Stat 225 - Fall 2011 Exam 2 Solutions

Your Name: _________________________________________________________

Your Section (Circle One):
Juan (7:30)           Mike C. (8:30)           Mike C. (9:30)           Yen-Ning (10:30)
Yen-Ning (11:30)      Chris (12:30)         Chris (1:30)           Jeremy (2:30)
                      Jeremy (3:30)         Mike L. (4:30)

Instructions:
- Show your work on ALL questions. Unsupported work will NOT receive full credit.
- Decimal answers should be exact, or to exactly 4 decimal places. (Examples: if it is .25 use .25, if it is say .57891234 then use .5789.)
- You are responsible for upholding the Honor Code of Purdue University. This includes protecting your work from other students.
- Please write legibly. If a grader cannot read your writing, NO credit will be given.
- You are allowed the following aids: 2 one-page 8.5" x 11" handwritten (in your handwriting) cheat sheets, a scientific calculator, and pencils or pens.
- Instructors will not interpret questions for you. If you do have questions, wait until you have looked over the whole exam so that you can ask all of your questions at one time.
- You must show your student ID (upon request), turn in your cheat sheet and sign the class roster when you turn in your exam to your instructor.
- Turn off your cell phone before the exam begins.

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1. Starsuck's makes and dispenses their drinks at a rate of 15 per hour. You order a drink from Starsuck's. Let W be the amount of time you wait for your order. (20 points)

a) Find the probability that Starsuck's dispenses 28 drinks in the first 2 hours and 17 drinks in the third hour. (5 points)

\[ X \sim \text{Poi}(\lambda = 15 / \text{hour}) \quad Y \sim \text{Poi}(\lambda = 30 / 2 \text{ hours}) \]

\[ P(Y = 28 \text{ and } X = 17) = P(Y=28)*P(X = 17) \text{ since non-overlapping intervals are independent in the Poisson Process.} = \exp(-30)*30^{28}/(28!) \times \exp(-15)*15^{17}/(17!) = .0060. \]

b) Name the distribution of W and the value of its parameter(s). (2 points) Find the probability that W is less than 7 minutes. (2 points)

\[ W \sim \text{Expo}(\lambda = 15 / \text{hour or } \mu = 1/15 \text{ hours}) \]

\[ P(W < 7 \text{ minutes}) = P(W < 7/60 \text{ hours}) = 1 - \exp(-15*(7/60)) = .8262. \]

c) Find the probability that W is more than 15 minutes given that it is more than 5 minutes? (4 points)

\[ P(W > 15 \mid W > 5) = P(W > 10 \text{ minutes}) \] by the memoryless property

\[ P(W > 10/60 \text{ hours}) = \exp(-15*(10/60)) = .0821. \]

d) Find E[W] in minutes. (2 points)

\[ E[W] = 1/15 \text{ hours} = 60 / 15 \text{ minutes} = 4 \text{ minutes} \]

e) Suppose that Starsuck's dispensed 20 drinks in the first hour. What is the probability that 8 of those drinks were dispensed in the first 20 minutes? (5 points)

The 20 drinks are Uniformly distributed over that hour. So probability a given drink is in the first 20 minutes is 20/60 = 1/3. Let X be the # of drinks in the first 20 minutes knowing 20 occurred in the 1st hour. \( X \sim \text{Bin}(n=20, p = 1/3) \).

\[ P(X =8) = "20 \text{ choose } 8" \times (1/3)^8 \times (2/3)^{12} = .1480. \]
2. Let $X$ be a continuous random variable with the following pdf (16 points):

$$f_X(x) = \begin{cases} 
  c(10 - 2x) & 2 \leq x \leq 4 \\
  0 & \text{everywhere else}
\end{cases}$$

a) Find the value of $c$ that makes this a legitimate pdf. (3 points)

Leave off $c$ and integrate from 2 to 4. This integral is 8, so $c = 1/8$.

b) Find the cdf (remember to include your $c$ from part a). (4 points)

$$F_X(x) = \begin{cases} 
  0 & x < 2 \\
  -x^2 + 10x - 16 & 2 \leq x \leq 4 \\
  1 & x > 4
\end{cases}$$

$F(x)$ is the integral from 2 to $x$ of $f(x)$.

c) Find $E[X]$. (3 points)

$$E[X] = \text{integral from 2 to 4 of } x \times f(x) = 17/6 = 2.83333333333.$$ 

d) Find the value such that 37.5% of the distribution is higher than that value. (3 points)

62.5th percentile. $0.625 = F(x)$ and solve for $x$. $x = 3$.

e) Find the probability that $X$ is at least 2.5. (3 points)

$$P(X \geq 2.5) = 1 - F(2.5) = 1 - 0.34375 = 0.65625.$$
3. Mr. DeFries' golf scores per 9 holes are Normally distributed with a mean of 50 strokes and a variance of 25 strokes. For this entire problem, use the Empirical Rules (aka Rules of Thumb). No points will be awarded if the Empirical Rules are not followed. (9 points)

a) Find the probability Mr. DeFries' scores between 45 and 60 on his next round. (3 points)

45 = μ - 1*σ, and 60 = μ + 2*σ. 16% is less than 45 and 2.5% is more than 60. So .815 = P(45 < X < 60).

b) Find the probability Mr. DeFries' scores between 55 and 65 on his next round. (3 points)

55 = μ + 1*σ and 65 = μ + 3*σ. 16% are higher than 55 and .15% are higher than 65. So .1585 = P(55 < X < 65)

c) Find the probability Mr. DeFries' scores less than 55 on his next round. (3 points)

55 = μ + 1 *σ, again 16% are higher than this, so .84 = P(X < 55)
4. Let $X$ be Normally distributed with a mean of 100 and a variance of 36. Take 80 independent samples from $X$. Let $Y$ denote the number of these samples that have a value less than 91. (12 points)

a) Name the distribution of $Y$ and the value of its parameter(s) (4 points). Then find the probability that $Y$ is 6. (2 points)

$Y \sim \text{Bin}(n=80, p = P(X < 91))$

\[ P(X < 91) = P(Z < -1.5) = 0.0668. \]

$Y \sim \text{Bin}(n=80, p = 0.0668)$

\[ P(Y=6) = \binom{80}{6} \times 0.0668^6 \times (1-0.0668)^{74} = 0.1602. \]

b) Let $Y^*$ be the approximation to $Y$. Is $Y^*$ a valid approximation to $Y$, why or why not? Calculate $P(3 \leq Y < 9)$. If $Y^*$ is appropriate use $Y^*$ for the calculation. (6 points)

$Y^* \sim \text{N}($mean$=5.344$, standard deviation $= 2.233164)$

$Y^*$ is appropriate because $np = 5.344 > 5$ and $n(1-p) = 74.656 > 5$.

\[ P(3 \leq Y < 9) = P(2.5 < Y^* < 8.5) = P(-1.27 < Z < 1.41) = P(Z < 1.41) - P(Z < -1.27) = 0.9207 - 0.1020 = 0.8187. \]

The CC is appropriate since this is the Normal approximation to the Binomial
5. Let \( X \) be the time that Bryan arrives at a party. \( X \) is equally likely over the interval 8 PM to 10 PM. Let \( Y \) be the time that Nick arrives at a party. \( Y \) is equally likely over the interval 8:45 PM to 10:15 PM. Suppose the party starts at 9:10 PM. Assume their arrival times are independent. (14 points)

a) Find the Probability that Bryan will be late to the party. (3 points)

\[
X \sim U(0,120) \quad P(X > 70) = (120-70)/(120-0) = .416666666666667.
\]

b) Find the Probability that at least one of Bryan or Nick will be late to the Party. (5 points)

\[
1 - P( \text{neither is late}) = 1 - P(\text{Bryan is not late}) \times P(\text{Nick is not late}) \text{ since Bryan and Nick's arrival times are independent.}
\]

\[
1 - (7/12) \times (P(Y < 25)) = 1 - (7/12) \times (25/90) = .8380.
\]

Or you can use a union statement and do \( P(\text{Bryan is Late}) + P(\text{Nick is Late}) - P(\text{both are late}).\)

c) Find the standard deviation of \( X \) (in hours). (3 points)

\[
\text{Var } X = 2^2 / 12 = 1/3. \quad \text{SD } X = \sqrt{1/3} = .5774 \text{ hours.}
\]

d) Find the 40th percentile of \( Y \). (3 points)

\[
.4 = (y-0)/(90-0) \text{ so } y = 36. \quad Y \text{ is the time in minutes are 8:45 PM, so the 40th percentile is 9:21 PM.}
\]
6. Let \( X \) be a continuous random variable with mean 50 and variance 900. Let us take a sample of size 225 from the distribution of \( X \). (15 points)

a) Find the distribution of the sample mean. (3 points)

\[ \bar{X} \sim N(\text{mean} = 50, \text{standard deviation} = 2) \]

b) Find the probability that the sample mean is at least 53.64. (3 points)

\[ P(\bar{X} > 53.64) = P(Z > 1.82) = .0344. \]

c) Find the probability that the sample mean is more than 48.4 given that it is less than 50? (4 points)

\[ P(X > 48.4 \mid X < 50) = \frac{P(48.4 < X < 50)}{P(X < 50)} = \frac{.5 - P(Z < -.8)}{.5} = (.5 - .2119)/.5 = .5763. \text{ Note 50 is the mean so } P(X < 50) = .5. \]

d) Find the 7.08th percentile of the sample mean. (3 points)

7.08th percentile gives a \( z \)-value of -1.47, so \( \bar{X} \) here is 47.06.

e) If one were to repeatedly take samples of size 225 from the distribution of \( X \), how long (in number of samples), on average, would it take to get a sample mean that was larger than 53.64? (2 points)

Let \( Y \) be the number of samples until the 1st one that has a sample mean > 53.64

\[ Y \sim \text{Geo}(p = .0344 \text{ which was calculated in 6b}). \text{ EY} = 1/p = 29.0698. \]
7. Glen is waiting for the Happy Hollow (HH) bus. It on average takes 5 minutes for Glen to catch the HH bus. Let T be the amount of time Glen waits. (13 points)

a) Name the distribution of T and the value of its parameter(s). (2 points)

T ~ Expo(\(\mu = 5\) minutes, or \(\lambda = 1/5\))

b) Find the 63rd percentile of T. (2 points)

0.63 = F(x), so x = 4.971261.

c) Suppose Glen waits for the HH bus 81 times this school year. Let S be the total amount of time that Glen waits for the HH bus this school year. Name the approximate distribution of S and the value of its parameter(s). (3 points)

approximation of \(S = S^* \sim N(\text{mean} = 405, \text{standard deviation} = 45)\)

d) Find the approximate probability that S is between 369 and 432 inclusive. (4 points)

\[P(369 \leq S^* \leq 432) = P(-0.8 < Z < 0.6) = 0.7257 - 0.2119 = 0.5139.\] Note no CC is used since the original distribution is the Exponential which is continuous.

e) Find the probability that S is exactly 405. (2 points)

Read the comment for 7d, so \(P(S = 405) = 0\).