1. Consider the model $z_t = (1 - .5B)(1 - B^{12})a_t$ with $\sigma_a^2 = 1$.

(a) Calculate $\gamma_k$, for all $k$.

(b) Calculate $\hat{z}_t(10)$, $\hat{z}_t(20)$, and the variances of the corresponding forecasting errors.

Solution:

(a) As $z_t = a_t - .5a_{t-1} - a_{t-12} + .5a_{t-13}$, one has
\begin{align*}
\gamma_0 &= 1 + (-.5)^2 + (-1)^2 + (.5)^2 = 2.5, \\
\gamma_1 &= 1(-.5) + (.5)(-1) = -1, \\
\gamma_{11} &= (-1)(-.5) = .5, \\
\gamma_{12} &= 1(-1) + (-.5)(.5) = -1.25, \\
\gamma_{13} &= 1(.5) = .5, \\
\gamma_{-k} &= \gamma_k, \quad k \neq 0, \pm 1, \pm 11, \pm 12, \pm 13.
\end{align*}

(b) $\hat{z}_t(10) = -a_{t-2} + .5a_{t-3}$ with error variance $V_{10} = 1 + (-.5)^2 = 1.25$. $\hat{z}_t(20) = 0$ with error variance $V_{20} = 1 + (.5)^2 + (-1)^2 + (.5)^2 = 2.5 = \gamma_0$.

2. Consider the model $(1 + .7B)(1 - B)z_t = (1 - .3B + .7B^2)a_t$ with $\sigma_a^2 = .36$.

(a) Given $z_{90} = 9$, $z_{89} = 7$, $a_{90} = .5$, $a_{89} = 1$, obtain $\hat{z}_{90}(l)$, $l = 1, 2, 3$.

(b) Find the $\psi$ weights $\psi_j$, $j = 1, 2, 3$. State the recursive formula one may use to calculate $\psi_j$ for $j > 3$.

(c) Calculate the variances of the forecasting errors $e_{90}(l) = z_{90+l} - \hat{z}_{90}(l)$, $l = 1, 2, 3$, and construct 95% prediction intervals for $z_{91}$, $z_{92}$, $z_{93}$.

(d) State the difference equation satisfied by the “eventual” forecasting function, and give an expression of the general form of the forecasting function $\hat{z}_t(l)$ with all deterministic constants specified.

Solution: $\varphi_1 = .3$, $\varphi_2 = .7$, $\theta_1 = .3$, $\theta_2 = -.7$.

(a) Using the difference equation, one has
\begin{align*}
\hat{z}_{90}(1) &= 0.3z_{90} + .7z_{89} - .3a_{90} + .7a_{89} = 8.15 \\
\hat{z}_{85}(2) &= 0.3\hat{z}_{90}(1) + .7z_{90} = 9.095 \\
\hat{z}_{85}(3) &= 0.3\hat{z}_{85}(2) + .7\hat{z}_{90}(1) = 8.4335
\end{align*}

(b) $\psi_j = \varphi_1 \psi_{j-1} + \varphi_2 \psi_{j-2}$, $j > 1$.
\begin{align*}
\psi_1 &= \varphi_1 \psi_0 - \theta_1 = 0 \\
\psi_2 &= \varphi_1 \psi_1 + \varphi_2 \psi_0 - \theta_2 = 1.4 \\
\psi_3 &= \varphi_1 \psi_2 + \varphi_2 \psi_1 = .42 \\
\psi_k &= \varphi_1 \psi_{k-1} + \varphi_2 \psi_{k-2}, \quad k \geq 2
\end{align*}
3. Let $x_t = 3 + 2t + u_t$, where $u_t$ satisfies $u_t = 0.5u_{t-1} + a_t$ with $a_t$ a white noise, $E[a_t^2] = \sigma_a^2$. (12 pts.)

(a) Show that $z_t = x_t - x_{t-1}$ is a stationary ARMA process (with $\mu_z \neq 0$), and specify its orders.

(b) Give the power spectra of $u_t$ and $z_t$ as real functions.

(c) Calculate $\psi_j$ for $\tilde{z}_t = z_t - \mu_z$, $j = 0, 1, 2$, and provide an explicit (non-recursive) expression for $j > 2$.

Solution:

(a) $z_t = 2 + (1 - B) u_t$, $(1 - 0.5B)(z_t - 2) = (1 - B)a_t$, so $z_t$ is an ARMA(1,1) process with $\mu_z = 2$.

(b) For $u_t$,

$$p_n(\omega) = \frac{\sigma_a^2}{|1 - 0.5e^{-i2\pi\omega}|^2} = \frac{\sigma_a^2}{1.25 - \cos 2\pi\omega}.$$ 

For $z_t$,

$$p_z(\omega) = |1 - e^{-i2\pi\omega}|^2 p_n(\omega) = \frac{2\sigma_a^2(1 - \cos 2\pi\omega)}{1.25 - \cos 2\pi\omega}.$$ 

(c) $\psi_0 = 1$, $\psi_1 = 0.5 - 1 = -0.5$. For $j > 1$, $\phi_j = 0.5\phi_{j-1} = -(0.5)^j$.

4. Consider an AR(1) model $z_t = \phi z_{t-1} + a_t$, where $|\phi| < 1$ and $a_t$ is a white noise with $E[a_t^2] = \sigma_a^2$. (6 pts.)

(a) Give an explicit expression of the conditional LS estimate $\hat{\phi}$ of $\phi$ that minimizes $\sum_{t=2}^{n}(z_t - \phi z_{t-1})^2$.

(b) Define $U = (\sum_{t=2}^{n} z_t^2) - (\hat{\phi} - \phi)$. Calculate $E[U]$ and $E[U^2]$. [Hint: $E[z_{t-1}a_t] = 0$.]

Solution:

(a) $\hat{\phi} = \sum_{t=2}^{n} z_{t-1}z_t / \sum_{t=2}^{n} z_{t-1}^2$.

(b) Since $\hat{\phi} = \sum_{t=2}^{n} z_{t-1}(\phi z_{t-1} + a_t) / \sum_{t=2}^{n} z_{t-1}^2 = \phi + \sum_{t=2}^{n} z_{t-1}a_t / \sum_{t=2}^{n} z_{t-1}^2$, $U = \sum_{t=2}^{n} z_{t-1}a_t$. 

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Using conditioning arguments, one has

\[ E[U] = \sum_{t=2}^{n} E[z_{t-1}a_t] = 0, \]

\[ E[U^2] = \sum_{t=2}^{n} \sum_{s=2}^{n} E[z_{t-1}z_{s-1}a_ta_s] = \sum_{t=2}^{n} E[z_{t-1}^2a_t^2] = (n-1)\sigma_a^4 E[z^2] = \frac{(n-1)\sigma_a^4}{1 - \phi^2}. \]