STAT 473. Homework 6 Solutions.  
Spring 2015.

Problem 1. 
(i)
\[ \Delta = e^{-\delta T} N(d_1) \]
\[ \delta = 0, \quad T = 1 \]
\[ d_1 = \frac{\ln \left( \frac{100}{95} \right) + (0.08 - 0 + \frac{1}{2} \times 0.3^2) \times 1}{0.3 \times 1} = 0.5376 \]
\[ N(d_1) = N(0.5376) = 0.7224 \]
\[ \Delta = 0.7224 \]

(ii)
\[ \Omega = \frac{0.7224 \times (100)}{18.3858} = 3.9291 \]

(iii)
We expect that the premium of the call will increase 0.36 dollars if the stock price increases 50 cents. Similarly, we will expect that the premium of the option will increase by 7.8582% if the stock price increases by 2%.

(iv)
\[ \sum_{\text{call}} = |\Omega| \sum_{\text{stock}} = (0.30)(3.9291) = 1.1787 \]

(v)
\[ \Delta = -e^{-\delta T} N(-d_1) = -e^0 \times (1 - N(-0.5376)) \]
\[ = -\left[ 1 - 0.7224 \right] = -0.2776 \]

(vi)
\[ \Omega = \frac{\Delta S}{C} = \frac{(-0.2776) \times (100)}{6.0819} = -4.5644 \]
(vii)
\[ \sigma_{option} = \sqrt{\Omega^2 \sigma_{stock}^2} = \sqrt{(4.5644)^2 \times 0.3} = 1.3693 \]

(viii)
The Sharpe ratio of the call coincides with the Sharpe ration of the stock, which is then given by \((\alpha - r) / \sigma = (0.1 - 0.08) / 0.3 = 0.066\).

**Problem 2.**
The idea is to use the fundamental formula:
\[ \gamma - r = \Omega(\alpha - r) \]
We know gamma from (vi). The Key formula to use here to find \(\alpha\) is
\[ S_T = S_0 e^{(\alpha - \delta - \sigma^2 / 2) T} \]
Then, in the Black-Scholes Framework, the instantaneous rate of return \(\ln(S_1 / S_0)\) is
normal distributed with mean \(\alpha - \delta - \sigma^2 / 2\) and variance \(\sigma^2\).
Thus, \(\sigma^2 = 0.14\) and \(\alpha - \delta - \sigma^2 / 2 = 0.09\), from which we obtained that \(\alpha = 0.19\).

Now, to find the elasticity, we first use that the replicating portfolio is 0.4, which coincides with the Delta. Then, \(\Omega = S \Delta / P = 50(-0.4) / 5 = -4\)

Therefore, \(-0.62 - r = (-4)(-0.19 - r)\), from which we can find easily \(r\).
Problem 3.

(i) Given $S_0 = 100, K = 95, r = 8\%, \delta = 0, \sigma = 0.3, T = 1$

$\Delta f_{\text{final}} = \Delta f_{\text{initial}} + n \Delta f_{\text{stock}} = 0$

$\Delta f_{\text{initial}} = -\Delta f_{\text{call}} = -e^{-rT} N(d_1) = -N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r - \delta)T}{\sigma \sqrt{T}}\right)$

$= -0.7216$

$\Rightarrow n = 0.7216$

$0.7216 \times 50 = 36.08 \text{ shares of stock}$

(ii) Stock: 100 $\rightarrow$ 101

$C_0 = S_0 e^{-rT} N(d_1) - Ke^{-rT} N(d_2) = 18.3848$

$C_0S_0 = 19.0946$

Transaction at time 0

$\begin{align*}
\text{CF} & \quad \text{Transaction} & \quad \text{CF} \\
\text{Short Call} & \quad (50) 0.3848 & \quad \text{buy back Call} & \quad (-50) (19.0946) \\
\text{long 36.08 shares} & \quad -(26.88) (100) & \quad \text{sell stock} & \quad (38.8) (101) \\
\text{borrow money} & \quad 2688.76 & \quad \text{pay loan} & \quad -2688.76 e^{0.08(1)} \\
\text{PIL} & \quad 0 & \quad \text{PIL} & \quad 0.0006
\end{align*}$

(iii) Maintain $\Delta - $Hedge

$\Delta_1 = e^{-\delta T} N(d_1) = e^{-0.08} N(0.62) = 0.7324$

$0.7324 \times 50 = 36.62$

$36.62 - 36.08 = 0.54 \text{ buy 0.54 shares}$
Problem 4.

\[ h = 1, \quad n = 2, \quad T = 2, \quad \beta = 0.065, \quad \gamma = 0.1, \quad \kappa = 300 \]

American Put:

\[ \frac{e^{-r \cdot T} - d}{1 - d} = \frac{e^{-0.065 \cdot 2} - \frac{37.5}{315}}{1 - \frac{315}{300}} = 0.0161022 \]

\[ q_0 = 1 - \frac{e^{-r \cdot T} - d}{1 - d} = 1 - 0.0161022 = 0.983878 \]

\[ t = 1, \quad S = 210 \]

\[ W_0 = e^{-0.1 \cdot (37.5 - 0.61022 + 153 - 0.387878)} = 12.667 \]

Exercise: \[ T_0 = 90 \]

\[ t = 1, \quad S = 315 \]

\[ W_0 = e^{-0.1 \cdot (315 - 0.387878)} = 13.226 \]

Exercise: \[ T_0 = 1 \]

\[ t = 2, \quad S = 200 \]

\[ W_0 = e^{-0.1 \cdot (13.226 - 0.61022 + 10 - 0.387878)} = 39.045 \]

Exercise: \[ T_0 = 0 \]

\[ \Delta(0) = \frac{C(0) - L_d}{S_0 - L_d} \cdot e^{-rT} = \frac{13.226 - 9.0}{315 - 210} \cdot e^{-0.065} = -0.4236 \]

\[ \Delta(1,2T) = \frac{C(2T) - L_d}{S_0 - L_d} \cdot e^{-0.065} = -0.0110376 \]

\[ \Delta(1,3T) = \frac{C(3T) - L_d}{S_0 - L_d} \cdot e^{-0.065} = -0.0093079 \]

\[ \left( T_0 \right)^{-1} = \frac{\Delta(0) + \Delta(1,2T)}{S_0 - L_d} = -0.70566(-0.0317) \]

\[ \frac{-0.0046}{375 - 210} = -0.0046 \]

To find the \( \theta \) we solve it from the \( \Delta - \Gamma - \theta \) approximation of \( C(2h, S_{d}) \) as follows:

\[ C(2h, S_{d}) = C(0, S) + \Delta_0 \epsilon + \frac{1}{2} \Gamma_0 \epsilon^2 + 2h \theta_0. \]

Here, \( \epsilon = S_{d} - S = 262.5 - 300 = -37.5 \) and, thus,

\[ 37.5 = 39.045 + (-0.436)(-37.5) + \frac{1}{2} (0.0046)(-37.5)^2 + 2\theta_0. \]

Finally, the \( \theta \) expresses in years is given by

\[ \theta_0 = \frac{37.5 - 39.045 - (-0.436)(-37.5) - \frac{1}{2} (0.0046)(-37.5)^2}{2} = -10.5647. \]
Problem 5.

\[
P/L = -\left[ \frac{1}{2} e^{2T_0} + \Theta_0 h + rh (\Delta_0 S_0 - C_0) \right]
\]
\[
= -\left[ \frac{1}{2} (0.5)^2 (0.031) + (-0.03) \frac{2}{365} + 0.06 \frac{2}{365} ((0.46)(57) - 2.50) \right]
\]
\[
= -0.01150897.
\]

Problem 6.

We write the transactions and portfolio positions at each day.

Day 0

<table>
<thead>
<tr>
<th>Transactions</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell Call</td>
<td>+6.50</td>
</tr>
<tr>
<td>Buy Delta</td>
<td>-0.4 x 35 = -22</td>
</tr>
<tr>
<td>Borrow from treasuries</td>
<td>-15.5</td>
</tr>
<tr>
<td>Net cash flow</td>
<td>0</td>
</tr>
</tbody>
</table>

Portfolio Positions
- Short 1 call
- Long 0.4 shares of stock
- -15.5 in treasuries

Day 1. Since now we want \( \Delta = 0.6 \) shares, we buy 0.2 extra shares

<table>
<thead>
<tr>
<th>Transactions</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 0.2 shares</td>
<td>-0.2 x 60</td>
</tr>
<tr>
<td>Borrow from treasuries</td>
<td>+12</td>
</tr>
<tr>
<td>Net cash flow</td>
<td>0</td>
</tr>
</tbody>
</table>

Portfolio Positions
- Short 1 call
- Long 0.6 shares of stock
- -15.5 x e^{0.05/365} = -12 = -27.502 dollars in treasuries

Day 2. Close out all our portfolio positions

<table>
<thead>
<tr>
<th>Transactions</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy back Call</td>
<td>-10</td>
</tr>
<tr>
<td>Sell all shares</td>
<td>0.6 x 61 = 36.6</td>
</tr>
<tr>
<td>Day Room</td>
<td>-27.502 x e^{0.05/365} = -27.5054</td>
</tr>
<tr>
<td>Net Marking-in</td>
<td>-0.905</td>
</tr>
</tbody>
</table>

Summary of answers from the previous tables:
(a) Need to buy 0.2 extra shares on day 1 to maintain the delta hedge.
(b) Cumulative cost = -27.502.
(c) M-to-M P/L on day 2 = -0.905.
Another method is to compute the overnight marking-to-market profit/loss from day 0 to day 1 and from day 1 to day 0.

\[ P/L(0,1) = \text{Overnight marking-to-market P/L from day 0 to 1} \]

\[ = -(C(\text{S}_0/365) - C(\text{S}_0/365)) + D_0(S_0/365 - S_0) - (E^{0.05/365} - 1)(D_1S_1 - C(0.5S)) \]

\[ = -(9.50 - 6.50) + 0.01(40.00 - 55) - (E^{0.05/365} - 1)(0.4 \times 55 - 6.50) \]

\[ = -1.0021 \]

\[ P/L(1,2) = \text{Overnight marking-to-market P/L from day 1 to 2} \]

\[ = -(C(\text{S}_0/365) - C(\text{S}_0/365)) + D_1(S_1/365 - S_1) - (E^{0.05/365} - 1)(D_2S_2 - C(1/365, S_2/365)) \]

\[ = -(10.00 - 9.50) + 0.6(61.00 - 55) - (E^{0.05/365} - 1)(0.6 \times 60 - 9.50) \]

\[ = 0.0964 \]

Finally,

\[ P/L(0,2) = \text{Marking-to-market P/L from day 0 to day 2} \]

\[ = P/L(0,1) E^{0.05/365} + P/L(0,2) = -1.0021 E^{0.05/365} + 0.0964 \]

\[ = -0.905 \]

Remark 2

In principle, the previous method is simpler than the one based on transaction tables, but it has two drawbacks. First, it is easy to apply if we have only a short position on a stock. Second, in practice, we don't close out the portfolio at day 1 and start a new portfolio at day 1. We simply rebalance the portfolio from day 0.
Problem 7.

To Delta Hedge, the portfolio must have \( \Delta = 0 \)

To Gamma Hedge, the portfolio must have \( \Gamma = 0 \)

\[(a)\] \[1000(-.7) + X(-.1) + Y(-.6) = 0\]
\[(b)\] \[1000(-.12) + X(.02) + Y(.09) = 0\]

\[(c)\] \[-700 - .1X - .6Y = 0\]
\[(d)\] \[-600 + .1X + .2Y = 0\]

\[\text{Add (c) + (d)}\]
\[-1300 + 0 - .4Y = 0\]

\[Y = -3250\]

\[\text{Sell 3250 Puts Y}\]

\[-700 - .1X - .6(-3250) = 0\]

\[X = 12500\]

So Buy 12,500 Put X