Lecture 5: Review of interest rate models

Xiaoguang Wang

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Outline

1. Bonds and Interest Rates
2. Short Rate Models
3. Forward Rate Models
4. LIBOR and Swaps
Bonds and interest rates

**Definition**

A zero coupon bond with maturity date $T$, also called a $T$-bond, is a contract which guarantees the holder 1 dollar to be paid on the date $T$. The price at time $t$ of a bond with maturity date $T$ is denoted by $p(t, T)$.

We assume the following:

- There exists a (frictionless) market for $T$-bonds for every $T > 0$.
- The relation $p(t, t) = 1$ holds for all $t$.
- For each fixed $t$, the bond price $p(t, T)$ is differentiable w.r.t time of maturity $T$. 
Interest Rates

At time $t$, we can make a contract guaranteeing a riskless rate of interest over the future interval $[S, T]$. Such an interest rate is called a forward rate.

**Definition**

The simple forward rate (or **LIBOR rate**) $L$, is the solution to the equation

$$1 + (T - S)L = \frac{p(t, S)}{p(t, T)}$$

whereas the **continuously compounded** forward rate $R$ is the solution to the equation

$$e^{R(T - S)} = \frac{p(t, S)}{p(t, T)}$$
Definition

- The simple forward rate for \([S, T]\) contracted at \(t\), henceforth referred to as the LIBOR forward rate, is defined as

\[
L(t; S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)}
\]

- The simple spot rate for \([S, T]\), or the LIBOR spot rate, is defined as

\[
L(S, T) = -\frac{p(S, T) - 1}{(T - S)p(S, T)}
\]
Interest Rates

- The continuously compounded forward rate for \([S, T]\) contracted at \(t\) is defined as

\[
R(t; S, T) = -\frac{\log p(t, T) - \log p(t, S)}{T - S}
\]

- The continuously compounded spot rate, \(R(S, T)\) is defined as

\[
R(S, T) = -\frac{\log p(S, T)}{T - S}
\]

- The instantaneous forward rate with maturity \(T\), contracted at \(t\), is defined by

\[
f(t, T) = -\frac{\partial \log p(t, T)}{\partial T}
\]

- The instantaneous short rate at time \(t\) is defined by

\[
r(t) = f(t, t)
\]
Some useful facts

The money account is defined by

\[ B_t = \exp \int_0^t r(s) \, ds \]

For \( t \leq s \leq T \) we have

\[ p(t, T) = p(t, s) \times \exp \left\{ - \int_s^T f(t, u) \, du \right\} \]

and in particular

\[ p(t, T) = \exp \left\{ - \int_t^T f(t, s) \, ds \right\} \]
Relations between short rates, forward rates and zero coupon bonds

Assume we have

\[ dr(t) = a(t)dt + b(t)dW(t) \]

\[ dp(t, T) = p(t, T)m(t, T)dt + p(t, T)v(t, T)dW(t) \]

\[ df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t) \]

Then we must have

\[
\begin{cases}
\alpha(t, T) = v_T(t, T)v(t, T) - m_T(t, T) \\
\sigma(t, T) = -v_T(t, T)
\end{cases}
\]

and

\[
\begin{cases}
a(t) = f_T(t, t) + \alpha(t, t) \\
b(t) = \sigma(t, t)
\end{cases}
\]
And we also should have

\[
dp(t, T) = p(t, T) \left\{ r(t) + A(t, T) + \frac{1}{2} \|S(t, T)\|^2 \right\} dt
\]

\[+ p(t, T) S(t, T) dW(t) \]

where

\[
\begin{align*}
A(t, T) &= - \int_t^T \alpha(t, s) ds \\
S(t, T) &= - \int_t^T \sigma(t, s) ds
\end{align*}
\]
Fixed Coupon Bonds

- Fix a number of dates, i.e. points in time, \( T_0, \ldots, T_n \). Here \( T_0 \) is interpreted as the emission date of the bond, whereas \( T_1, \ldots, T_n \) are coupon dates.
- At time \( T_i, i = 1, \ldots, n \), the owner of the bond receives the deterministic coupon \( c_i \).
- At time \( T_n \) the owner receives the face value \( K \).

Then the price of the fixed coupon bond at time \( t < T_1 \) is given by

\[
p(t) = K \times p(t, T_n) + \sum_{i=1}^{n} c_i \times p(t, T_i)
\]

And the return of the \( i \)th coupon is defined as \( r_i \):

\[
c_i = r_i(T_i - T_{i-1})K
\]
Floating rate bonds

If we replace the coupon rate $r_i$ with the spot LIBOR rate $L(T_{i-1}, T_i)$:

$$c_i = (T_i - T_{i-1})L(T_{i-1}, T_i)K$$

If we set $T_i - T_{i-1} = \delta$ and $K = 1$, then the value of the $i$th coupon at time $T_i$ should be

$$c_i = \delta \frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)} = \frac{1}{p(T_{i-1}, T_i)} - 1$$

which further discounted to time $t < T_0$ should be

$$p(t, T_{i-1}) - p(t, T_i)$$

Summing up all the terms we finally obtain the price of the floating coupon bond at time $t$

$$p(t) = p(t, T_n) + \sum_{i=1}^{n} [p(t, T_{i-1}) - p(t, T_i)] = p(t, T_0)$$

This also means that the entire floating rate bond can be replicated through a self-financing portfolio. (Exercise for you)
An interest rate swap is a basically a scheme where you exchange a payment stream at a fixed rate of interest, known as the swap rate, for a payment stream at a floating rate (typically a LIBOR rate). Denote the principal by $K$, and the swap rate by $R$. By assumption we have a number of equally spaced dates $T_0, \ldots, T_n$, and payment occurs at the dates $T_1, \ldots, T_n$ (not at $T_0$). If you swap a fixed rate for a floating rate (in this case the LIBOR spot rate), then at time $T_i$, you will receive the amount

$$K \delta L(T_{i-1}, T_i)$$

and pay the amount

$$K \delta R$$

where $\delta = T_i - T_{i-1}$. 
The price, for \( t < T_0 \), of the swap above is given by

\[
\Pi(t) = Kp(t, T_0) - K \sum_{i=1}^{n} d_i p(t, T_i)
\]

where

\[
d_i = R \delta, \quad i = 1, \ldots, n - 1,
\]

\[
d_n = 1 + R \delta
\]

If, by convection, we assume that the contract is written at \( t = 0 \), and the contract value is zero at the time made, then

\[
R = \frac{p(0, T_0) - p(0, T_n)}{\delta \sum_{i=1}^{n} p(0, T_i)}
\]
In most cases, the yield of an interest rate product is the “internal rate of interest” for this product. For example, the continuously compounded zero coupon yield $y(t, T)$ should solve

$$ p(t, T) = e^{-y(T-t) \times 1} $$

which is given by

$$ y(t, T) = -\frac{\log p(t, T)}{T - t} $$

For a fixed $t$, the function $T \rightarrow y(t, T)$ is called the (zero coupon) yield curve.
The **yield to maturity**, \( y(t, T) \), of a fixed coupon bond at time \( t \), with market price \( p \), and payments \( c_i \), at time \( T_i \) for \( i = 1, \cdots, n \) is defined as the value of \( y \) which solves the equation

\[
p(t) = \sum_{i=1}^{n} c_i e^{-y(T_i-t)}
\]

For the fixed coupon bond above, with price \( p \) at \( t = 0 \), and yield to maturity \( y \), the **duration**, \( D \) is defined as

\[
D = \frac{\sum_{i=1}^{n} T_i c_i e^{-yT_i}}{p}
\]

which can be interpreted as the "weighted average of the coupon dates". With the notations above we have

\[
\frac{dp}{dy} = -D \times p
\]
Yield Curve

Definition

The zero-coupon curve (sometimes also referred to as "yield curve") at time $t$ is the graph of the function

$$
T \mapsto \begin{cases}
L(t, T), & t \leq T \leq t + 1 \text{ (years)} \\
Y(t, T), & T > t + 1 \text{ (years)}
\end{cases}
$$

where the $L(t, T)$ is the spot LIBOR rate and $Y(t, T)$ is the annually compounded spot interest rate.

Such a zero-coupon curve is also called the term structure of interest rates at time $t$. Under different economic environments, the shape of zero-coupon curve can be very different, such as the "normal curve", "flat curve", "inverted curve", "steep curve" and so on.
Zero-bond curve

Definition

The zero-bond curve at time $t$ is the graph of the function

$$T \mapsto P(t, T), \quad T > t$$

which, because of the positivity interest rates, is a $T$-decreasing function starting from $P(t, t) = 1$. Such a curve is also referred to as term structure of discount factors.
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In this section we turn to the problem of how to model an arbitrage free family of zero coupon bond price process \( \{ p(\cdot, T); T \geq 0 \} \). To model that, we first assume that short rate under the objective probability measure \( P \), satisfies the SDE

\[
dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))d\bar{W}(t)
\]

And the only exogenously given asset is the money account, with price process \( B \) defined by the dynamics

\[
 dB(t) = r(t)B(t)dt
\]

We further assume that there exists a market for zero coupon \( T \)-bond for every value of \( T \).
Question: Are bond prices uniquely determined by the $P$-dynamics of the short rate $r$?
Answer: No! The market is incomplete. The short rate, as an ”underlying”, is not tradable.

Pricing ideas:

- Prices of bonds with 
  - different maturities will have to satisfy certain internal consistency relations in order to avoid arbitrage possibilities on the bond market.

- If we take the price of one particular ”benchmark” bond as given then the prices of all other bonds (with maturity prior to the benchmark) will be uniquely determined in terms of the price of the benchmark bond (and the $r$-dynamics).
Term Structure Equation: Assumptions

Assumption: We assume that there is a market for $T$-bonds for every choice of $T$ and that the market is arbitrage free. We assume furthermore that, for every $T$, the price of a $T$-bond has the form

$$p(t, T) = F(t, r(t); T)$$

where $F$ is a smooth function of three real variables. Sometimes we write $F^T(t, r)$ instead of $F(t, r(t); T)$. Apply Ito formula, then

$$dF^T = F^T \alpha_T dt + F^T \sigma_T d\bar{W}$$

where

$$\alpha_T = \frac{F^T_t + \mu F^T_r + \frac{1}{2} \sigma^2 F^T_{rr}}{F^T}$$

$$\sigma_T = \frac{\sigma F^T_r}{F^T}$$
Theorem

Assume that the bond market is free of arbitrage. Then there exists a process $\lambda$ such that the relation

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t)$$

holds for all $t$ and for every choice of maturity time $T$. 
Term Structure Equation

**Theorem**

In an arbitrage free bond market, $F^T$ will satisfy the term structure equation

$$
\begin{cases}
F^T_t + (\mu - \lambda \sigma)F^T_r + \frac{1}{2} \sigma^2 F^T_{rr} - rF^T = 0, \\
F^T(T, r) = 1
\end{cases}
$$

For a general contingent claim $\chi = \Phi(r(T))$. The price $F(t, r(t))$ will then satisfy

$$
\begin{cases}
F_t + (\mu - \lambda \sigma)F_r + \frac{1}{2} \sigma^2 F_{rr} - rF = 0, \\
F^T(T, r) = \Phi(r)
\end{cases}
$$
Risk neutral valuation

Bond prices are given by the formula \( p(t, T) = F(t, r(t); T) \) where

\[
F(t, r; T) = E_{t,r}^Q \left[ e^{-\int_t^T r(s)ds} \right]
\]

Here the martingale measure \( Q \) and the subscripts \( t, r \) denote that the expectation shall be taken given the following dynamics for the short rate

\[
dr(s) = (\mu - \lambda \sigma)ds + \sigma dW(s),
\]

\[r(t) = r\]

For a general contingent claim, the valuation becomes

\[
F(t, r; T) = E_{t,r}^Q \left[ e^{-\int_t^T r(s)ds} \times \Phi(r(T)) \right]
\]

The term structure, as well as the prices of all other interest rate derivatives, are completely determined by specifying the \( r \)-dynamics under the martingale measure \( Q \).
Popular r-dynamics

- **Vasicek**
  
  \[ dr = (b - ar)dt + \sigma dW, \quad (a > 0) \]

- **Cox-Intersoll-Ross (CIR)**
  
  \[ dr = a(b - r)dt + \sigma \sqrt{r} dW \]

- **Black-Derman-Toy**
  
  \[ dr = \Theta(t)rdt + \sigma(t)rdW \]

- **Ho-Lee**
  
  \[ dr = \Theta(t)dt + \sigma dW \]

- **Hull-White**
  
  \[ dr = (\Theta(t) - a(t)r)dt + \sigma(t)dW, \quad a(t) > 0 \]

- **Hull-White (extended CIR)**
  
  \[ dr = (\Theta(t) - a(t)r)dt + \sigma(t)\sqrt{r}dW \quad (a(t) > 0) \]
Invert the yield curve

It is of key importance to pin down the \( r \)-dynamics under the martingale measure \( Q \) since all the derivatives pricing problems in a short rate model framework depend on that. In order to accomplish that, we can use the so-called inverting yield curve approach:

- Choose a particular model involving one or more parameters and denote the entire parameter vector as \( \alpha \).

- Solve, for every conceivable time of maturity \( T \), the term structure equation for the T-bonds. Thus we have the theoretic term structure as

\[
p(t, T; \alpha) = F^T(t, r; \alpha)
\]

- Collect price date from the bond market. Denote the empirical term structure by \( \{p^*(0, T); T \geq 0\} \).

- Now choose the parameter vector \( \alpha \) in such a way that the theoretical curve fits the empirical curve as well as possible. This gives us the estimated parameter \( \alpha^* \).
Invert yield curve: Continued

- Insert $\alpha^*$ into $\mu$ and $\sigma$. Now we have pinned down exactly which martingale measure we are working with. Let us denote the result by $\mu^*$ and $\sigma^*$ respectively.

- We now can compute an interest rate derivative with final payoff $\mathcal{X} = \Gamma(r(T))$. The price process $\Pi(t; \Gamma) = G(t, r(t))$ solves

$$\begin{aligned}
G_t + \mu^* G_r + \frac{1}{2} [\sigma^*]^2 G_{rr} - rG &= 0 \\
G(T, r) &= \Gamma(r)
\end{aligned}$$

It is of great importance that the PDEs involved are easy to solve. And it turns out that some of the models above are much easier to deal with analytically than the others, and this leads us to the subject of so called affine term structures.
Affine Term Structure

If the term structure \( \{ p(t, T); 0 \leq t \leq T, T > 0 \} \) has the form

\[
p(t, T) = F(t, r(t); T),
\]

where \( F \) has the form

\[
f(t, r; T) = e^{A(t,T) - B(t,T)r}
\]

and where \( A \) and \( B \) are deterministic functions, then the model is said to possess an **affine term structure** (ATS).

It turns out that the ATS is general enough to include lots of popular models. If the drift \( \mu(t, r) \) and diffusion part \( \sigma(t, r) \) have the form

\[
\begin{align*}
\mu(t, r) &= \alpha(t)r + \beta(t) \\
\sigma(t, r) &= \sqrt{\gamma(t)r + \delta(t)}
\end{align*}
\]

then the model admits an affine term structure.
Summary of short rate models

The main advantages with such models are as follows:

- Specifying \( r \) as the solution of an SDE allows us to use Markov process theory, so we may work within a PDF framework.
- In particular it is often possible to obtain analytical formulas for bond prices and derivatives.

The main drawbacks of short rate models:

- From an economic point of view it seems unreasonable to assume that the entire money market is governed by only one explanatory variable.
- It is hard to obtain a realistic volatility structure for the forward rates without introducing a very complicated short rate model.
- As the short rate model becomes more realistic, the inversion of the yield curve described above becomes increasingly more difficult.
An obvious extending idea would, for example, be to present a prior model for the sort rate as well as for some long rate, and one could of course model one or several intermediary interest rates. The method proposed by Heath-Jarrow-Morton is at the far end of this spectrum: they choose the entire forward rate curve as their (infinite dimensional) state variable. We assume that, for every fixed \( T > 0 \), the forward rate has a stochastic differential which under the objective measure \( P \) is given by

\[
df(t, T) = \alpha(t, T)dt + \sigma(t, T)d\bar{W}(t)
\]

\[
f(0, T) = f^*(0, T)
\]

where \( \bar{W} \) is a \( d \)-dimensional \( P \)-Wiener process whereas the \( \alpha \) and \( \sigma \) are adapted process.
Theorem

Assume that the family of forward rates is given as above and that the induced bond market is arbitrage free. Then there exists a d-dimensional column-vector process

$$\lambda(t) = [\lambda_1(t), \cdots, \lambda_d(t)]'$$

with the property that for all $T \geq 0$ and for all $t \leq T$, we have

$$\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s)' ds - \sigma(t, T)\lambda(t)$$
HJM drift condition under martingale measure

**Theorem**

*Under the martingale measure \( Q \), the process \( \alpha \) and \( \sigma \) must satisfy the following relation, for every \( t \) and every \( T \geq t \).*

\[
\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s)' ds
\]

Thus we know that when we specify the forward rate dynamics (under \( Q \)) we may freely specify the volatility structure. The drift parameters are then uniquely determined.
A simple Example

Now we illustrate a simple example. Set \( \sigma(t, T) = \sigma \).

Then

\[
\alpha(t, T) = \sigma \int_t^T \sigma ds = \sigma^2 (T - t)
\]

Then we have

\[
f(t, T) = f^*(0, T) + \int_0^t \sigma^2 (T - s) ds + \int_0^t \sigma dW(s),
\]

i.e.

\[
f(t, T) = f^*(0, T) + \sigma^2 t \left( T - \frac{t}{2} \right) + \sigma W(t)
\]

In particular we see

\[
r(t) = f(t, t) = f^*(0, t) + \sigma^2 \frac{t^2}{2} + \sigma W(t)
\]

so the short rate dynamics is

\[
dr(t) = (f_T(0, t) + \sigma^2 t) dt + \sigma dW(t)
\]

which is exactly the Ho-Lee model, fitted to the initial term structure.
The interest rate models based on infinitesimal interest rates like the instantaneous short rate and the instantaneous forward rates are nice to handle from a mathematical point of view, but they have two main disadvantages:

- The instantaneous short and forward rates can never be observed in real life.
- If you would like to calibrate your model to cap or swaption data, then this is typically very complicated from a numerical point of view if you use one of the "instantaneous" models.
- For a very long time, the market practice has been to value caps, floors, and swaptions by using a formal extension of the Black (1976) model. Such an extension typically on one hand assumes that the short rate at one point to be deterministic, while on the other hand the LIBOR rate is assumed to be stochastic, which is of course logically inconsistent.
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Logically consistent models for LIBOR rates, caps, and swaptions

- Instead of modeling the instantaneous interest rates, we model discrete market rates like LIBOR rates in the LIBOR market models, or forward swap rates in the swap market models.

- Under a suitable choice of numeraire, these market rates can in fact be modeled log normally.

- The market models will thus produce pricing formulas for caps and floors (LIBOR models), and swaptions (the swap market models) which are of the Black-76 type and this confirming with market practice.

- By construction the market models are thus very easy to calibrate to market data for caps/floors and swaptions respectively.
Consider a fixed set of increasing maturities $T_0, T_1, \cdots, T_N$ and we define $\alpha_i$, by

$$\alpha_i = T_i - T_{i-1}, \quad i = 1, \cdots, N$$

The number $\alpha_i$ is known as the **tenor**, which often is a quarter of a year in practice. Let $p_i(t)$ denote the zero coupon bond price $p(t, T_i)$ and let $L_i(t)$ denote the LIBOR forward rate contracted at $t$, for period $[T_{i-1}, T_i]$. A *cap* or **cap rate** $R$ and resettlement dates $T_0, \cdots, T_N$ is a contract which at time $T_i$ gives the holder of the cap amount

$$X_i = \alpha_i \cdot \max[L_i(T_{i-1}) - R, 0]$$

for each $i = 1, \cdots, N$. The cap is thus a portfolio of the individual **caplets** $X_1, \cdots, X_N$. 
Theorem

The Black-76 formula for the caplet $X_i = \alpha_i \cdot \max[L(T_{i-1}, T_i) - R, 0]$ is given by the expression

$$\text{Capl}_B^i(t) = \alpha_i \cdot p_i(t)\{L_i(t)N[d_1] - RN[d_2]\}, \quad i = 1, \ldots, N$$

where

$$d_1 = \frac{1}{\sigma_i \sqrt{T_i - t}} \left[ \ln \left( \frac{L_i(t)}{R} \right) + \frac{1}{2} \sigma_i^2 (T - t) \right],$$

$$d_2 = d_1 - \sigma_i \sqrt{T_i - t}$$

The constant $\sigma_i$ is known as the **Black Volatility** for caplet No. $i$.

Sometimes we also write $\text{Capl}_B^i(t; \sigma_i)$. 
Implied Black Volatilities

In the market, cap prices are not quoted in monetary terms but instead in terms of **implied Black volatilities**. And there are two types of implied Black volatilities, the **flat volatilities** and the **spot volatilities** (also known as **forward volatilities**). First of all, it is easy to see that

$$\text{Cap}_i(t) = \text{Cap}_i(t) - \text{Cap}_{i-1}(t), \quad i = 1, \ldots, N$$

Then the implied flat volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as the solutions of the equations

$$\text{Cap}^m_i(t) = \sum_{k=1}^{i} \text{Cap}^B_k(t; \bar{\sigma}_i), \quad i = 1, \ldots, N$$

The implied spot volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as solutions of the equations

$$\text{Cap}^m_i(t) = \text{Cap}^B_i(t; \bar{\sigma}_i), \quad i = 1, \ldots, N$$

A sequence of implied volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ (flat or spot) is called a volatility term structure.
The standard risk neutral valuation for $\text{C}ap_l(t)$ should be

$$\text{C}ap_l(t) = \alpha_l E^Q \left[ e^{-\int_0^{T_l} r(s)ds} \max[L_l(T_{i-1}) - R, 0] | F_t \right]$$

But it is much more natural to use the $T_i$ forward martingale measure to obtain

$$\text{C}ap_l(t) = \alpha_l p_i(t) E^{T_i} \left[ \max[L_l(T_{i-1}) - R, 0] | F_t \right]$$

Furthermore, we have for every $i = 1, \cdots, N$, the LIBOR process $L_l$ is a martingale under the corresponding forward measure $Q^{T_i}$, on the interval $[0, T_{i-1}]$. 
LIBOR Market Model

Set up:

- A set of resettlement dates $T_0, \ldots, T_N$.
- An arbitrage free market bond with maturities $T_0, \ldots, T_N$.
- A $k$-dimensional $Q^N$-Wiener process $W^N$.
- For each $i$ a deterministic function of time $\sigma_i(t)$.
- An initial nonnegative forward rate term structure $L_1(0), \ldots, L_N(0)$.
- For each $i$, we define $W^i$ as the $k$-dimensional $Q^i$-Wiener process generated by $W^N$ under the Girsanov transformation $Q^N \to Q^i$.

If the LIBOR forward rates have the dynamics

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), \; i = 1, \ldots, N$$

where $W^i$ is $Q^i$-Wiener as described above, then we say we have a discrete tenor LIBOR market model with volatilities $\sigma_1, \ldots, \sigma_N$. 
Consider a given volatility structure $\sigma_1, \cdots, \sigma_N$, where each $\sigma_i$ is assumed to be bounded, a probability measure $Q^N$ and a standard $Q^N$-Wiener process $W^N$. Define $L_1, \cdots, L_N$ by

$$dL_i(t) = -L_i(t) \left( \sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1 + \alpha_k L_k(t)} \sigma_k(t)\sigma_k^*(t) \right) dt + L_i(t)\sigma_i(t)dW^N(t),$$

for each $i$ where we use the convention $\sum_{N}^{N} (\cdots) = 0$. Then the $Q^i$-dynamics of $L_i$ are given as above in the LIBOR market model. Thus there exists a LIBOR model with the given volatility structure.
Theorem

In the LIBOR market model, the caplet prices are given by

$$\text{Capl}_i(t) = \alpha_i \cdot p_i(t) \{ L_i(t) N[d_1] - RN[d_2] \}, \quad i = 1, \ldots, N$$

where

$$d_1 = \frac{1}{\Sigma_i(t, T_{i-1})} \left[ \ln \left( \frac{L_i(t)}{R} \right) + \frac{1}{2} \Sigma_i^2(t, T_{i-1}) \right],$$

$$d_2 = d_1 - \Sigma_i(t, T_{i-1})$$

with $\Sigma_i$ defined below

$$\Sigma_i^2(t, T) = \int_t^T \| \sigma_i(s) \|^2 ds$$

We see that each caplet price is given by a Black type formula.
Suppose we want to price some exotic (not a cap or a floor) interest rate derivative, performing this with a LIBOR model means that we typically carry out the following two steps:

- Use implied Black volatilities in order to calibrate the model parameters to market data.
- Use Monte Carlo (or some other numerical methods) to price the exotic instrument.

Assume that we are given an empirical term structure of implied forward Black volatilities $\bar{\sigma}_1, \cdots, \bar{\sigma}_N$ for all caplets. In order to calibrate the model we have to choose the deterministic LIBOR volatilities $\sigma_1(\cdot), \cdots, \sigma_N(\cdot)$ such that

$$\bar{\sigma}_i = \frac{1}{T_i} \int_0^{T_i-1} ||\sigma_i(s)||^2 ds, \quad i = 1, \cdots, N$$

which is obviously a highly undetermined system. So in practice it is common to make some structural assumptions about the shape of the volatility functions.
Shape of volatility functions

Popular specifications on the shape of volatility functions:

- $\sigma_i(t) = \sigma_i$, $0 \leq t \leq T_{i-1}$
- $\sigma_t(t) = \sigma_{ij}$, $T_{j-1} \leq t \leq T_j$, for $j = 0, \cdots, i$.
- $\sigma_t(t) = \sigma_{ij} = \beta_{i-j}$, $T_{j-1} \leq t \leq T_j$, for $j = 0, \cdots, i$
- $\sigma_i(t) = q_i(T_{i-1} - t)e^{\beta_i(T_{i-1} - t)}$ where $q_i(\cdot)$ is some polynomial and $\beta_i$ is a real number.

After the model has been calibrated, Monte Carlo simulation is the standard tool for computing the prices of exotics. Since the SDEs in the LIBOR model are generally too complicated to allow analytical solutions, we have to resort to simulation of discretized versions of the equations using methods like Euler scheme.