Note: 1) Please show your work legibly, clearly, and with rigorous proofs to be eligible for credit.
2) The test is open notes, but closed book.
3) You can get a maximum of 5 free bonus points. Answer as much as you can.
Good luck.

mean = \frac{177.3}{3}
median = 49
SD = 8.5
1. Suppose $X \sim Bin(n, p_1)$ and $Y \sim Bin(n, p_2)$, where $X, Y$ are independent, and it is known that $p_1 \leq p_2$. Give an exact analytical formula for the MLE of $(p_1, p_2)$. No proof is required in this problem.

6 points

The unconstrained MLEs of $p_1, p_2$ are

$$\hat{p}_1 = \frac{X}{n} \quad \text{if} \quad X > 0 \quad \text{and} \quad \hat{p}_2 = \frac{Y}{n} \quad \text{if} \quad Y > 0.$$ 

If $(H) = \{ (\hat{p}_1, \hat{p}_2) : 0 < \hat{p}_1 < 1, \hat{p}_2 < 1 \}$, then

$$\hat{p}_1 \leq \hat{p}_2 \} \cap \{ (\hat{p}_1, \hat{p}_2) : 0 < \hat{p}_1 \leq 1 \} \cap \{ (\hat{p}_1, \hat{p}_2) : 0 < \hat{p}_2 \leq 1 \}.$$

Then $(\hat{p}_1, \hat{p}_2) = (\frac{X}{n} \quad \text{if} \quad 0 < X \leq Y).$

If $X + Y > 0$ and $X > Y$, then

the MLE of $(\hat{p}_1, \hat{p}_2)$ is attained

on the boundary of $(H)$, and is

$$\hat{p}_1 = \frac{X + Y}{2n} \quad \text{and} \quad \hat{p}_2 = \frac{X + Y}{2n}.$$ 

There are some $(X, Y)$ configurations

for which an MLE does not exist.
2. Suppose we have iid observations $X_1, X_2, \ldots$ from the $U[\theta, 3\theta]$ distribution.

Provide, with rigorous proofs, four different consistent estimates of $\theta$.

12 points

1. \[ \frac{X(n)}{3} \xrightarrow{\text{p}} \theta \quad \text{such that} \quad P \left( \frac{1}{X(n)} - \theta \right) > \epsilon \right) = P \left( X(n) < 3\theta - 3\epsilon \right) \]

\[ = \left( \frac{3\theta - 3\epsilon - \theta}{3\theta - \theta} \right)^n \xrightarrow{n \to \infty} 0 \quad \text{as} \quad n \to \infty \quad \neq \theta, \]

since $1 - \frac{3\epsilon}{2\theta} < 1$ for small enough $\epsilon$.

2. Similarly, $X(1) \overset{\text{p}}{\to} \theta$.

3. By the WLLN $\bar{X} \overset{\text{p}}{\to} 2\theta \Rightarrow \frac{\bar{X}}{2} \overset{\text{p}}{\to} \theta$.

4. Take any convex combination of

1, 2, and 3.
3. Suppose $X_1, \ldots, X_n \simiid f(x | \alpha) = \frac{e^{-x} \alpha^{-1}}{\Gamma(\alpha)}, x \geq 0$, and $\alpha > 0$ an unknown parameter.

Find, with complete proof, the UMVUE of $\alpha$.

12 points

$$f(x | \alpha) = \frac{e^{-x} \alpha^{-1}}{\Gamma(\alpha)}, \quad x \geq 0.$$ 

$$\mathcal{I} = \{ \alpha : \alpha > 0 \}.$$ 

By theorem proved in class, $\sum \log X_i$

$$= \log (\prod X_i)$$

is complete and sufficient.

$$\Rightarrow \prod X_i$$

is complete and sufficient.

Also, $E_{\alpha}(X_i) = x \quad \forall x > 0$.

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by the Lehmann–Scheffe theorem.

the ! UMVUE of $\alpha$ is

$$\hat{\alpha} = E_{\alpha \leftarrow 1}(X_i | \prod X_i)$$

is standard exponential.
4. It is desired to estimate \( \mu^2 \) from iid observations \( X_1, \ldots, X_n \sim \text{iid } N(\mu, 1) \).

a) Derive the CRLB on the variance of an unbiased estimate of \( \mu^2 \).

b) Prove that the UMVUE of \( \mu^2 \) does not attain the CRLB of part a).

\[ 6+10 = 16 \text{ points} \]

(a) Using notation used in class,

the CRLB is

\[
\frac{\left[ \hat{y}'(\mu) \right]^2}{\eta I(\mu)} = \left[ \frac{d}{d\mu} \mu^2 \right]^2
\]

\[
= \frac{4\mu^2}{n}
\]

(b) By the Lehmann–Scheffe theorem,

the UMVUE of \( \mu^2 \) is \( \left( \bar{X} \right)^2 - \frac{1}{n} \).

\[
\operatorname{Var}_\mu \left( \left( \bar{X} \right)^2 - \frac{1}{n} \right) = \operatorname{Var}_\mu \left( \bar{X}^2 \right)
\]

\[
= \mathbb{E}_\mu \left( \bar{X}^4 \right) - \left[ \mathbb{E}_\mu \left( \bar{X}^2 \right) \right]^2
\]

\[
= \mathbb{E}_\mu \left( \bar{X} - \mu + \mu \right)^4 - \left[ \mathbb{E}_\mu \left( \bar{X} - \mu \right) \right]^2
\]

\[
= \frac{3}{n^2} + \frac{6\mu^2}{n} + \frac{\mu^4}{n} - \frac{\mu^2}{n} - \frac{2\mu^2}{n} - \frac{1}{n^2}
\]

\[
= \frac{4\mu^2}{n} + \frac{2}{n^2} > \frac{4\mu^2}{n} + \mu.
\]
10 points
6. Suppose $X_1, \ldots, X_n \iid \text{Geometric}(p)$ with the pmf $f(x \mid p) = p(1-p)^{x-1}, x = 1, 2, 3, \ldots$. Derive the Fisher information function and plot it accurately.

12 points

$$\log f(x \mid p) = \log p + (x-1) \log (1-p)$$

$$\frac{d}{dp} \log f = \frac{1}{p} - \frac{x-1}{1-p}$$

$$\operatorname{Var} \left(\frac{d}{dp} \log f\right) = \operatorname{Var} \left(\frac{1}{p} - \frac{x-1}{1-p}\right)$$

$$= \frac{\operatorname{Var}(X)}{1-p} = \frac{\frac{1-p}{p^2}}{(1-p)^2}$$

$$= \frac{1}{\frac{1}{p^2} + (1-p)} \quad 0 \leq p < 1.$$
7. Write the full legal name of the instructor of this class.

2 points

Huh?
Let me check.

I will get back on this one.