A Review of Matrix Algebra

Basic notation and concepts

vectors

\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \]

matrices

\[ A_{n \times k} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} = (a_{ij}) \]

Basic Concepts

vectors:
linear independent, length, angle, inner product, perpendicular, projection, Gram-Schimdt procedure, etc.

matrices:
determinant, rank of a matrix, nonsingular, inverse, orthogonal, symmetric, positive definite

Definition: Let \( A = (a_{ij}) \) be a \( k \times k \) matrix. The trace of the matrix \( A \), written \( tr(A) \), is the sum of the diagonal elements. That is \( tr(A) = \sum_{i=1}^{k} a_{ii} \).

Results: Let \( A_{k \times k} \) and \( B_{k \times k} \) be two matrices. Let \( c \) be a scalar.

(a) \( tr(A) = tr(B) \)
(b) \( tr(A \pm B) = tr(A) \pm tr(B) \)
(c) \( tr(AB) = tr(BA) \)
(d) $\text{tr}(B^{-1}AB) = \text{tr}(A)$

(e) $\text{tr}(A^2) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij}^2$

**Definition** Let $A$ be a $k \times k$ matrix. Let $I$ be the $k \times k$ identity matrix.

(a) $f(\lambda) = |A - \lambda I|$ is call the characteristic polynomial of $A$.

(b) $\lambda_1, \lambda_2, \ldots, \lambda_k$, satisfying $f(\lambda) = 0$, are called the eigenvalues of $A$.

**Definition** Let $A$ be a $k \times k$ matrix and $\lambda$ an eigenvalue of $A$. If $\vec{x}$ is a nonzero vector ($\vec{x} \neq 0$) such that $A\vec{x} = \lambda \vec{x}$, then $\vec{x}$ is said to be an eigenvector of $A$ associated with the eigenvalue $\lambda$. Let $\vec{e} = \frac{\vec{x}}{||\vec{x}||}$, then $(\lambda, \vec{e})$ is called a pair of eigenvalue-eigenvector of $A$.

**Definition** A quadratic function $Q(x)$ in $(x_1, x_2, \ldots, x_k)$ is

$$Q(x) = \vec{x}' A \vec{x},$$

where $\vec{x}' = (x_1, x_2, \ldots, x_k)$ and $A$ is a $k \times k$ symmetric matrix.

**Results**

1. Eigenvalues of real symmetric matrix are real and so are their corresponding eigenvectors.
2. A symmetric positive definite matrix has real positive eigenvalues.

**Spectral Decomposition:**

Let $A$ be a $k \times k$ symmetric matrix. Then $A$ can be expressed in terms of its $k$ eigenvalue-eigenvector pairs $(\lambda_i, e_i)$ as

$$A = \sum_{i=1}^{k} \lambda_i e_i e_i'$$

If $A$ is positive definite, then

$$A^{-1} = \sum_{i=1}^{k} \frac{1}{\lambda_i} e_i e_i'$$
and

\[ A^{1/2} = \sum_{i=1}^{k} \sqrt{\lambda_i} e_i e_i' \]

**Singular Value Decomposition**

Let \( A \) be an \( n \times k \) matrix, then there exists an \( n \times n \) orthogonal matrix \( U \) and an \( k \times k \) matrix \( V \) such that

\[ A = U \Lambda V \]

where the \( n \times k \) matrix \( \Lambda \) has \((i, i)\) entry \( \lambda_i \geq 0 \), for \( i = 1, 2, \ldots, \min(n, k) \) and the other entries are zero. The positive constants \( \lambda_i \) are called the singular values of \( A \).

Assume that the rank of \( A \) is \( r \). There exist \( r \) positive constants \( \lambda_1, \lambda_2, \ldots, \lambda_r \), \( r \) orthogonal \( n \times 1 \) unit vectors \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r \) and \( r \) orthogonal \( k \times 1 \) unit vector \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r \) such that

\[ A = \sum_{i=1}^{k} \lambda_i \vec{u}_i \vec{v}_i' = U_r \Lambda_r V_r \]

where \( U_r = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r) \), \( V_r = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r) \), and \( \Lambda_r \) is an \( r \times r \) diagonal matrix with diagonal entries \( \lambda_i \).

**Rayleigh-Ritz Theorem**

Let \( B_{k \times k} \) be a positive definite matrix with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \) and associated normalized eigenvectors \( \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_k \). Then,

\[
\max_{\vec{x} \neq 0} \frac{\vec{x}' B \vec{x}}{\vec{x}' \vec{x}} = \lambda_1, \text{ (attained when } \vec{x} = \vec{e}_1) \\
\min_{\vec{x} \neq 0} \frac{\vec{x}' B \vec{x}}{\vec{x}' \vec{x}} = \lambda_k, \text{ (attained when } \vec{x} = \vec{e}_k) 
\]

Moreover

\[
\max_{\vec{x} \perp \vec{e}_1, \ldots, \vec{e}_i} \frac{\vec{x}' B \vec{x}}{\vec{x}' \vec{x}} = \lambda_i + 1, \text{ (attained when } \vec{x} = \vec{e}_{i+1}, \ i = 2, \ldots, k - 1) 
\]