Lecture 13: Linear Systems
Linear Systems in the Time Domain

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• Identify a model for the input and output.

\[ \text{Input } X_t \quad \rightarrow \quad \text{Physical System} \quad \rightarrow \quad \text{Output } Y_t \]

**Example**

\( X_t : \) Temperature at which reactor is kept.
\( Y_t : \) Yield from a chemical reactor.

• Important applications in automatic control theory, signal processing and telecommunications.
Special Cases of Linear Systems

Definition

If \( X_1,t \rightarrow Y_1,t \) and \( X_2,t \rightarrow Y_2,t \) then \( \lambda_1 X_1,t + \lambda_2 X_2,t \rightarrow \lambda_1 Y_1,t + \lambda_2 Y_2,t. \)

\[
\begin{align*}
\text{Input} & : \quad \lambda_1 X_1,t + \lambda_2 X_2,t \\
\text{Physical System} & : \quad \rightarrow \\
\text{Output} & : \quad \lambda_1 Y_1,t + \lambda_2 Y_2,t
\end{align*}
\]

1. \( \lambda_1 = \lambda_2 = 1 : \quad X_1,t + X_2,t \rightarrow Y_1,t + Y_2,t. \)
   
   Sum of inputs \( \Rightarrow \) Sum of outputs.
   
   Principle of Superposition.

2. \( \lambda_2 = 0 : \quad \lambda X_t \rightarrow \lambda Y_t. \)

Example

\( 2X_t \rightarrow 2Y_t \)

Double inputs \( \Rightarrow \) Double outputs.

Homogeneity Proportionality or Scale Invariant.
**Time Invariant and Linear Systems**

**Definition**

If $X_t \rightarrow Y_t$, then the system is said to be time-invariant if a delay of time $\tau$ in $X_t$ produces the same delay in $Y_t$ that is

$$X_{t-\tau} \rightarrow Y_{t-\tau}.$$  

Any equation with constant coefficients defines a *time-invariant* system.

**Example**

$$Y_t = 0.5X_t + 0.25X_{t-1}.$$  

$$X_t \quad \rightarrow \quad \text{System} \quad \rightarrow \quad Y_t = 0.5X_t + 0.25X_{t-1}$$

$$X_{1,t} \quad \rightarrow \quad Y_{1,t} = 0.5X_{1,t} + 0.25X_{1,t-1} \quad \text{multiply by } \lambda_1.$$  

$$X_{2,t} \quad \rightarrow \quad Y_{2,t} = 0.5X_{2,t} + 0.25X_{2,t-1} \quad \text{multiply by } \lambda_2.$$
Now,

\[ \lambda_1 Y_{1,t} = 0.5\lambda_1 X_{1,t} + 0.25\lambda_1 X_{1,t-1}. \]
\[ \lambda_2 Y_{2,t} = 0.5\lambda_2 X_{2,t} + 0.25\lambda_2 X_{2,t-1}. \]

Add to get,

\[ \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t} = 0.5(\lambda_1 X_{1,t} + \lambda_2 X_{2,t}) + 0.25(\lambda_1 X_{1,t} + \lambda_2 X_{2,t-1}). \]

\[ \lambda_1 X_{1,t} + \lambda_2 X_{2,t} \rightarrow \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t}. \]

So, the system is linear.
Example

\[ Y_t = 0.5Y_{t-1} + 0.25X_t. \]

\[ X_{1,t} \rightarrow Y_{1,t} = 0.5Y_{1,t-1} + 0.25X_{1,t}. \]

\[ X_{2,t} \rightarrow Y_{2,t} = 0.5Y_{2,t-1} + 0.25X_{2,t}. \]

Multiply by \( \lambda_1, \lambda_2 \) and then add:

\[ \lambda_1 Y_{1,t} = 0.5\lambda_1 Y_{1,t-1} + 0.25\lambda_1 X_{1,t} \]

\[ \lambda_2 Y_{2,t} = 0.5\lambda_2 Y_{2,t-1} + 0.25\lambda_2 X_{2,t} \]

We get

\[ \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t} = 0.5(\lambda_1 Y_{1,t-1} + \lambda_2 Y_{2,t-1}) + 0.25(\lambda_1 X_{1,t} + \lambda_2 X_{2,t}). \]
\[ Y'_t = 0.5Y'_{t-1} + 0.25X'_t. \]

\[ X'_t \rightarrow Y'_t. \]

\[ \lambda_1 X_{1,t} + \lambda_2 X_{2,t} \rightarrow \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t}. \]

It is linear.

**Definition**

For the systems \( X_{1,t} \Rightarrow Y_{1,t} \) and \( X_{2,t} \Rightarrow Y_{2,t} \) if

\[ \lambda_1 X_{1,t} + \lambda_2 X_{2,t} \Rightarrow \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t}. \]

Then the system is called linear.

**Definition**

Any equation with constant coefficients defines a time-invariant system.
1. \( Y_t = 0.5X_t + 0.25X_{t-1} \). Defines a linear system, also time invariant.

2. \( Y_t = 0.5Y_{t-1} + 0.25X_t \). Defines a linear system, also time invariant.

3. \( Y_t = 0.5X_t + 2 \). Does not define a linear system although it is time-invariant.

**Example**

\( Y'_t = 0.5\lambda X_t + 2 \), where \( \lambda X_t \) is the input (scale-invariant).

According to scale invariant property \( \lambda X_t \Rightarrow \lambda Y_t \).

\[
\lambda Y_t = 0.5\lambda X_t + 2\lambda, \quad \text{but} \quad Y'_t = 0.5\lambda X_t + 2.
\]

\( 0.5\lambda X_t + 2\lambda \neq 0.5\lambda X_t + 2 \).

\( Y'_t \neq \lambda Y_t \) so the scale invariant property does not hold then it is not linear.
Check for Time Invariance

Example

\[ Y_t = 0.5X_t + 0.25X_{t-1} \]. Let \( t \to t - \tau \). Now,

\[ Y_{t-\tau} = 0.5X_{t-\tau} + 0.25X_{t-\tau-1}, \text{ and } Y'_t = 0.5X_{t-\tau} + 0.25X_{t-\tau-1}. \]

Clearly, \( Y'_t = Y_{t-\tau} \). So, \( X_{t-\tau} \Rightarrow Y_{t-\tau} \), i.e. time invariant.

Example

\[ Y_t = 0.5Y_{t-1} + 0.25X_t \]. Let \( t \to t - \tau \). Now,

\[ Y_{t-\tau} = 0.5Y_{t-\tau-1} + 0.25X_{t-\tau}, \text{ and } Y'_t = 0.5Y_{t-\tau-1} + 0.25X_{t-\tau}. \]

Clearly, \( Y'_t = Y_{t-\tau} \). So, \( X_{t-\tau} \Rightarrow Y_{t-\tau} \), i.e. time invariant.
**Example**

$Y_t = 0.5X_t + 2$. First of all, it is not a linear system. Moreover,

$$Y_{t-\tau} = 0.5X_{t-\tau} + 2, \quad \text{and} \quad Y'_t = 0.5X_{t-\tau} + 2.$$ 

Clearly, $Y' = Y_{t-\tau}$. So, $X_{t-\tau} \Rightarrow Y_{t-\tau}$, i.e. time invariant.

**Example**

$Y_t = 0.5 \cdot t \cdot X_t$. At time $t$, given $X_{t-\tau}$ we have

$$Y_{t-\tau} = 0.5 \cdot (t - \tau) \cdot X_{t-\tau}, \quad \text{and} \quad Y'_t = 0.5 \cdot t \cdot X_{t-\tau}.$$ 

Clearly, $Y' \neq Y_{t-\tau}$. So, $X_{t-\tau} \not\Rightarrow Y_{t-\tau}$, i.e. time variant.
Example
Let $t = 9$, $\tau = 3 \rightarrow t - \tau = 6$. At $t = 9$, we input $X_6$.

$$Y_6 = 0.5 \times 6 \times X_6 = 3X_6,$$
and
$$Y_9' = 0.5 \times 9 \times X_6 = 4.5X_6.$$

$$3X_6 \neq 4.5X_6.$$

Clearly, $Y_9 \neq Y_6$ and $Y_t = 0.5 \cdot t \cdot X_t$ is time varying system.

Example
Consider, $Y_t = 0.5X_t$. Clearly,

$$Y_6 = 0.5X_6$$
and
$$Y_9' = 0.5X_6.$$

So, $Y_9 = Y_6$, i.e. time invariant.

Show that $Y_t = 0.5 \cdot t \cdot X_t$ is a linear system.
Exercise

Consider $Y_t = \frac{1}{X_t}$. Show that the system

1. is scale invariant.
2. is time invariant.
3. is not linear.
Exercise

\[ Y_t = X_t(t - \tau) \text{ for } \tau > 0. \]

Is it linear? time invariant? scale invariant?

\[(\lambda_1 X_{1,t} + \lambda_2 X_{2,t})(t - \tau) = \lambda_1 X_{1,t}(t - \tau) + \lambda_2 X_{2,t}(t - \tau) = \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t}.\]

\[\lambda_1 X_{1,t} + \lambda_2 X_{2,t} \Rightarrow \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t}.\]

Hence, it is a linear system.

Since \((t - \tau)\) is not constant (depends on \(t\)), the system is not time-invariant.
A time-invariant linear system may generally be written in the form

\[ Y_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k} \]

in discrete time, where \( h_k \) is constant.

A linear system is said to be **physically realizable** or **causal** if \( h_k = 0 \) for all \( k < 0 \), that is

\[ Y_t = \sum_{k=0}^{\infty} h_k X_{t-k} = h_0 X_t + h_1 X_{t-1} + h_2 X_{t-2} + \cdots. \]

The present \( Y_t \) depends only on the past and current \( X_t, X_{t-1}, \cdots \), but not on \( X_{t+1}, X_{t+2}, \cdots \).