Problem 1.
a)

Since 
\[ u = \frac{55}{50} = 1.1 \quad \text{and} \quad d = \frac{40}{50} = 0.8, \]
we have, by (10.5),

\[ p^* = \frac{e^{(r-g)t} - d}{u - d} = \frac{e^{0.05} - 0.8}{1.1 - 0.8} = 0.5041. \]

The no-arbitrage price of the call is
\[ C_0 = e^{-rT} \left[ p^* C_u + (1 - p^*) C_d \right] = e^{-0.05} (0.5041 \times 5) = 2.3976 > 1.9. \]

b)

Since the market call is underpriced, the arbitrageur will have to buy the market call and sell the replicating portfolio of the call. To determine the latter, we need to find \( \Delta \):

\[ \Delta = e^{-rT} \frac{C_u - C_d}{S_0 u - S_0 d} = e^{-0.1} \frac{5 - 0}{55 - 40} = e^{-0.1} \frac{1}{3} = 0.3016. \]

So, the transaction at time 0 will be:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy the market call</td>
<td>-1.9</td>
</tr>
<tr>
<td>Short sell 0.3016 shares of the stock</td>
<td>0.3016 \times 50 = 15.0806</td>
</tr>
<tr>
<td>Lend money</td>
<td>-(15.0806 - 1.9) = 13.1806</td>
</tr>
<tr>
<td>Net P/L</td>
<td>0</td>
</tr>
</tbody>
</table>

At expiration, the profit/loss is given as follows:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flow if ( S_T = 55 )</th>
<th>Cash Flow if ( S_T = 40 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Receive the payoff from the call</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Buy back ( 0.3016e^{0.1} = \frac{1}{3} ) shares</td>
<td>(-\frac{1}{3} \times 55 = 18.3333)</td>
<td>(-\frac{1}{3} \times 40 = 13.3333)</td>
</tr>
<tr>
<td>Receive money from treasuries</td>
<td>13.1806e^{0.05} = 13.8563</td>
<td>13.1806e^{0.05} = 13.8563</td>
</tr>
<tr>
<td>Net P/L</td>
<td>0.523</td>
<td>0.523</td>
</tr>
</tbody>
</table>
Problem 2.

The time-0 price of the call option is

\[
C_0 = e^{-rT} \left[ p^* \times C_u + (1 - p^*) \times C_d \right] = e^{-r} \left[ \frac{e^r - 0.8}{1.2 - 0.8} \times 2 + (1 - p^*) \times 0 \right] = \frac{1 - 0.8e^{-r}}{0.2}.
\]

Setting \( \frac{1 - 0.8e^{-r}}{0.2} = 1.13 \), we get \( e^{-r} = 0.9675 \) (or \( r = 3.3\% \)).

If \( S_d = 6 \), then \( d = 0.6 \), and

\[
C_0 = e^{-r} \times \frac{e^r - 0.6}{1.2 - 0.6} \times 2 = \frac{1 - 0.6e^{-r}}{0.3} = \frac{1 - 0.6 \times 0.9675}{0.3} = 1.398.
\]

Problem 3.

\[
u = e^{(r-d)h + \sigma \sqrt{h}} = e^{0.04/4 \times (0.3/2)} = e^{0.16} = 1.173511
\]

\[
d = e^{(r-d)h - \sigma \sqrt{h}} = e^{0.04/4 \times (0.3/2)} = e^{-0.14} = 0.869358
\]

\( S \) = initial stock price = 100

\[
p^* = \frac{1}{1 + e^{0.3/2}} = \frac{1}{1 + e^{0.15}} = \frac{1}{1 + 1.1618} = 0.46257.
\]

Then, inequality that needs to be satisfied is

\[
K - 100 > e^{-0.01} \times \{ 0.4626 \times (K - 117.35), + 0.5374 \times (K - 86.94) \}.
\]

and we check three cases: \( K \leq 86.94, K \geq 117.35, \) and \( 86.94 < K < 117.35 \).

For \( K \leq 86.94 \), inequality (5) cannot hold, because its LHS < 0 and its RHS = 0.

For \( K \geq 117.35 \), (5) always holds, because its LHS = \( K - 100 \) while its RHS = \( e^{-0.01} K - 100 \).

For \( 86.94 < K < 117.35 \), inequality (5) becomes

\[
K - 100 > e^{-0.01} \times 0.5374 \times (K - 86.94),
\]

or

\[
K > \frac{100 - e^{-0.01} \times 0.5374 \times 86.94}{1 - e^{-0.01} \times 0.5374} = 114.85.
\]

Smallest integer-valued strike price for which an investor will early exercise is \( 115 \).
Problem 4.

a) 
\[
\begin{align*}
\sigma &= 0.1 \\
h &= 0.50 \\
r &= 0.04 \\
delta &= 0 \\
S_0 &= 42 \\
u &= 1.094952166 \\
d &= 0.950553647 \\
p^* &= 0.482329692 \\
q^* &= 0.517670308 \\
K &= 42
\end{align*}
\]

Binomial Tree for the future price

\[
\begin{array}{ccc}
42 & 45.98799096 & 50.35465031 \\
39.92325316 & 43.71405252 & 37.94919388
\end{array}
\]

Binomial Tree for the European Option Price

\[
\begin{array}{ccc}
1.0430 & 0.0000 & 0.0000 \\
2.0555 & 0.0000 & 0.0000 \\
4.0508 & &
\end{array}
\]

Binomial Tree for American Option Price (yellow represents early exercise opportunity)

\[
\begin{array}{ccc}
1.0538 & 0.0000 & 0.0000 \\
2.0767 & 0.0000 & 0.0000 \\
4.0508 & &
\end{array}
\]

Immediate Payoff

\[
\begin{array}{ccc}
0 & 0 & 0 \\
2.076746838 & 0 & 0 \\
4.050806118 & &
\end{array}
\]

b) Final Profit = (2.07674 - 2.0555) * exp(0.04/2) = 0.02171.
Problem 5.
From (i) and (ii), it follows that

\[ u = \frac{4}{3}d, \quad p^* = \frac{1}{3} = \frac{1 - d}{u - d} = \frac{1 - d}{\frac{4}{3}d - d}. \]

Then, we get that \( d = 0.9 \) and \( u = 1.2 \).

**Binomial Tree for the future price**

<table>
<thead>
<tr>
<th>80</th>
<th>96</th>
<th>115.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>72</td>
<td></td>
<td>86.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>64.8</td>
</tr>
</tbody>
</table>

**Binomial Tree for the European Option Price**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4551</td>
<td>1.4000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

**Immediate Payoff**

<table>
<thead>
<tr>
<th>0</th>
<th>11</th>
<th>30.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.4</td>
<td>0</td>
</tr>
</tbody>
</table>

**Binomial Tree for American Option Price (yellow represents early exercise opportunity)**

<table>
<thead>
<tr>
<th>3.8721</th>
<th><strong>11.0000</strong></th>
<th>30.2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4551</td>
<td>1.4000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

**Delta for the replicating portfolio**

<table>
<thead>
<tr>
<th>0.4281</th>
<th>1.0000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0648</td>
<td></td>
</tr>
</tbody>
</table>

(a) The difference between the American and European call prices is about 0.08829.

(b) Since the European market call is less than the synthetic call price, we should buy the market call and sell the replicating portfolio, which means that we should sell 0.42 future contracts (see delta above) and borrow 3 dollars. The following table illustrates the transactions and corresponding cash flows:
Problem 6.
The option is exercised if $S(T) > K$. So, we want to compute

$$P(S(T) > K).$$

Recall that, under the Black-Scholes framework,

$$S(T) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})T + \sigma \sqrt{T}Z}.$$

Equivalently,

$$\ln S(T) = \ln S(0) + (\alpha - \delta - \frac{\sigma^2}{2})T + \sigma \sqrt{T}Z.$$

Using the previous information,

$$\Pr(S(T) > K)$$

$$= \Pr(\ln S(T) > \ln K)$$

$$= \Pr(Z > \frac{\ln K - [\ln S(0) + (\alpha - \delta - \frac{\sigma^2}{2})T]}{\sigma \sqrt{T}})$$  where $Z \sim N(0, 1)$

$$= \Pr(Z > \frac{\ln 1.25 - \ln 100 - (0.1 - 0 - \frac{0.3^2}{2}) \times 0.75}{0.3 \times 0.75})$$

$$= \Pr(Z > \frac{\ln 1.25 - 0.055 \times 0.75}{0.3 \times 0.75})$$

$$= \Pr(Z > 0.700109)$$

$$= 1 - 0.7580$$

$$= 0.242$$
Problem 7.
By (12.7), the price of the put option is

\[ P = e^{-rT} [KN(-d_2) - FN(-d_1)], \]

where \[ d_1 = \frac{\ln(F/K) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \]
and \[ d_2 = d_1 - \sigma \sqrt{T}. \]

With \( F = K \), we have \( \ln(F/K) = 0 \),
\[ d_1 = \frac{\frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} = \frac{1}{2} \sigma \sqrt{T}, \]
\[ d_2 = -\frac{1}{2} \sigma \sqrt{T}, \]
and
\[ P = Fe^{-rT} [N(\frac{1}{2} \sigma \sqrt{T}) - N(-\frac{1}{2} \sigma \sqrt{T})] = Fe^{-rT} [2N(\frac{1}{2} \sigma \sqrt{T}) - 1]. \]

Putting \( P = 1.6, r = 0.1, T = 0.75, \) and \( F = 20 \), we get
\[ 1.625 = 20e^{-0.1 \times 0.75} [2N(\frac{1}{2} \sigma \sqrt{0.75}) - 1] \]
\[ N(\frac{1}{2} \sigma \sqrt{0.75}) = 0.5438 \]
\[ \frac{1}{2} \sigma \sqrt{0.75} = 0.11 \]
\[ \sigma = 0.254 \]

After 3 months, we have \( F = 17.7 \) and \( T = 0.5 \); hence
\[ d_1 = \frac{\ln(F/K) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} = \frac{\ln(17.7/20) + \frac{1}{2} \times 0.254^2 \times 0.5}{0.254 \times 0.5} = -0.5904 \approx -0.59 \]
\[ N(-d_1) \approx 0.7224 \]
\[ d_2 = d_1 - \sigma \sqrt{T} = -0.5904 - 0.254 \times 0.5 = -0.7700 \]
\[ N(-d_2) \approx 0.7794 \]

The put price at that time is
\[ P = e^{-rT} [KN(-d_2) - FN(-d_1)] \]
\[ = e^{-0.1 \times 0.5} [20 \times 0.7794 - 17.7 \times 0.7224] \]
\[ = 2.66489 \]

Problem 8.
See Solution to Quiz 3.