Assignment 9 Answer Keys

Problem 0

\[(c) \hat{\sigma}_\beta^2 = (11.002 - 0.6062)/(2 \times 3) = 1.733.\]

Using Satterthwaite’s method the \(df_\beta = (MSB - MSAB)^2/(MSB^2/df_b + MSAB^2/df_{ab}) = (11.002 - 0.602)^2/((11.002)^2/9 + (0.602)^2/9) = 8.018.\)

Also from SAS \(\chi^2_{0.025,8.018} = 17.5616, \chi^2_{0.975,8.018} = 2.189.\)

Hence the 95\% approximate CI on \(\hat{\sigma}_\beta^2\) can be given by:

\(\left(\hat{\sigma}_\beta^2 / \chi^2_{0.05/2,8.018}, \hat{\sigma}_\beta^2 / \chi^2_{1-0.05/2,8.018}\right) = (8.018 \times 1.733/17.5616, 8.018 \times 1.733/2.189) = (0.791, 6.348).\)

For Problem 3 of HW#8, SAS uses the unrestricted model, or you can analyze the data yourself assuming an unrestricted model.

Problem 1

a) Note that SAS uses the unrestricted model. Here we analyze the data assuming a restricted model.

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Squares</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>19</td>
<td>104.8500000</td>
<td>5.5184211</td>
<td>3.68</td>
<td>0.0003</td>
</tr>
<tr>
<td>Error</td>
<td>40</td>
<td>60.0000000</td>
<td>1.5000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corrected Total</td>
<td>59</td>
<td>164.8500000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Type III SS</th>
<th>Mean Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>operator</td>
<td>1</td>
<td>0.41666667</td>
<td>0.41666667</td>
<td>0.28</td>
<td>0.6011</td>
</tr>
<tr>
<td>part</td>
<td>9</td>
<td>99.01666667</td>
<td>11.00185185</td>
<td>7.33</td>
<td>&lt;.0001</td>
</tr>
<tr>
<td>operator*part</td>
<td>9</td>
<td>5.41666667</td>
<td>0.60185185</td>
<td>0.40</td>
<td>0.9270</td>
</tr>
</tbody>
</table>

Here we assume both ‘operators’ and ‘parts’ are random effects.

**Test for ‘operators’** \(H_0 : \sigma^2_\tau = 0.\)

\[F_0 = MSA/MSAB = 0.4167/0.6018 = 0.692.\]

P-value is 0.4269. As the P-value is large we fail to reject null hypothesis. The random effect ‘operators’ is not significant.

**Test for ‘parts’** \(H_0 : \sigma^2_\beta = 0.\)

\[F_0 = MSB/MSAB = 11.00185/0.6018 = 18.2816.\]

P-value is 9.377e-05 and we reject null hypothesis. The random effect ‘parts’ is significant.
Test for ‘interaction’ $H_0: \sigma^2_{\tau\beta} = 0$. 

$$F_0 = \text{MSAB}/\text{MSE} = 0.40.$$ 

As the P-value is very big (much greater than 0.05) we fail to reject $H_0$ and conclude that the effect due to ‘interaction’ is not significant.

The variance component estimates are:

\[
\hat{\sigma}^2_{\tau} = \frac{(0.41667 - 0.60185)/(10 \times 3)}{10} = -0.0062 (\approx 0).
\]

\[
\hat{\sigma}^2_{\beta} = \frac{(11.00185 - 0.60185)/(2 \times 3)}{2} = 1.7333.
\]

\[
\hat{\sigma}^2_{\tau\beta} = \frac{(0.60185 - 1.5)/3}{3} = -0.299 (\approx 0).
\]

b)-d): please refer to page 50-53 of lecture 10.

### Problem 2

(a) For the restricted model, we have the following EMS table.

<table>
<thead>
<tr>
<th>Term</th>
<th>F</th>
<th>F</th>
<th>R</th>
<th>R</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>l</th>
<th>EMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_i$</td>
<td>0</td>
<td>b</td>
<td>c</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2 + bn\sigma^2_{\tau\gamma} + \frac{bcn}{a-1} \sum_{i=1}^{a-1} \tau^2_i$</td>
</tr>
<tr>
<td>$\beta_j$</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2 + an\sigma^2_{\beta\gamma} + \frac{acn}{b-1} \sum_{j=1}^{b-1} \beta^2_j$</td>
</tr>
<tr>
<td>$\gamma_k$</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2 + abn\sigma^2_{\gamma}$</td>
</tr>
<tr>
<td>$(\tau\beta)_{ij}$</td>
<td>0</td>
<td>0</td>
<td>c</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + \frac{cn}{(a-1)(b-1)} \sum_{i=1}^{a-1} \sum_{j=1}^{b-1} (\tau\beta)^2_{ij}$</td>
</tr>
<tr>
<td>$(\tau\gamma)_{ik}$</td>
<td>0</td>
<td>b</td>
<td>1</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2 + bn\sigma^2_{\tau\gamma}$</td>
</tr>
<tr>
<td>$(\beta\gamma)_{jk}$</td>
<td>a</td>
<td>0</td>
<td>1</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2 + an\sigma^2_{\beta\gamma}$</td>
</tr>
<tr>
<td>$(\tau\beta\gamma)_{ijk}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2 + n\sigma^2_{\tau\beta\gamma}$</td>
</tr>
<tr>
<td>$\epsilon_{ijkl}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2$</td>
</tr>
</tbody>
</table>

The F-tests for the main effects are

Test ‘$H_0$: all $\tau_i$ are zero’ vs. ‘$H_1$: at least one of $\tau_i$ is not zero’ with $F_\tau = MSA/MSAC$, which follows $F_{a-1,(a-1)(c-1)}$ under the null hypothesis.

Test ‘$H_0$: all $\beta_j$ are zero’ vs. ‘$H_1$: at least one of $\beta_j$ is not zero’ with $F_\beta = MSB/MSBC$, which follows $F_{b-1,(b-1)(c-1)}$ under the null hypothesis.

Test ‘$H_0$: $\sigma^2_{\gamma} = 0$’ vs. ‘$H_1$: $\sigma^2_{\gamma} > 0$’ with $F_\gamma = MSC/MSE$, which follows $F_{c-1,abc(n-1)}$ under the null hypothesis.

(b) For the unrestricted model, we have the following EMS table.
<table>
<thead>
<tr>
<th>Term</th>
<th>F</th>
<th>F</th>
<th>R</th>
<th>R</th>
<th>EMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau_i)</td>
<td>0 b c n</td>
<td>(\sigma^2 + b n \sigma^2_{\tau \gamma} + n \sigma^2_{\tau \beta \gamma} + \frac{b c n \sum_{i=1}^{a} \tau_i^2}{a-1})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta_j)</td>
<td>a 0 c n</td>
<td>(\sigma^2 + a n \sigma^2_{\beta \gamma} + n \sigma^2_{\tau \beta \gamma} + \frac{a c n \sum_{j=1}^{b} \beta_j^2}{b-1})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\gamma_k)</td>
<td>a b 1 n</td>
<td>(\sigma^2 + b n \sigma^2_{\gamma} + b n \sigma^2_{\tau \gamma} + an \sigma^2_{\beta \gamma} + n \sigma^2_{\tau \beta \gamma})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((\tau \beta)_{ij})</td>
<td>0 0 c n</td>
<td>(\sigma^2 + n \sigma^2_{\tau \beta \gamma} + \frac{c n \sum_{i=1}^{a} (\tau \beta)_{ij}^2}{(a-1)(b-1)})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((\tau \gamma)_{ik})</td>
<td>1 b 1 n</td>
<td>(\sigma^2 + b n \sigma^2_{\gamma} + n \sigma^2_{\tau \beta \gamma})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((\beta \gamma)_{jk})</td>
<td>a 1 1 n</td>
<td>(\sigma^2 + a n \sigma^2_{\beta \gamma} + n \sigma^2_{\tau \beta \gamma})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((\tau \beta \gamma)_{ijk})</td>
<td>1 1 1 n</td>
<td>(\sigma^2 + n \sigma^2_{\tau \beta \gamma})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\epsilon_{ijkl})</td>
<td>1 1 1 1</td>
<td>(\sigma^2)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The F-tests for the main effects are:
Test \('H_0\): all \(\tau_i\) are zero’ vs. ‘\(H_1\): at least one of \(\tau_i\) is not zero’ with \(F_{\tau} = \frac{MSA}{MSAC}\), which follows \(F_{a-1,(a-1)(c-1)}\) under the null hypothesis.
Test \('H_0\): all \(\beta_j\) are zero’ vs. ‘\(H_1\): at least one of \(\beta_j\) is not zero’ with \(F_{\beta} = \frac{MSB}{MSBC}\), which follows \(F_{b-1,(b-1)(c-1)}\) under the null hypothesis.
Test \('H_0\): all \(\sigma^2_{\gamma} = 0\) ’ vs. ‘\(H_1\): \(\sigma^2_{\gamma} > 0\)’ with \(F_{\gamma} = \frac{(MSC + MSABC)/(MSAC + MSBC)}{MSAC^2/((a-1)(c-1)) + MSBC^2/((b-1)(c-1))}\) (you may have a different approximate F-test here), which follows \(F_{p,q}\) under the null hypothesis with

\[
p = \frac{(MSC + MSABC)^2}{MSC^2/((c-1) + MSBC^2/((a-1)(b-1)(c-1)))}
\]

\[
q = \frac{(MSAC + MSBC)^2}{MSAC^2/((a-1)(c-1)) + MSBC^2/((b-1)(c-1))}
\]

**Problem 3**

(a) The reason is: (i) “batch” has four levels at each level of “process”; (ii) under the same level of “process”, the levels of “batch” are comparable; (iii) under a level of “process”, the levels of “batch” can be arbitrarily numbered (i.e., the levels of “batch” from different levels of “process” are not comparable).

(b) Ignored.