STAT 473. Homework 3 Solutions.
Spring 2015.
Problem 1.

I. By put-call parity \( P(S, 35, T) - C(S, 35, T) = 35e^{-rT} - 35e^{-\delta T} \), and \( C(S, 35, T) \geq 0 \), so \( P(S, 35, T) \geq 35e^{-rT} - 35e^{-\delta T} \). ✓

II. By put-call parity \( P(S, 30, T) - C(S, 30, T) = 30e^{-rT} - 35e^{-\delta T} \), and \( P(S, 35, T) \geq P(S, 30, T) \). ✓

III. Since \( P(S, 35, T) - C(S, 35, T) = 35e^{-rT} - 35e^{-\delta T} \) and \( C(S, 30, T) \geq C(S, 35, T) \), the inequality should be reversed: \( P(S, 35, T) - C(S, 30, T) \leq 35e^{-rT} - 35e^{-\delta T} \). ✗

**Hint:** Alternatively, all these kind of problems can be resolved by checking whether the corresponding inequality holds at expiration or not. For instance, the inequality (ii) are expiration read s as Max\{35-S,0\}-max\{30-S,0\}>=30-S.

It is not hard to check that the above inequality holds for any value of \( S \geq 0 \). Therefore, this inequality holds at time 0.
Problem 2.


\[
\frac{C(K_2) - C(K_1)}{K_2 - K_1} < \frac{C(K_3) - C(K_2)}{K_3 - K_2}
\]

\[
\frac{9 - 22}{100 - 80} < \frac{5 - 9}{105 - 100}
\]

\[-0.75 < -0.8 \leftarrow \text{Not satisfied!}
\]

Since \[
\frac{C(K_2) - C(K_1)}{20} > \frac{C(K_3) - C(K_2)}{5}
\]

\[
\Rightarrow 5C(K_2) > 4C(K_3) + C(K_1)
\]

\[
\Rightarrow \text{Sell } \text{Profit} \quad \Rightarrow \text{Buy } \text{Loss}
\]

<table>
<thead>
<tr>
<th>Transaction</th>
<th>( t = 0 )</th>
<th>( S_T &lt; 80 )</th>
<th>( 80 \leq S_T &lt; 100 )</th>
<th>( 100 \leq S_T \leq 105 )</th>
<th>( 105 \leq S_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell 5 100-share call</td>
<td>45</td>
<td>0</td>
<td>0</td>
<td>(- (S_T - 100)5)</td>
<td>(- (S_T - 100)5)</td>
</tr>
<tr>
<td>Buy 4 105-share call</td>
<td>-20</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>((S_T - 105)4)</td>
</tr>
<tr>
<td>Buy 1 80-share call</td>
<td>-22</td>
<td>0</td>
<td>( S_T - 80 )</td>
<td>( S_T - 80 )</td>
<td>( S_T - 80 )</td>
</tr>
<tr>
<td>Net Gain/Loss</td>
<td>$3</td>
<td>0</td>
<td>( S_T - 80 &gt; 0 )</td>
<td>( S_T - 5S_T + 500 - 80 )</td>
<td>(- 5S_T + 4S_T + 80 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(- 4S_T + 420)</td>
<td>(- 80)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( = 4(105 - S_T))</td>
<td>( = 0 ).</td>
</tr>
</tbody>
</table>
Exercise 9.11

\[
\frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}
\]

\[
\frac{21 - 41}{100 - 80} \leq \frac{24.80 - 21}{105 - 100}
\]

0.85 \leq 0.76 \quad \text{Not satisfied}

\[\text{As before, } \quad 5P(K_2) > 4P(K_3) + P(K_1)\]

\[
\begin{array}{c|c|c|c|c}
\text{Transaction} & t = 0 & S_t > 105 & 100 \leq S_t \leq 105 & 80 \leq S_t \leq 100 & S_t < 80 \\
\hline
\text{Sell 5 100-strike puts} & 105 & 0 & 0 & -5(100 - S_t) & -5(100 - S_t) \\
\hline
\text{Buy 4 105-strike puts} & -87.2 & 0 & 4(105 - S_t) & 4(105 - S_t) & 4(105 - S_t) \\
\hline
\text{Buy 4 80-strike puts} & -4 & 0 & S_t & 0 & 0 & (80 - S_t) \\
\hline
\end{array}
\]
Problem 3.

(a) \( C(90) > C(95) \) \\
\[ C(90) - C(95) \leq 95 - 90 \times \]

<table>
<thead>
<tr>
<th>Transaction</th>
<th>( t = 0 )</th>
<th>( S_t \leq 90 )</th>
<th>( 90 \leq S_t \leq 95 )</th>
<th>( S_t &gt; 95 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell 90-Strike Call</td>
<td>+10</td>
<td>0</td>
<td>( -(S_t - 90) )</td>
<td>( -(S_t - 90) )</td>
</tr>
<tr>
<td>Buy 95-Strike Call</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>( S_t - 95 )</td>
</tr>
<tr>
<td>Total</td>
<td>5</td>
<td>0</td>
<td>( -5 )</td>
<td>Net profit \geq 1</td>
</tr>
</tbody>
</table>

(b) \( 0 \leq C(K, T) - C(K_2, T) \leq e^{-rT}(K_2 - K_1) \)
\[ 0 \leq 10 - 5.25 \leq e^{-0.1(2)}(95 - 90) = 4.09 \]
\[ \geq 4.75 \quad \text{Not satisfied} \]

<table>
<thead>
<tr>
<th>Transaction</th>
<th>( t = 0 )</th>
<th>( S_t &lt; 90 )</th>
<th>( 90 \leq S_t \leq 95 )</th>
<th>( S_t &gt; 95 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell 90-Strike Call</td>
<td>10</td>
<td>0</td>
<td>( -(S_t - 90) )</td>
<td>( -(S_t - 90) )</td>
</tr>
<tr>
<td>Buy 95-Strike Call</td>
<td>-5.25</td>
<td>0</td>
<td>0</td>
<td>( S_t - 95 )</td>
</tr>
<tr>
<td>Buy 4.09 dollars in bonds</td>
<td>4.09</td>
<td>[ 4.09 \times \frac{0.1}{2} = 0.2 ]</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Total</td>
<td>0.66</td>
<td>5</td>
<td>( -(S_t - 90) + 5 )</td>
<td>0</td>
</tr>
</tbody>
</table>

(c) \( \frac{C(K_2) - C(K_1)}{K_2 - K_1} \leq \frac{C(K_2) - C(K_1)}{K_2 - K_1} \)
\[ \frac{10 - 15}{100 - 90} \quad \frac{6 - 10}{105 - 100} \]
\[ \leq -0.5 \quad \leq -0.8 \quad \text{Not true} \]

Solve similar to exercise 9.11.
Problem 4.

(a) \( K_1 = 40 < K_2 = 45 < K_3 = 50 \)

Directionality:
\[ C(K_1,T) > C(K_2,T) > C(K_3,T) \]
\[ 8 > C(K_2,T) > 6 \]

Maximum Difference:
\[ \begin{cases} C(K_2,T) - C(K_3,T) < K_2 - K_1 \\ C(K_1,T) - C(K_2,T) < K_2 - K_1 \end{cases} \]
\[ \Rightarrow \begin{cases} C(K_2,T) < K_3 - K_2 + C(K_3,T) = 11 \\ C(K_2,T) > C(K_1,T) - K_2 + K_1 = 3 \end{cases} \]

Convexity:
\[ \frac{C(K_2,T) - C(K_1,T)}{K_2 - K_1} \leq \frac{C(K_3,T) - C(K_2,T)}{K_3 - K_2} \]
\[ \frac{C(K_2,T) - 8}{5} \leq \frac{6 - C(K_2,T)}{5} \]
\[ \Rightarrow C(K_2,T) \leq 7 \]

Overall, the \( C(K_2,T) \in [6, 7] \)
(b) \( K_1 = 40 < K_2 = 50 < K_3 = 55 \)

Convexity fails.

\[
\frac{C(K_2, T) - C(K_1, T)}{K_2 - K_1} \leq \frac{C(K_3, T) - C(K_2, T)}{K_3 - K_2}
\]

\[
LHS = \frac{6 - 8}{50 - 40} = -0.2 \text{ (Overpriced)}
\]

\[
RHS = \frac{4 - 6}{55 - 50} = -0.4 \text{ (Underpriced)}
\]

Since

\[
\frac{C(K_2, T) - C(K_1, T)}{K_2 - K_1} > \frac{C(K_3, T) - C(K_2, T)}{K_3 - K_2}
\]

\[
\frac{C(K_2, T) - C(K_1, T)}{50 - 40} > \frac{C(K_3, T) - C(K_2, T)}{55 - 50}
\]

\[
C(K_2, T) - C(K_1, T) > 2 \left[ C(K_3, T) - C(K_2, T) \right]
\]

\[
3C(K_1, T) > C(K_1, T) + 2C(K_3, T)
\]

Sell High & Buy Low

<table>
<thead>
<tr>
<th>( CF_0 )</th>
<th>IF ( S_T \leq 40 )</th>
<th>IF ( 40 &lt; S_T \leq 50 )</th>
<th>IF ( 50 &lt; S_T \leq 55 )</th>
<th>IF ( S_T &gt; 55 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-8)</td>
<td>( S_T = 40 )</td>
<td>( S_T = 40 )</td>
<td>( S_T = 40 )</td>
<td>( S_T = 40 )</td>
</tr>
<tr>
<td>( +3 \times 6 = +18 )</td>
<td>( -3 \times (S_T - 50) )</td>
<td>( -3 \times (S_T - 50) )</td>
<td>( -3 \times (S_T - 50) )</td>
<td>( -3 \times (S_T - 50) )</td>
</tr>
<tr>
<td>(-2 \times 4 = -8)</td>
<td>( +2e^{rT} )</td>
<td>( +2e^{rT} )</td>
<td>( +2e^{rT} )</td>
<td>( +2e^{rT} )</td>
</tr>
<tr>
<td>( +2e^{rT} )</td>
<td>( S_T - 40 + 2e^{rT} )</td>
<td>( 110 - 2S_T + 2e^{rT} )</td>
<td>( 110 - 2S_T + 2e^{rT} )</td>
<td>( 110 - 2S_T + 2e^{rT} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Transaction</th>
<th>@ t=0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long one 40-K Call</td>
<td>Not Exercised</td>
</tr>
<tr>
<td>Short three 50-K Call</td>
<td>Not Exercised</td>
</tr>
<tr>
<td>Long two 55-K Call</td>
<td>Not Exercised</td>
</tr>
</tbody>
</table>
### Problem 5.

The only times at which it may be rational to exercise a call option is right before the dividends are paid. In this case, this option pays only one divided on day 91. Furthermore, on this day, the option may be rational to exercise if the $PV(Div)$ exceeds $K(1-\exp(-rT))$. Thus, we need

$$PV_{t,T}(Div) > (1 - e^{-r(T-t)})$$

Upon substitution,

$$D > 100 \left(1 - e^{-0.04 \left(\frac{91}{365}\right)}\right) = 0.9923.$$
Problem 6.

Exercise 9.15

The idea is to sell the 1-year call & buy the 1.5 year call.
The net initial profit is $11.924 - $13.50 = 0.424.

At t=1 year, there are two possibilities: $S_t \leq 100 e^{0.05}$ or $S_t > 100 e^{0.05}$. In the first case, this option does not exercise & we don't worry about it. In the second case, the option is exercised.

We do the following: We short sell one-share of the stock (see last paragraph in page 12, McDonald) & get $100 e^{-0.05}$ that we put in bonds. At t=1.5 year, our position is

$$-S_{1.5} + 100 e^{0.05} e^{-0.05} (t) = -S_{1.5} + 100 e^{1.5(0.05)}$$

which is always less than the payoff of the other call:

$$\max\{ -S_{1.5} - 100 e^{1.5(0.05)}, 0 \}.$$