Lecture 14: State-Space Models and The Kalman Filter

Walid Sharabati
Purdue University

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State-Space Models

Observation = Signal + Noise.

In statistical language, this is equivalent to

Data = Fit + Residual.

Fit = Explained variation. Residual = Unexplained variation.

- Assume that the **signal is a linear combination** of a set of variables, called **state variable**, which constitutes state vector.
- Univariate time series $X_t$.

$$X_t = h_t^T \cdot \theta_t + n_t,$$ (1)

where $X_t$ is a scalar observation, $h_t$ is an $m \times 1$ column vector (assumed to be known), $\theta_t$ is an $m \times 1$ state vector (can’t be observed), and $n_t$ is the observation error (noise).
State-Space Models

- Use \( X_t \) to estimate \( \theta_t \).
- It is often reasonable to assume how \( \theta_t \) changes with time

\[
\theta_t = G_t \cdot \theta_{t-1} + \omega_t, \tag{2}
\]

where \( G_t \) is assumed to be known, \( \omega_t \) is an \( m \times 1 \) vector of noise.
- (1) and (2) constitute the general form of the univariate state-space model.
  - \( X_t = h_t^T \cdot \theta_t + n_t \) is the observation (or measurement) equation.
  - \( \theta_t = G_t \cdot \theta_{t-1} + \omega_t \) is the transition (or state or system) equation.
- \( n_t \sim N(0, \sigma_n^2) \) and \( \omega_t \sim MVN(0, W_t) \), where \( W_t(i, j) = \text{Cov}(\omega_{i,t}, \omega_{j,t}) \).
- \( n_t \) and \( \omega_t \) are uncorrelated.
Random Walk Plus Noise Model

**Example**

Consider

\[ X_t = \mu_t + n_t \] is the observation equation \((n_t \sim N(0, \sigma^2_n))\).

\[ \mu_t = \mu_{t-1} + \omega_t \] is the transition equation \((\omega_t \sim N(0, \sigma^2_\omega))\).

Compare with

\[ X_t = h_t^T \cdot \theta_t + n_t. \]

\[ \theta_t = G_t \cdot \theta_{t-1} + \omega_t \quad \rightarrow \quad \mu_t = \mu_{t-1} + \omega_t. \]

\[ \theta_t = \mu_t, \ h_t = 1, \ G_t = 1. \]

If \( \sigma^2_\omega = 0, \ \mu_t = \mu_{t-1} = \mu. \)

So, \( X_t = \mu + n_t \) reduces to a trivial constant-mean model.
The Linear Growth Model

Example

Consider

\[ X_t = \mu_t + n_t. \]
\[ \mu_t = \mu_{t-1} + \beta_{t-1} + \omega_{1,t}. \]
\[ \beta_t = \beta_{t-1} + \omega_{2,t}. \]

Let,

\[ \theta_t = \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix}, \quad h_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_t = (1 \ 0) \cdot \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} + n_t. \]

\[ X_t = h_t^T \cdot \theta_t + n_t. \]

and

\[ \theta_t = \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix}_{2 \times 1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}_{2 \times 2} \cdot \begin{pmatrix} \mu_{t-1} \\ \beta_{t-1} \end{pmatrix}_{2 \times 1} + \begin{pmatrix} \omega_{1,t} \\ \omega_{2,t} \end{pmatrix}_{2 \times 1}. \]

\[ \theta_t = G_t \cdot \theta_{t-1} + \omega_t. \]
Consider \( AR(2) : \) \( X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t. \) Let,

\[
\theta_t = \begin{pmatrix} X_t \\ \phi_2 X_{t-1} \end{pmatrix}, \quad h_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

\[
X_t = (1 \ 0) \cdot \begin{pmatrix} X_t \\ \phi_2 X_{t-1} \end{pmatrix} + \underbrace{0}_{\sigma_n^2=0}.
\]

Then,

\[
\theta_t = \begin{pmatrix} X_t \\ \phi_2 X_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_{t-1} \\ \phi_2 X_{t-2} \end{pmatrix} + \begin{pmatrix} Z_t \\ 0 \end{pmatrix}.
\]

\[
\theta_t = G_t \cdot \theta_{t-1} + \omega_t,
\]

where \( \omega_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix} Z_t. \)
Why do we write AR(2) into a state-space model?

Replaces **two-stage dependence** with two equations involving **one-stage dependence**.

Allows us to use the general results relating to state-space models such as the recursive estimation of parameters.

Not unique, we can have $\theta_t^T = (X_t \ X_{t-1})$ or $\theta_t^T = (X_t \ \hat{X}_{t+1})$. $\hat{X}_{t+1}$ is the one-step ahead forecast.
A Regression Model with Time-Varying Coefficients

\[ X_t = a_t + b_t \cdot \mu_t + n_t, \]

where \( a_t \) and \( b_t \) are constants. \( a_t \) and \( b_t \) are allowed to evolve through time according to a random walk.

\[
\begin{align*}
    a_t &= a_{t-1} + \omega_{1,t}. \\
    b_t &= b_{t-1} + \omega_{2,t}.
\end{align*}
\]

Let \( \theta_t^T = (a_t \ b_t) \). Then,

\[
X_t = (1 \ \mu_t) \cdot \begin{pmatrix} a_t \\ b_t \end{pmatrix} + n_t.
\]

\[
X_t = h_t^T \cdot \theta_t + n_t.
\]

and

\[
\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \begin{pmatrix} \omega_{1,t} \\ \omega_{2,t} \end{pmatrix}
\]

\[
\theta_t = G_t \cdot \theta_{t-1} + \omega_t.
\]
State-Space Model for MA(1) Process

\[ X_t = Z_t + \beta Z_{t-1}. \]

Let \( \theta_t^T = (X_t \ X_{t+1}) = (X_t \ \beta Z_t). \) Then,

\[
X_t = (1 \ 0) \cdot \begin{pmatrix} X_t \\ \beta Z_t \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma_n^2 = 0 \end{pmatrix}.
\]

\[ X_t = h_t^T \cdot \theta_t + n_t. \]

and

\[
\begin{pmatrix} X_t \\ \beta Z_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_{t-1} \\ \beta Z_{t-1} \end{pmatrix} + \begin{pmatrix} Z_t \\ \beta Z_t \end{pmatrix}.
\]

\[ \theta_t = G_t \cdot \theta_{t-1} + \omega_t, \]

where \( \omega_t = \begin{pmatrix} 1 \\ \beta \end{pmatrix} Z_t. \)
State-Space Model for MA(2) Process

\[ X_t = Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}. \]

Let \( \theta_t^T = (X_t, \hat{X}_{t+1}, \hat{X}_{t+2}) = (X_t, \beta_1 Z_t + \beta_2 Z_{t-1}, \beta_2 Z_t). \) Then,

\[
X_t = (1 \ 0 \ 0) \cdot \begin{pmatrix} X_t \\ \beta_1 Z_t + \beta_2 Z_{t-1} \\ \beta_2 Z_t \end{pmatrix} + \underbrace{0}_{n_t=0}.
\]

\[ X_t = h_t^T \cdot \theta_t + n_t. \]

and

\[
\begin{pmatrix} X_t \\ \beta_1 Z_t + \beta_2 Z_{t-1} \\ \beta_2 Z_t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_{t-1} \\ \beta_1 Z_{t-1} + \beta_2 Z_{t-2} \\ \beta_2 Z_{t-1} \end{pmatrix} + \begin{pmatrix} Z_t \\ \beta_1 Z_t \\ \beta_2 Z_t \end{pmatrix}.
\]

\[ \theta_t = G_t \cdot \theta_{t-1} + \omega_t, \]

where \( \omega_t = \begin{pmatrix} 1 \\ \beta_1 \\ \beta_2 \end{pmatrix} \cdot Z_t. \)