, 1) \\
\Gamma_4' &= (1, -4, 6, -4, 1)
\end{align*}
\]

\( C_2 \): includes:
• Suppose $C_1, C_2, \ldots, C_{a-1}$ are the estimates of the contrasts in a complete set of contrasts $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{a-1}\}$, then

$$
SS_{\text{Treatment}} = SS_{C_1} + SS_{C_2} + \cdots + SS_{C_{a-1}}
$$

$$
F_0 = \frac{MS_{\text{Treatment}}}{MSE} = \frac{F_{10} + F_{20} + \cdots + F_{(a-1)0}}{a - 1}
$$

where $F_{i0}$ is the test statistic used to test contrast $\Gamma_i$.

• Orthogonal contrasts (estimates) are independent with each other
  – The results follow Cochran’s Theorem, so comparisons are independent
  – Example on Slide 39

• Can also use orthogonal contrasts to study trend
  – Only interesting if treatments are quantitative (ordered): $X_1, \ldots, X_a$
  – For equally spaced treatments and $n_i = n$, $c_i$ in Table IX
  – Breakdown of polynomial regression $\mu(x) = \beta_0 + \beta_1 x + \cdots + \beta_{a-1} x^{a-1}$

$$
\implies \mu(x) = \tilde{\beta}_0 + \tilde{\beta}_1 P_1(x) + \cdots + \tilde{\beta}_{a-1} P_{a-1}(x)
$$

– $P_k(x)$ is of k-th order, and $\sum_{i=1}^{a} P_k(X_i) = 0$ (contrast)
Determining Orthogonal Polynomial Coefficients Using SAS

- Complete set: \( \{(P_k(X_1), \cdots, P_k(X_a)) : k = 1, \cdots, a - 1 \} \)
- Often the levels of the treatments are not equally spaced
- Can use PROC IML to determine coefficients \( (P_k(X_1), \cdots, P_k(X_a)) \)

```sas
proc iml;
levels={1 2 5 10 20}; /* Consider these 5 levels */
print levels;
coef=ORPOL(levels,4); /* Gives all coefs up through 4th order */
coef=t(coef); /* Puts coefs in rows instead of cols */
coef=coef[2:5,]; /* Eliminates the first row of coef matrix*/
print coef; run;
```

<table>
<thead>
<tr>
<th>LEVELS</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>COEF</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.424967</td>
<td>-0.360578</td>
<td>-0.167411</td>
<td>0.1545335</td>
<td>0.798423</td>
<td></td>
</tr>
<tr>
<td>0.4348974</td>
<td>0.2072899</td>
<td>-0.325207</td>
<td>-0.711616</td>
<td>0.3946361</td>
<td></td>
</tr>
<tr>
<td>-0.433125</td>
<td>0.1365799</td>
<td>0.7252914</td>
<td>-0.510844</td>
<td>0.0820972</td>
<td></td>
</tr>
<tr>
<td>0.4926155</td>
<td>-0.779975</td>
<td>0.3743878</td>
<td>-0.093597</td>
<td>0.0065682</td>
<td></td>
</tr>
</tbody>
</table>
Testing Multiple Contrasts (Multiple Comparisons)

• One contrast:

\[ H_0 : \Gamma = \sum c_i \mu_i = \Gamma_0 \quad \text{vs} \quad H_1 : \Gamma \neq \Gamma_0 \] at \( \alpha \)

100(1-\( \alpha \)) Confidence Interval (CI) for \( \Gamma \):

\[
\text{CI} : \sum c_i \bar{y}_i \pm t_{\alpha/2,N-a} \sqrt{MSE \sum \frac{c_i^2}{n_i}}
\]

\[ P(\text{CI does not contain } L_0 | H_0) = \alpha (= \text{type I error rate}) \]

• Decision Rule: Reject \( H_0 \) if CI does not contain \( \Gamma_0 \).
- Multiple contrasts, for \( i = 1, 2, \cdots, m \),

\[
H_0^{(i)} : \Gamma^{(i)} = \Gamma_0^{(i)}, \quad \text{vs} \quad H_1^{(i)} : \Gamma^{(i)} \neq \Gamma_0^{(i)}
\]

If we construct CI_1, CI_2, ..., CI_m, each with 100(1-\( \alpha \)) level, then for each CI_i,

\[
P(\text{reject} H_0^{(i)} \mid H_0^{(i)}) = P(\text{CI}_i \text{ does not contain } \Gamma_0^{(i)} \mid H_0^{(i)}) = \alpha
\]

- But the **overall Type I error rate** (probability of any type I error in testing \( H_0^{(i)} \) vs \( H_1^{(i)} \), \( i = 1, \cdots, m \)) is inflated and much larger than \( \alpha \), that is,

\[
P(\text{reject at least one of } \{ H_0^{(i)}, i = 1, \cdots, m \} \mid H_0^{(1)}, \cdots, H_0^{(m)})
= P(\text{at least one CI}_i \text{ do not contain } \Gamma_0^{(i)} \mid H_0^{(1)}, \cdots, H_0^{(m)})
\gg \alpha
\]

- One way to achieve small overall error rate, we require much smaller error rate (\( \alpha' \)) of each individual CI_i.
Bonferroni Method for Testing Multiple Contrasts

- Bonferroni Inequality

\[
P(\text{reject at least one of } \{H^{(i)}_0, i = 1, \ldots, m\} | H^{(1)}_0, \ldots, H^{(m)}_0) \\
= P(\text{reject } H^{(1)}_0 \text{ or reject } H^{(2)}_0 \text{ or } \cdots \text{ or reject } H^{(m)}_0 | H^{(1)}_0, \ldots, H^{(m)}_0) \\
\leq P(\text{reject } H^{(1)}_0 | H^{(1)}_0) + \cdots + P(\text{reject } H^{(m)}_0 | H^{(m)}_0) = m\alpha'
\]

- In order to control overall error rate (or, overall confidence level), let

\[
m\alpha' = \alpha, \text{ we have, } \alpha' = \alpha/m
\]

- Bonferroni CIs:

\[
\text{CI}_i : \sum c_{ij} \bar{y}_j. \pm t_{\alpha/2m}(N - a) \sqrt{\frac{\text{MS}_E}{\sum c_{ij}^2/n_j}}
\]

- When m is large, Bonferroni CIs are too conservative (overall type II error too large).
Scheffe’s Method for Testing All Contrasts

• Consider all possible contrasts: \( \Gamma = \sum c_i \mu_i \)

  Estimate: \( C = \sum c_i \bar{y}_i \),  St. Error: \( \text{S.E.}_C = \sqrt{MSE \sum \frac{c_i^2}{n_i}} \)

• Critical value: \( \sqrt{(a - 1) F_{\alpha, a-1, N-a}} \)

• Scheffe’s simultaneous CI: \( C \pm \sqrt{(a - 1) F_{\alpha, a-1, N-a}} \text{ S.E.}_C \)

• Overall confidence level and error rate for \( m \) contrasts

  \[
  P(\text{CIs contain true parameter for any contrast}) \geq 1 - \alpha \\
  P(\text{at least one CI does not contain true parameter}) \leq \alpha
  \]

Remark: Scheffe’s method is also conservative, too conservative when \( m \) is small
Methods for Pairwise Comparisons

- There are $a(a - 1)/2$ possible pairs: $\mu_i - \mu_j$ (contrast for comparing $\mu_i$ and $\mu_j$). We may be interested in $m$ pairs or all pairs.

- Standard Procedure:
  1. Estimation: $\bar{y}_i. - \bar{y}_j.$
  2. Compute a **Critical Difference** (CD) (based on the method employed)
  3. If
     \[
     | \bar{y}_i. - \bar{y}_j. | > CD
     \]
     or equivalently if the interval
     \[
     (\bar{y}_i. - \bar{y}_j. - CD, \ \bar{y}_i. - \bar{y}_j. + CD)
     \]
     does not contain zero, declare $\mu_i - \mu_j$ significant.
Methods for Calculating CD

- Least significant difference (LSD):
  \[
  CD = t_{\alpha/2, N-a} \sqrt{MS_E \left(\frac{1}{n_i} + \frac{1}{n_j}\right)}
  \]
  not control overall error rate

- Bonferroni method (for \( m \) pairs)
  \[
  CD = t_{\alpha/2m, N-a} \sqrt{MS_E \left(\frac{1}{n_i} + \frac{1}{n_j}\right)}
  \]
  control overall error rate for the \( m \) comparisons.

- Tukey’s method (for all possible pairs)
  \[
  CD = \frac{q_{\alpha}(a, N-a)}{\sqrt{2}} \sqrt{MS_E \left(\frac{1}{n_i} + \frac{1}{n_j}\right)}
  \]
  \( q_{\alpha}(a, N-a) \) from studentized range distribution (Table VII). Control overall error rate (exact for balanced experiments). (Example 3.7).
Comparing Treatments with Control (Dunnett’s Method)

1. Assume $\mu_1$ is a control, and $\mu_2, \ldots, \mu_a$ are (new) treatments

2. Only interested in $a - 1$ pairs: $\mu_2 - \mu_1, \ldots, \mu_a - \mu_1$

3. Compare $|\bar{y}_i - \bar{y}_1|$ to

$$\text{CD} = d_\alpha (a - 1, N - a) \sqrt{\text{MSE}(1/n_i + 1/n_1)}$$

where $d_\alpha (p, f)$ from Table VIII: critical values for Dunnett’s test.

4. Remark: control overall error rate. Read Example 3.9

For pairwise comparison, which method should be preferred? LSD, Bonferroni, Tukey, Dunnett or others?
Tensile Strength Example

data one;
   infile 'c:\saswork\data\tensile.dat';
   input percent strength time;

proc glm data=one;
   class percent;
   model strength=percent;

/* Construct CI for Treatment Means*/
means percent /alpha=.05 lsd clm;
means percent / alpha=.05 bon clm;

/* Pairwise Comparison*/
means percent /alpha=.05 lines lsd;
means percent /alpha=.05 lines bon;
means percent /alpha=.05 lines scheffe;
means percent /alpha=.05 lines tukey;
means percent /dunnett;
run;
The GLM Procedure

t Confidence Intervals for $y$

<table>
<thead>
<tr>
<th>Alpha</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Degrees of Freedom</td>
<td>20</td>
</tr>
<tr>
<td>Error Mean Square</td>
<td>8.06</td>
</tr>
<tr>
<td>Critical Value of $t$</td>
<td>2.08596</td>
</tr>
<tr>
<td>Half Width of Confidence Interval</td>
<td>2.648434</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>trt</th>
<th>N</th>
<th>Mean</th>
<th>Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>5</td>
<td>21.600</td>
<td>18.952</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>17.600</td>
<td>14.952</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>15.400</td>
<td>12.752</td>
</tr>
<tr>
<td>35</td>
<td>5</td>
<td>10.800</td>
<td>8.152</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>9.800</td>
<td>7.152</td>
</tr>
</tbody>
</table>
The GLM Procedure

Bonferroni t Confidence Intervals for y

<table>
<thead>
<tr>
<th>Alpha</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Degrees of Freedom</td>
<td>20</td>
</tr>
<tr>
<td>Error Mean Square</td>
<td>8.06</td>
</tr>
<tr>
<td>Critical Value of t</td>
<td>2.84534</td>
</tr>
<tr>
<td>Half Width of Confidence Interval</td>
<td>3.612573</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>trt</th>
<th>N</th>
<th>Mean</th>
<th>Simultaneous 95% Confidence Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>5</td>
<td>21.600</td>
<td>17.987 (17.987, 25.213)</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>15.400</td>
<td>11.787 (11.787, 19.013)</td>
</tr>
<tr>
<td>35</td>
<td>5</td>
<td>10.800</td>
<td>7.187 (7.187, 14.413)</td>
</tr>
</tbody>
</table>
t Tests (LSD) for y

NOTE: This test controls the Type I comparisonwise error rate, not the experimentwise error rate.

Alpha 0.05
Error Degrees of Freedom 20
Error Mean Square 8.06
Critical Value of t 2.08596
Least Significant Difference 3.7455

Means with the same letter are not significantly different.

<table>
<thead>
<tr>
<th>t Grouping</th>
<th>Mean</th>
<th>N</th>
<th>trt</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>21.600</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>B</td>
<td>17.600</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>B</td>
<td>15.400</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>10.800</td>
<td>5</td>
<td>35</td>
</tr>
<tr>
<td>C</td>
<td>9.800</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>
Bonferroni (Dunn) t Tests for y

NOTE: This test controls the Type I experimentwise error rate, but it generally has a higher Type II error rate than REGWQ.

<table>
<thead>
<tr>
<th>Alpha</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Degrees of Freedom</td>
<td>20</td>
</tr>
<tr>
<td>Error Mean Square</td>
<td>8.06</td>
</tr>
<tr>
<td>Critical Value of t</td>
<td>3.15340</td>
</tr>
<tr>
<td>Minimum Significant Difference</td>
<td>5.6621</td>
</tr>
</tbody>
</table>

Means with the same letter are not significantly different.

<table>
<thead>
<tr>
<th>Bon Grouping</th>
<th>Mean</th>
<th>N</th>
<th>trt</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>21.600</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>B A</td>
<td>17.600</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>B C</td>
<td>15.400</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>10.800</td>
<td>5</td>
<td>35</td>
</tr>
<tr>
<td>C</td>
<td>9.800</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>
Scheffe’s Test for $y$

**NOTE:** This test controls the Type I experimentwise error rate.

- **Alpha:** 0.05
- **Error Degrees of Freedom:** 20
- **Error Mean Square:** 8.06
- **Critical Value of F:** 2.86608
- **Minimum Significant Difference:** 6.0796

Means with the same letter are not significantly different.

<table>
<thead>
<tr>
<th>Scheffe Grouping</th>
<th>Mean</th>
<th>N</th>
<th>trt</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>21.600</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B A</td>
<td>17.600</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B C</td>
<td>15.400</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>10.800</td>
<td>5</td>
<td>35</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>9.800</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>
Tukey’s Studentized Range (HSD) Test for y

NOTE: This test controls the Type I experimentwise error rate, but it generally has a higher Type II error rate than REGWQ.

<table>
<thead>
<tr>
<th>Alpha</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Degrees of Freedom</td>
<td>20</td>
</tr>
<tr>
<td>Error Mean Square</td>
<td>8.06</td>
</tr>
<tr>
<td>Critical Value of Studentized Range</td>
<td>4.23186</td>
</tr>
<tr>
<td>Minimum Significant Difference</td>
<td>5.373</td>
</tr>
</tbody>
</table>

Means with the same letter are not significantly different.

<table>
<thead>
<tr>
<th>Tukey Grouping</th>
<th>Mean</th>
<th>N</th>
<th>trt</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>21.600</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>A</td>
<td>17.600</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>B</td>
<td>15.400</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>10.800</td>
<td>5</td>
<td>35</td>
</tr>
<tr>
<td>D</td>
<td>9.800</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>
Dunnett’s t Tests for y

NOTE: This test controls the Type I experimentwise error for comparisons of all treatments against a control.

Alpha 0.05
Error Degrees of Freedom 20
Error Mean Square 8.06
Critical Value of Dunnett’s t 2.6511
Minimum Significant Difference 4.7602

Comparisons significant at the 0.05 level are indicated by ***.

<table>
<thead>
<tr>
<th>Difference</th>
<th>Between Means</th>
<th>Simultaneous 95% Confidence Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 - 15</td>
<td>11.800</td>
<td>7.040 16.560 ***</td>
</tr>
<tr>
<td>25 - 15</td>
<td>7.800</td>
<td>3.040 12.560 ***</td>
</tr>
<tr>
<td>20 - 15</td>
<td>5.600</td>
<td>0.840 10.360 ***</td>
</tr>
<tr>
<td>35 - 15</td>
<td>1.000</td>
<td>-3.760  5.760</td>
</tr>
</tbody>
</table>