THE RIEMANN ZETA FUNCTION AND
THE RANGE OF BROWNIAN BRIDGE

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Technical Report #00-10

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December 2000
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Abstract

In this preliminary report, we show that certain properties of the roots of the Riemann zeta function follow from a connection between the zeta function and the moments of the range of a Brownian Bridge on \([0,1]\). The Hadamard product formula drives this connection between the zeta function and the Brownian Bridge. The moment connection enables one to make certain convexity statements. We show that these convexity statements permit construction of bounds on number-theoretic sums. It also appears that two specific properties (listed as A and B in section 4) of the roots of the zeta function that are on the critical line and the Riemann hypothesis cannot simultaneously hold. At the time of this writing, we do not know about the truth of those two properties.

1. Introduction

In this preliminary note, we record certain interesting connections between the range of a Brownian Bridge on the interval \([0,1]\) and the roots of the classic Riemann zeta function. The proven results establish a class of inequalities that the roots must satisfy. In addition, from the Brownian Bridge connection, one can make certain conclusions about the validity of the Riemann hypothesis, provided the roots that are on the critical line satisfy two specific properties. At the time of this writing, we do not know if those two specific properties are true, or false, or not known to be either true or false. We are currently investigating the status of these two properties of the roots on the critical line.

In the remainder of this note, \(X(t)\) will denote the standard Brownian Bridge on \([0,1]\), and \(W\) its range, i.e., \(W = \sup(X(t)) - \inf(X(t))\). In addition, \(\varphi(t)\) will denote the
Riemann zeta function
\[ \varphi(t) = \sum_{n=1}^{\infty} \frac{1}{nt}; \]

recall that the zeta function can be defined analytically all through the complex plane, and its roots are either the trivial roots \(-2, -4, -6, \ldots\), or they lie in the critical strip \( \{s : 0 \leq Re(s) \leq 1\} \). Actually certain zero-free regions near \( \{s : Re(s) = 1\} \) (and so by symmetry, near \( \{s : Re(s) = 0\} \)) are known; the Riemann hypothesis asserts that all the roots in the critical strip are exactly on the critical line \( \{s : Re(s) = 1/2\} \). It is well known that the roots on the critical line have positive density, and also that at least the first 1.5 billion roots are indeed on the critical line. This numerical evidence is somewhat supportive of the validity of the Riemann hypothesis; but one may recall that the prime counting function \( \pi(x) \) is numerically less than the logarithmic integral function \( Li(x) \) for any \( x \) that has ever been attempted, even \( x \) of the order of \( 10^{18} \), and yet we know that \( \pi(x) - Li(x) \) switches sign infinitely often, as was proved by Littlewood.

Section 2 collects together the results for which there is a proof. Actually, the Brownian Bridge connection has the potential to imply inequalities for other number-theoretic functions; the reason is that the Riemann zeta function and its derivatives are connected to various number-theoretic functions. This is available in any standard text; e.g., see Baker, or Schoibengeler and Taschner. In this preliminary note, we will show only one such inequality, in section 3. Section 4 indicates what one can say about the Riemann hypothesis if certain specific properties of the roots on the critical line are true. Certain algebraic details have been omitted in this write-up.

2. 2.1. The Range of a Brownian Bridge

A formula for an arbitrary moment of the range of a Brownian Bridge is first provided. For this, we will use the following expression for the tail distribution function of the range. This is available in a number of places; e.g., see pp 367 in Dudley.

**Lemma 1.** Let \( W \) denote the range of a Brownian Bridge on \([0,1]\). Then for \( x > 0 \),

\[ P(W > x) = \sum_{n=1}^{\infty} (8n^2x^2 - 2)e^{-2n^2x^2} \]

(2)

The density of \( W \) is plotted in Figure 1.
Proposition 1. Let $s > 1$ be a real number. Then

$$E(W^s) = \frac{s(s - 1)\Gamma(\frac{s}{2})\varphi(s)}{2^{s/2}}$$  \hspace{1cm} (3)

Proof:

$$E(W^s) = -\int_{0}^{\infty} x^s d(1 - F(x))$$

$$= s \int_{0}^{\infty} x^{s-1}(1 - F(x))dx \quad \text{(on integration by parts;)}$$

$$\quad \text{the constant term drops out)}$$

$$= -s \int_{0}^{\infty} x^{s-1} \frac{d}{dx} \left[ 2x \sum_{n=1}^{\infty} e^{-2n^2x^2} \right] dx$$

$$= 2s(s - 1) \int_{0}^{\infty} x^{s-1} \left( \sum_{n=1}^{\infty} e^{-2n^2x^2} \right) dx \quad \text{(another integration by parts)}$$

$$= 2s(s - 1)\varphi(s) \frac{1}{2^{s/2+1}} \Gamma(\frac{s}{2}), \text{ giving (3)}.$$

2.2. The Roots of Zeta

The moment formula (3) of Proposition 1 implies that the roots of the zeta function must satisfy certain properties. The bridge between the moment formula (3) and the roots of the zeta function is the Hadamard Product formula for integral holomorphic functions of any finite order. The general theorem can be seen in many places; e.g. see appendix A5 in Patterson.

Corollary 1. Let $S$ denote the set of zeroes of the zeta function in the critical strip $\{s: 0 \leq \text{Re}(s) \leq 1\}$ that have a positive imaginary part. Then for all real $s > 1$,

$$E(W^s) = \left(\frac{\pi}{2}\right)^{s/2} \prod_{\rho \in S} \{(1 - s/\rho)(1 - s/(1 - \rho))\},$$  \hspace{1cm} (4)

where the product includes the multiplicities of the roots.

Proof: The Hadamard Product formula says that for any $s$,

$$s(s - 1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\varphi(s) = \prod_{\rho \in S} \{(1 - s/\rho)(1 - s/(1 - \rho))\};$$  \hspace{1cm} (5)

see, e.g., Patterson.
(4) follows on combining (5) with (3).

**Proposition 2.** For any nonnegative integer \( k \), define the sequence

\[
\lambda_k = \prod_{\rho \in S} \{(1 - k/\rho)(1 - k/(1 - \rho))\}
\]  

(6)

Let \( n \) be any positive integer. Then the determinants

\[
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_n \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_n & \lambda_{n+1} & \cdots & \lambda_{2n}
\end{vmatrix},
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{n+1} \\
\lambda_2 & \lambda_3 & \cdots & \lambda_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n+1} & \lambda_{n+2} & \cdots & \lambda_{2n+1}
\end{vmatrix}
\]  

(7)

are nonnegative.

**Proof:** Equation (4) implies that the sequence \( \{\lambda_k\} \) in (6) is the moment sequence of a nonnegative random variable. So one can make the general statement that any property of the moment sequence of a nonnegative random variable is inherited by the sequence \( \{\lambda_k\} \).

From the general theory of moments, one knows that a sequence \( \{\lambda_k\} \) corresponds to the moment sequence of a nonnegative random variable if and only if the two determinants in (7) are nonnegative; see, e.g., pp. 107, 173, and 199 in Karlin and Studden, and pp. 5, 6, and 77 in Shohat and Tamarkin. So the Proposition follows.

**Corollary 2.** Suppose the Riemann hypothesis is true. Let \( \rho_i = 1/2 + t_i \), \( t_i > 0 \), denote the roots of the zeta function with positive imaginary part. Then the determinants in (7) are nonnegative, with the adjusted expression for \( \lambda_k \) given as

\[
\lambda_k = \prod \{1 + k(k - 1)/(1/4 + t_i^2)\},
\]  

(8)

with the multiplicities duly counted.

**Proof:** If the Riemann hypothesis holds, the roots \( \rho \) in (6) satisfy \( 1 - \rho = \bar{\rho} \), the complex conjugate of \( \rho \). Simple algebra then reduces expression (6) to expression (8), and the corollary follows.

**Remark.** Obviously, a way to think about Corollary 2 is that if for any \( n \), either of the two determinants in (7) referred to in Corollary 2 becomes negative, then the Riemann hypothesis would be false.
Corollary 3. Suppose the Riemann hypothesis is true. Then, with the notation as in Corollary 2, the real function
\[ f(s) = \sum_{i=1}^{\infty} \log(1 + s(s - 1)/(1/4 + t_i^2)), \quad s > 1, \] (9)
is convex.

Proof: Equation (4) implies that \( f(s) \) defined in (9) is the logarithm of \( E(W^s) \) under the Riemann hypothesis. It is known that for any nonnegative random variable, say \( Y \), \( \log(E(Y^s)) \) is a convex function; this is an easy consequence of Holder's inequality. Therefore, Corollary 3 follows.

Remark. The assertion of Corollary 3 will be discussed in a bit more detail in Section 4.

3. Number-Theoretic Sums

The moment formula connecting the zeta function to \( W \), the range of the Brownian Bridge, and the convexity result of Corollary 3 lead to inequalities on various number theoretic sums. This is because a variety of number theoretic sums admit representations in terms of the zeta function. In this preliminary report, we record just one illustration.

Proposition 3. Let \( \Lambda(n) \) denote the von Mangoldt function which takes the value \( \log p \) if \( n \) is a power of the prime \( p \), and is otherwise 0. Then, for \( s > 1 \),
\[ \sum \Lambda(n) \log n/n^s > 1/s^2 + 1/(s - 1)^2 - 1/4 \psi_2(s/2), \] (10)
where \( \psi \) denotes the digamma function.

Proof: From the moment formula (3), we get
\[ -\log \varphi(s) = -s/2 \log 2 + \log s + \log(s - 1) + \log \Gamma(s/2) - \log \mu(s), \] (11)
where \( \mu(s) \) is \( E(W^s) \).

Differentiating both sides of (11) twice with respect to \( s \), and using \( d^2/ds^2 \log \mu(s) \geq 0 \) due to the aforementioned convexity of \( \log \mu(s) \), one will get
\[ d/ds(-\varphi'(s)/\varphi(s)) \leq -1/s^2 - 1/(s - 1)^2 + 1/4 \psi_2(s/2). \] (12)
If one now uses the representation $\Sigma \Lambda(n)/n^s = -\varphi'(s)/\varphi(s)$ for $s > 1$, the inequality in (10) will follow on reversing the sign in each side of (12).

Remark. Inequality (10) can be usefully sharp. For example, if $s = 1.5$, the left side is 3.863 and the lower bound given by the right side is 3.809; if $s = 2$, they are, respectively, .884 and .839.

Similar bounds involving other number-theoretic sums will be reported in an update of this report.

4. The Riemann Hypothesis

It appears that the logarithmic convexity result in Corollary 3 enables one to say certain things about the Riemann hypothesis if one knew certain properties of the roots on the critical line to be true. These properties are:

Property A. Let $\{\rho_i\}$ denote the roots of the zeta function on the critical line $\{s : \text{Re}(s) = 1/2\}$. Then $\sum 1/|\rho_i|^2 < \infty$.

Property B. $\lim_{s \to \infty} \sum (2s - 1)/(s(s - 1) + |\rho_i|^2)$ exists and equals zero.

Remark. As was said before, at the time of this writing we do not know about the truth or otherwise of Property A or Property B. Property B is basically an interchange of the limit and the sum. The truth of each property should depend on the growth of the roots on the critical line. We are enquiring about both properties at this time.

Corollary 3 says that if the Riemann hypothesis is true, then $f(s)$ as it was defined in (9) would be convex. The derivative of the individual logarithmic terms within the sum is $(2s - 1)/(s(s - 1) + |\rho|^2)$. So if the derivative of $f$ at a fixed $s > 1$ could be evaluated by term by term differentiation, then $f'(s)$ would equal

$$f'(s) = \sum (2s - 1)/(s(s - 1) + |\rho_i|^2)$$

(13)

The interchange of the order of the derivative and the sum is correct if Property A holds, as can be seen from the dominated convergence theorem. Now, convexity of $f(s)$ tells us that $\lim \inf_{s \downarrow 1} f'(s)$ has to be less than or equal to $\lim \sup_{s \to \infty} f'(s)$. If Property B were
to hold, then \( \limsup_{s \to \infty} f'(s) \) would be zero, while by Fatou's lemma, \( \liminf_{s \to 1} f'(s) \) would be greater than or equal to \( \sum 1/|\rho_i|^2 > 0 \), which would contradict the previous sentence. Therefore, it appears that if, per chance, Property A and B were both true, then the Riemann hypothesis would be false.

**Acknowledgement.** We are glad to thank Herman Rubin and Burgess Davis for helpful conversations over an extended period of time.

**References**


Fig. 1: Density of Range of Brownian Bridge