Problem 1 (9.4 from McDonald)

\[ X_{\theta 1} = 0.95 \quad r_{\theta} = 0.04 \quad r_{\phi} = 0.06 \quad K = 0.93 \quad T = 1 \]

\[ C^\phi_f(0.93, 1) = 0.0571 \]

\[ C^\phi_f(0.93, 1) - P^\phi_f(0.93, 1) = X_{\theta 1} (0) e^{-K_\theta T} - K e^{-K_\phi T} \]

\[ \Rightarrow 0.0571 - P^\phi_f(0.93, 1) = 0.95 e^{-0.04} - 0.93 e^{-0.06} \Rightarrow P^\phi_f(0.93, 1) = 0.0202 \]

Problem 2.

\[ r_{\phi} = 0.05 \quad r_{\theta} = 0.01 \quad X_0 = 0.00941 \]

\[ C^\phi_f(0.009, 1) = 0.0006 = X_{\theta 1} \cdot 0.009 \cdot P^\phi_f\left(\frac{1}{0.009}, 1\right) \]

\[ \Rightarrow 0.0006 = 0.009 \cdot 0.009 \cdot P^\phi_f\left(\frac{1}{0.009}, 1\right) \Rightarrow P^\phi_f\left(\frac{1}{0.009}, 1\right) = \frac{0.0006}{0.00081} = 7.407 \]

\[ C^\phi_f\left(\frac{1}{0.009}, 1\right) = 7.407 = \frac{1}{0.09} e^{-0.05} - \frac{1}{0.09} e^{-0.01} \]

\[ \Rightarrow C^\phi_f\left(\frac{1}{0.09}, 1\right) = \frac{3.09403}{0.09} \]

Problem 3

Given: \[ X_0 = 110 \% \quad r_{\phi} = 2\% \quad r_{\theta} = 4\% \]

\[ C^\phi_f(K, 1) = 3 \quad P^\phi_f(K, 1) = 2 \]

\[ (a) \quad C^\phi_f(K, 1) - P^\phi_f(K, 1) = X_0 e^{-K_\theta T} - K e^{-K_\phi T} \]

\[ \Rightarrow 3 - 2 = 110 e^{-0.04} - K e^{-0.02} \Rightarrow K = 106.80165 \% \]

\[ (b) \quad C^\phi_f(K, 1) = X_{\theta 1} (0) \cdot K \cdot P^\phi_f(K, 1) \]

\[ \Rightarrow P^\phi_f(K, 1) = \frac{3}{110 \cdot (106.80165)} = 0.002555395 \% \]
Problem 4

\[ \text{Given } S_r(0) = 100 \quad \text{(No Div)} \quad S_t(0) = 100 \]
\[ C(S_r, S_t) = 15 \quad \tau = 5\% \quad P(S_r, S_t) = ? \]
\[ C(S_r, S_t) - P(S_r, S_t) = S_r(0) - [S_t(0) - PV(DIV)] \]
\[ \Rightarrow 15 - P(S_r, S_t) = 100 - [100 - (2e^{-0.06 \cdot 0.5} + 2e^{-0.06 \cdot 0.25} + 2e^{-0.06 \cdot 0.125} + 2e^{-0.06 \cdot 0.0625})] \]
\[ \Rightarrow P(S_r, S_t) = 7.213 \]

Problem 5.

\[ \text{Given } S_r(0) = 180 \quad S_t(0) = 90 \quad \tau = 0.06 \]
\[ \text{DIV for } R: \quad \begin{array}{c|cccc}
0 & 2 & 5 & 8 & 11 \\
\hline
3 & 6 & 9 & 12 & \\
\end{array} \]
\[ \text{DIV for } T: \quad \begin{array}{c|cccc}
0 & 4 & 7 & 10 & 13 \\
\hline
6 & 9 & 12 & 15 & \\
\end{array} \]
\[ C(x_{ST}, S_r, x) = 4.6 \quad C(S_r, x_{ST}, x) = 7.04 = P(x_{ST}, S_r, x) \]
\[ \Rightarrow C(x_{ST}, S_r, x) - P(x_{ST}, S_r, x) = x(S_t(0) - PV(DIV_T)) - [S_r(0) - PV(DIV_R)] \]
\[ PV(DIV_T) = 1 \left( e^{-0.06 \cdot 0.75} + e^{-0.06 \cdot 0.45} \right) = 1.97521 \]
\[ PV(DIV_R) = 2 \left( e^{-0.06 \cdot 1.5} + e^{-0.06 \cdot 0.9} \right) = 5.91084 \]
\[ \Rightarrow 4.6 - 7.04 = x(90 - 1.97521) - (180 - 5.91084) \]
\[ \Rightarrow x = 1.95 \]
\[ \Rightarrow P(S_r, x_{ST}, x) = C(x_{ST}, S_r, x) = 4.6 \]
\[ \Rightarrow C(x_{ST}, S_r, x) = \frac{1}{1.95} \cdot 7.04 = 3.6102 \]
Problem 6.

Solution. We need to check the no-arbitrage properties of directionality and maximal difference.

Call Premiums: We should check the property

\[ 0 \leq C(50, T) - C(55, T) \leq 55 - 50, \]

which is clearly satisfied in this case. However, since \( C(50, T) < C(55, T) \), the directionality property fails. The inequality \( C(50, T) < C(55, T) \) also suggests that the 50-strike call is underpriced (hence, we should buy it) and the 55-strike call is overpriced (so, we should sell it). The following table illustrates the arbitrage (below, \( \tau \) represents the expiration time \( T \) if the options in the problem were European options or the exercise time of the 55-strike call if they were American style options).

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash flow at ( t = 0 )</th>
<th>Cash flow at ( \tau ) if ( S_\tau \leq 50 )</th>
<th>Cash flow at ( \tau ) if ( 50 &lt; S_\tau \leq 55 )</th>
<th>Cash flow at ( \tau ) if ( S_\tau &gt; 55 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy the 50-strike call</td>
<td>-9</td>
<td>0</td>
<td>( S_\tau - 50 )</td>
<td>( S_\tau - 50 )</td>
</tr>
<tr>
<td>Sell the 55-strike call</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>( -(S_\tau - 55) )</td>
</tr>
<tr>
<td>Buy Treasuries</td>
<td>-1</td>
<td>( e^{r\tau} )</td>
<td>( e^{r\tau} )</td>
<td>( e^{r\tau} )</td>
</tr>
<tr>
<td>Profit/Loss</td>
<td>0</td>
<td>( e^{r\tau} &gt; 0 )</td>
<td>( e^{r\tau} + (S_\tau - 50) &gt; 0 )</td>
<td>( e^{r\tau} + 5 &gt; 0 )</td>
</tr>
</tbody>
</table>

Put Premiums: Clearly,

\[ 0 \leq P(55, T) - C(50, T) \leq 55 - 50, \]

and, thus, the maximal difference property is satisfied. But, given that \( P(55, T) < P(50, T) \), the direction property fails. Moreover, the inequality \( P(55, T) < P(50, T) \) suggests that the 55-strike put is underpriced (hence, we should buy it) and the 50-strike put is overpriced (so, we should sell it). The following table illustrates the arbitrage (as above, \( \tau \) below is the expiration time \( T \) in the case of European put options or the exercise time of the 50-strike put in the case of American put options).

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash flow at ( t = 0 )</th>
<th>Cash flow at ( \tau ) if ( S_\tau &lt; 50 )</th>
<th>Cash flow at ( \tau ) if ( 50 \leq S_\tau &lt; 55 )</th>
<th>Cash flow at ( \tau ) if ( 55 \leq S_\tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy the 55-strike put</td>
<td>-6</td>
<td>( 55 - S_\tau )</td>
<td>( 55 - S_\tau )</td>
<td>0</td>
</tr>
<tr>
<td>Sell the 50-strike put</td>
<td>7</td>
<td>( -(50 - S_\tau) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Buy Treasuries</td>
<td>-1</td>
<td>( e^{r\tau} )</td>
<td>( e^{r\tau} )</td>
<td>( e^{r\tau} )</td>
</tr>
<tr>
<td>Profit/Loss</td>
<td>0</td>
<td>( e^{r\tau} + 5 &gt; 0 )</td>
<td>( e^{r\tau} + (55 - S_\tau) &gt; 0 )</td>
<td>( e^{r\tau} &gt; 0 )</td>
</tr>
</tbody>
</table>