MATH FOR CREDIT
Purdue University, Feb 6\textsuperscript{th}, 2004

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Outline

- The space of credit products
- Key drivers of value
- Mathematical models
- Pricing
- Trading strategies
- Research areas
The space of credit products

• The basic securities: Credit default swaps (CDSs)
• The usual suspects: Corporate bonds
• Some unusual products: Syndicated loans
• A lot of product innovation!
  – Basket default swaps
  – Synthetic Collateralized Debt Obligations (CDOs)
  – CDS spread options
  – Index CDS products
  – < More coming every day >
Key drivers of value

- Three main quantities
  1. Interest rate dynamics
  2. Default time(s) distribution
  3. Loss or Recovery given default process

- Correlation between all above
Mathematical models

- Default probability models
  - Reduced form
  - Structural
- Recovery
  - Deterministic
  - Stochastic with or without correlation to default probabilities
- Correlation
  - Factor models
  - Copula models
- The simplest models are often the most used
Pricing credit default swaps

Notional: 100MM
Maturity: 3 years
Reference Entity: GM

How much should “S” be?
CDS cash flows

Fixed Leg for unit notional

Premium Payments Until Default Occurs

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Default Leg

\[(1-R)\]

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CDS pricing equations

- Under certain risk neutral measure, the expected present value of fixed leg payments equals the expected present value of a possible default event payment.

- Both sides of the equation can be calculated as follows:

\[
PV_{Fixed\_Leg} = \sum_{t < t_i < T} c \times df(t_i) \times sp(t_i)
\]

\[
PV_{Floating\_Leg} = -\int_{t}^{T}(1 - R) \times df(u) \times dp(u)du
\]

\[
PV_{CDS} = PV_{Fixed\_Leg} + PV_{Floating\_Leg}
\]

- No simulation is required for a large class of models.

\[\begin{array}{ll}
 s & \text{Spread} \\
 df & \text{Discount factor} \\
 sp & \text{Survival probability} \\
 dp & \text{Default probability} \\
 R & \text{Recovery} \\
 t_i & i^{th} \text{ cash flow date} \\
 T & \text{Maturity}
\end{array}\]
Credit curve calibration remarks

Example (Qualitative Picture):

1. **Curve Mark**
   - 100bp
   - 200bp
   - 150bp

2. **Survival Prob.**
   - Survival Prob:
     - 0
     - 0.002
     - 0.004
     - 0.006
     - 0.008
     - 0.01
     - 0.012
     - 0.014
     - 0.016
     - 0.018
     - 0.02

3. **Hazard Rate**
   - Hazard Rate:
     - 0.0179
     - 0.0037

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Pricing corporate bonds

- The same mechanism for pricing CDS can be used for pricing bonds that have default risk
- Coupons and Notional payments are treated like CDS premium cash flows
- Receive default payout of recovery + $R$ if default (if long bond)
Bond cash flows

\[ PV_{Fixed\_Leg} = \sum_{t < t_i \leq T} c \times df(t_i) \times sp(t_i) + df(T) \times sp(T) \]

\[ PV_{Floating\_Leg} = \int_{t}^{T} R \times df(u) \times dp(u) du \]

\[ PV_{BOND} = PV_{Fixed\_Leg} + PV_{Floating\_Leg} \]
Example

Bond Price / PV$_{01}$ vs. Curve Mark (up to 10000bp), $R = 25\%$

Price

 PV$_{01}$

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Pricing syndicated loans

• Funded
  - Receive coupon and principal until default / maturity
  - Coupon = LIBOR + Spread + Facility Fee + Utilization Fee
  - Receive recovery at default
  - Spread may be rating dependent

• Unfunded
  - Receive fee until default / maturity
  - Total fee = Commitment fee + Facility fee
  - Pay loss amount on default = draw – recovery amount

• Credit Line = Funded + Unfunded
  - Funded and unfunded amounts are time dependent
Loan risks

• Default risks
  – Default likelihood
  – Draw at default
  – Recovery at default

• Other risks
  – Rating migration
  – Lending spread and fees
  – Draw and prepayment option
  – Interest rate sensitivity
Simple loan valuation: Replicating portfolio

- **Funded**
  - Identical to bond with notional equal to funded amount plus expected draw

- **Unfunded**
  - Equivalent to default swap with default payoff adjusted for draw down
Simple loan valuation: Assumptions

- Loan recovery via loan LGD ratio
- Draw on default given maturity
- Expected draw during life

\[\begin{align*}
&\text{Determined by covenants} \\
&\text{Determined by rating}
\end{align*}\]

- Bond recovery
  - Used to calibrate CDS marks or bond yield curve
Basket default swaps: Introduction

- Provide a way to buy / sell protection on “multiple” names
- Highly illiquid markets but lucrative
- Structurally much more complex than CDS, Bonds, and Loans
- Non-linear cash flows that depend on multiple credit events
- Correlation between the credit events is key valuation driver
- A dream for mathematical / computational finance research.
Basket default swaps: Structure

- The reference portfolio is sliced into multiple “tranches”
- Three tranche structure is typical
  1. Equity
     - Junior most, Most risky for protection seller
  2. Mezzanine
  3. Senior
     - Least risky for protection seller
- Tranche cash flows are similar to credit default swaps
Basket default swaps: Tranche cash flows

- In case of a default event the protection buyer receives a payoff dependent on the cumulative loss amount and the tranche attachment points.
- Protection seller receives regular premium proportional to the amount of notional remaining in the tranche.
Basket CDS pricing

Loss process

\[ L(t) = \sum_{i=1}^{K} (1 - R_i) \times N_i \times 1\{\tau_i < t\} \]

Attachment times

\[ T_a = \inf\{t > 0 | L(t) > a\} \]
\[ T_b = \sup\{t < T | L(t) \leq b\} \]

Tranche loss process

\[ L_{ab}(t) = (L(t) - a)^+ \land (b - a) \]
Basket CDS pricing

\[ PV_{defaul} = E \left\{ \int_{T_a}^{T_b} dL_{ab}(t) \times df(t) \times dt \right\} \]

\[ PV_{premium} = E \left\{ \sum_{t < t_i < T} \rho_{ab}[(b - a) - L_{ab}(t_i)]df(t_i) \right\} \]

\[ p_{tranche} = \{ p_{ab} | PV_{premium} = PV_{defaul} \} \]
Research areas

• Fast simulation and variance reduction techniques
  – Absolutely essential for risk management of portfolio products

• Semi-analytical methods for pricing
  – Provide accurate* prices
  – Allow much better risk / sensitivity analysis

• Dynamic spread models
  – How to update spreads after defaults?

• Econometric research for better models for:
  – Recovery, Correlation, Spread volatility etc.