Last time (Session 2.0)  
Joint densities $f(x,y)$  
marginal density $f_x(x) = \int_{-\infty}^{\infty} f(x,y) dy$

This time (Session 2.1)  
Independence  
Density of a sum (convolution)  
Independent normals  
circular symmetry  
sums  
chi-square dist.

Convolution Example (p.373)  
$X, Y$ independent, densities $f_X(x), f_Y(y)$  
$Z = X + Y$ has density  

$$f_z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$$

Discrete analog:  
$$P(X+Y = z) = \sum_x P(X=x) P(Y=z-x)$$

Pitman's (p.373) proof of  
$$f_z(z) = \int_{-\infty}^{\infty} f_X(x) f_X(z-x) dx, \quad Z = X + Y$$

$$f_z(z) dz = P\{Z \leq dz\} = P\{X+Y \leq z+dz\}$$

$$= P\{(X,Y) \in \text{strip}\}$$

$$= \sum_{dx} P\{(X,Y) \in \text{strip}, X \leq dx\}$$

$$= \sum_{dx} f(x, z-x) \cdot \text{(volume of strip)}$$

$$= \sum_{dx} f(x, z-x) (dx/2dz)$$

$X, Y$ independent, $\exp(\lambda = 1)$:  
$$f_X(x) = e^{-x} I\{x > 0\}, \quad f_Y(y) = e^{-y} I\{y > 0\}$$

$Z = X + Y$ has density $f_z(z) = ze^{-z} I\{z > 0\}$.  

$$f(x,y) = e^{-(x+y)} I\{x > 0\} I\{y > 0\}$$
Base of slab:

$$(x, y)$$ points with $z < x + y < z + dz$

$$\text{Area} = z \cdot dz + \frac{1}{2}(dz)^2$$

Height of slab = $e^{-z}$

Joint density and Circular Symmetry

$X, Y \text{ independent } \mathcal{N}(0, 1)$

$$f(x, y) = \phi(x)\phi(y) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot \frac{1}{2\pi} e^{-\frac{y^2}{2}}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2 + y^2)}$$

Joint density = surface of revolution

See p 357
A. $P(-1 < Z < 1) = .6826$

B. $P(0 < Z < 2) = .4772$

C. $P(1 < Z < 3) = .1574$

**Remarks**

1. Can get density of $X+Y$ from Convolution Formula

   \[ f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx \]

2. For $n_1, n_2$ large

   \[
   X \sim \text{Bin}(n_1, p) \sim \mathcal{N}(n_1 p, n_1 pq) \]
   \[
   Y \sim \text{Bin}(n_2, p) \sim \mathcal{N}(n_2 p, n_2 pq) \]

   Then

   \[
   X+Y \sim \text{Bin}(n_1+n_2, p) \sim \mathcal{N}(n_1+n_2 p, (n_1+n_2) pq) \]

---

**Fact**

If $\frac{x^2}{\sigma^2} + \frac{y^2}{\beta^2} = 1$, $X$ and $Y$ independent $\mathcal{N}(0, 1)$
then $\frac{x + \beta y}{\sqrt{x^2 + \beta^2}} \sim \mathcal{N}(0, 1)$

**Consequence**

If $X \sim \mathcal{N}(\lambda, \sigma^2)$, $Y \sim \mathcal{N}(\mu, \tau^2)$

$X, Y$ independent

$X + Y \sim \mathcal{N}(\lambda + \mu, \text{Var} \sigma^2 + \tau^2)$

**5. rev. 15** $Z_1, Z_2, Z_3$ independent $\mathcal{N}(0, 1)$

6. $P(Z_1 + Z_2 + Z_3 < 2)$
5. rev. 15

(i) \( P(3Z_1 - 2Z_2 < 4Z_3 + 1) \)

\[ P(3Z_1 - 2Z_2 < 4Z_3 + 1) \]

\[ \Rightarrow T_r + T_s \sim \text{Gamma}(r + s, \lambda) \]

True for all \( r > 0, s > 0 \).

**Proof:**

**Convolution Formula**

For \( \text{Gamma}(r, \lambda) \)

\[ \text{mean} = r \frac{1}{\lambda} \quad \text{var} = r \frac{1}{\lambda^2} \]

Poisson Processes, Gamma r.v.'s, Normal r.v.'s

\( W_1 \sim \text{Exp}(\lambda) \quad W_2 \sim \text{Exp}(\lambda) \quad W_3 \sim \text{Exp}(\lambda) \quad W_4 \sim \text{Exp}(\lambda) \quad W_5 \sim \text{Exp}(\lambda) \)

\( T_3 \sim \text{Gamma}(r = 3, \lambda) \)

\( T_2' \sim \text{Gamma}(r = 2, \lambda) \)

\( T_5 = T_3 + T_2' \sim \text{Gamma}(r = 5, \lambda) \)

\( f(t) = \frac{t^{r-1} e^{-\lambda t}}{\Gamma(r)} \quad \text{for } t > 0 \)

\( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \quad y > 0 \)

\( \text{Gamma}(r = \frac{1}{2}, \lambda = \frac{1}{2}) \)

"Chi-square, 1 degree of freedom"
If $Z_i$'s independent $\mathcal{N}(0,1)$,

$$Z_1^2 + \ldots + Z_n^2 \sim \text{Gamma} \left( \frac{n}{2}, \frac{1}{2} \right) \quad (3.6.5)$$

\sim \text{Chi-square, n degrees of freedom}

Special Case: $n=2$

$$Z_1^2 + Z_2^2 \sim \text{Gamma} \left( \frac{1}{2}, \frac{1}{2} \right)$$

\sim \text{Exp} (\lambda = \frac{1}{2})

---

Ex. 1. p 320

Target hit at $(X, Y)$

$X, Y$ independent $\mathcal{N}(0, \sigma^2)$

$R = \sqrt{X^2 + Y^2}$

$P(R > r) = \frac{1}{2}$

$P(R > 2r) = ?$

Sol':

$$R^2 = X^2 + Y^2 \sim \text{Exp} (\lambda = \frac{1}{2\sigma^2})$$

$P(R > 2r) = P(R^2 > 4r^2)$

$$= \left[ P(R^2 > r^2) \right]^{1/2} = \left( \frac{1}{2} \right)^{1/2} = \frac{1}{\sqrt{2}}$$
1. Show your work if you want partial credit for wrong answers.
2. Give answers either as fractions or as decimal numbers to at least three significant digits.
3. Write neatly and clearly. Remember, good penmanship is the key to success in life.
4. There is a normal table at the end of this exam.
5. If you need more space, use the back of the preceding page. Write “See back of preceding page” in the answer space.
6. All parts of all problems are worth 10 points each.

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1. Male students at Purdue have heights which are normally distributed, with mean 70 inches and standard deviation 3 inches.

(a) What is the 20th percentile of heights of Purdue men?

(b) If you pick three Purdue men at random, what is the probability that the middle height of the three heights is less than 71 inches?

(c) If you pick nine Purdue men at random, what is the approximate probability that the middle height of the nine heights is between 69.9 and 70.1 inches?
2. Telephone calls come into a telephone exchange according to a Poisson process, with a rate of two calls per minute. Let $t = 0$ be noon, with time measured in minutes.

(a) What is the probability that the second call to arrive after noon arrives between $t = 1$ and $t = 1.5$ (that is, more than 60 seconds but less than 90 seconds after noon)?

(b) Given that exactly two calls arrive during the first minute, what is the conditional probability that they both arrived during the first twenty seconds?
3. An insurance company has sold term insurance policies to 1000 forty-seven year old men and to 1000 forty-seven year old women. Forty-seven year old men have a 0.007 probability of dying during the next year, and forty-seven year old women have a 0.005 probability of dying during the next year. Assuming independence between the 2000 people, what is the approximate probability that exactly 15 of them die during the next year?
4. Let $X$ be a random variable with the density below.

(a) Find the cdf of $X$. 
(b) Find the density of $X^2$
5. In July, dandelion plants on the Purdue athletic fields constitute a Poisson scatter with rate one dandelion per square meter. Pick a random point on the fields, and let \( R \) be the distance from your point to the nearest dandelion. Find the median of \( R \).
6. Let $X$ be uniformly distributed on the unit interval $[0,1]$. Let $Y$ be a unit exponential random variable (with mean 1), independent of $X$. Find $P(Y > X)$. 
Appendix 5

Normal Table

Table shows values of $\Phi(z)$ for $z$ from 0 to 3.59 by steps of .01. Example: to find $\Phi(1.23)$, look in row 1.2 and column .03 to find $\Phi(1.2 + .03) = \Phi(1.23) = .8907$.
Use $\Phi(z) = 1 - \Phi(-z)$ for negative $z$.

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1. Heights, $N(\mu = 70, \sigma = 3)$
   (a) By Table, 20th percentile is 0.84 SD below $\mu$ so, $20^{th} \text{ percentile} = 70 - (0.84)(3) = 67.48$
   
   (b) 3 independent $N(70, 3)$ has $X_1, X_2, X_3$
   $P(X_{(2)} = \text{middle height} < 71)$
   $= P\{2 \leq 3 \text{ of } X_1, X_2, X_3 \text{ are } < 71\}$
   Since $P\{X_i < 71\} = P\{Z < \frac{71 - 70}{3}\} = \Phi\left(\frac{1}{3}\right) \approx 0.6293$
   # $X_1, X_2, X_3$ less than 71 is Binomial $n=3, p=0.6293$
   and above probability is
   $\approx (\binom{3}{2})(0.6293)^2(0.3707) + (\binom{3}{3})(0.6293)^3(0.3707)^0$
   $= 0.5896$

   (c) $P\{69.9 < X_{(5)} < 70.1\}$
   $\approx 9 \times \binom{8}{4} P\{X_1, X_2, X_3, X_4 < 70, X_5 = 70 \mid \text{ and } X_6, X_7, X_8 > 70\}$
   $\approx 9 \cdot \binom{8}{4} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^0 \frac{1}{3^{12/5}} \approx 0.06545$

2. (a) $P(1 < T_2 < 1.5) = P(T_2 > 1) - P(T_2 > 1.5)$
   $= P\left(N(0, 1) = 0 \text{ or } 1\right) - P\left(N(0, 1) = 0 \text{ or } 1\right)$
   $= e^{-2} + 2e^{-2} - e^{-3} - 3e^{-3} = 3e^{-2} - 4e^{-3} \approx 0.2069$
Given that \( N[0, 1] = 2 \), each of the 2 calls has a \( \frac{1}{3} \) probability that it arrived during the first 3 minutes, so 

\[
P(\text{both in } 1^{st} \text{ 3 minutes}) = \frac{1}{9}
\]

defined as

\[
P\left\{ N[0, \frac{1}{3}] = 2 \mid N[0, 1] = 2 \right\}
\]

\[
P \left\{ N[0, \frac{1}{3}] = 2 \text{ and } N[\frac{1}{3}, 1] = 0 \right\}
\]

\[
\frac{P \{ N[0, 1] = 2 \}}{P \{ N[0, 1] = 2 \}}
\]

\[
e^{-\frac{2}{3}} \frac{(\frac{2}{3})^2}{2!} - e^{-\frac{4}{3}} \frac{(\frac{4}{3})^2}{2!}
\]

\[
e^{-\frac{2}{3}} \frac{2^2}{2!}
\]

\[
= \binom{2}{2} \left( \frac{1}{3} \right)^2 \left( \frac{2}{3} \right)^0 = \frac{1}{9}
\]

\[
Y = \# \text{ male deaths } \sim \text{ Bin } (n = 10000, p_m = .007)
\]

\[
X = \# \text{ female deaths } \sim \text{ Bin } (n = 10000, p_f = .005)
\]

Since \( p_m \) and \( p_f \) are very small,

\[
Y \sim \text{ Poisson} \left( \mu_m = np_m = 7 \right)
\]

\[
X \sim \text{ Poisson} \left( \mu_f = np_f = 5 \right)
\]

and

\[
T = X + Y \sim \text{ Poisson} \left( \mu_T = \mu_m + \mu_f = 12 \right)
\]
or, putting it another way,

\[ T = \text{total # deaths} = \text{sum of 2000 independent Bernoulli r.v.'s, all with small } p \text{'s,} \]
so \( T \approx \text{Poisson} (\mu_T = \text{sum of } 2p \text{'s} = 12) \).

**Punchline:**

\[
P(T = 15) \approx e^{-\mu_T} (\mu_T)^{15} \cdot \frac{e^{-12} (12)^{15}}{15!} = 0.07239
\]

The height above must be \( \frac{2}{3} \) to make the total area under \( F \) equal to 1.

By geometry,

\[
F(x) = P\{X \leq x\} = \text{area under } F \text{ to left of } x
\]

\[
= \begin{cases} 
0 & \quad x < 0 \\
\frac{1}{3} x^2 & \quad 0 \leq x \leq 1 \\
\frac{1}{3} + \frac{2}{3}(x-1) = \frac{2}{3} x - \frac{1}{3} & \quad 1 \leq x \leq 2 \\
1 & \quad x > 2 
\end{cases}
\]
or, if you insist on using calculus,

\[ f(x) = \begin{cases} 
  cx & 0 < x < 1 \\
  c & 1 \leq x \leq 2 \\
  0 & \text{else}
\end{cases} \]

and

\[ 1 = \int_{0}^{\infty} f(x) \, dx = \int_{0}^{1} cx \, dx + \int_{1}^{2} c \, dx = \frac{3}{2} c \quad \Rightarrow \quad c = \frac{2}{3} \]

and

\[ F(x) = \int_{-\infty}^{x} f(t) \, dt \]

\[ = \begin{cases} 
  \int_{0}^{x} \frac{2}{3} t \, dt = \frac{1}{3} x^2 & 0 \leq x \leq 1 \\
  \int_{0}^{1} \frac{2}{3} t \, dt + \int_{1}^{x} \frac{2}{3} \, dt = \frac{2}{3} x - \frac{1}{3} & 1 \leq x \leq 2 \\
  1 & x > 2 \\
  0 & x < 0
\end{cases} \]

(b) CDF method: (With \( Y = X^2 \))

\[ F_Y(y) = P(Y \leq y) = P(X^2 \leq y) \quad (\Rightarrow y \geq 0) \]

\[ = P(X \leq \sqrt{y}) = F_X(\sqrt{y}) \]

\[ = \begin{cases} 
  0 & y < 0 \\
  \frac{1}{3} \sqrt{y}^2 = \frac{1}{3} y & 0 \leq \sqrt{y} \leq 1, \quad \text{i.e. 0 \leq y \leq 1} \\
  \frac{2}{3} \sqrt{y} - \frac{1}{3} & 1 < \sqrt{y} \leq 2, \quad \text{i.e. 1 < y \leq 4} \\
  1 & \sqrt{y} > 2, \quad \text{i.e. y > 4}
\end{cases} \]
Now differentiate:
\[ f_y(y) = \frac{d}{dy} f_x(x(y)) = \begin{cases} \frac{1}{3} & 0 < y < 1 \\ \frac{1}{3y^2} & 1 < y < 4 \\ 0 & \text{else} \end{cases} \]

Density method: \( y = x^2 \) so \( x = \sqrt{y} \)

\[ f_y(y) = f_x(x(y)) \frac{dx}{dy} = f_x(\sqrt{y}) \frac{1}{2\sqrt{y}} \]

\[ \begin{align*}
\frac{1}{3} (\sqrt{y}) \frac{1}{2\sqrt{y}} &= \frac{1}{3} \\
\frac{2}{3} \frac{1}{2\sqrt{y}} &= \frac{1}{3\sqrt{y}} & 0 < \sqrt{y} < 1, \text{i.e., } 0 < y < 1 \\
0 & & \text{else}
\end{align*} \]

5. For any number \( t > 0 \)

\[ P\{ R > t \} = P\{ \text{no dandelion in circle } \mathcal{Q} \text{ radius } t \text{ around random pt.} \} \]

\[ = e^{-\pi t^2}, \text{ since circle } \mathcal{Q} \text{ radius } t \text{ has area } \pi t^2, \text{ and, for any region } A \]

\[ P(\text{no dandelion in } A) = e^{-\lambda \text{ area}(A)} = e^{-\pi t^2} \text{ for any region } A \]

Solve \( \frac{1}{2} = P\{ R > m \} = e^{-\pi m^2} \) for median \( m \).

Median \( m \) of \( R = \sqrt{\frac{2\ln^2}{\pi}} = 0.4697 \)
\( X \sim U[0,1] \) \( Y \sim \exp(1) \) independent, so

\[ f(x,y) = f_x(x) f_y(y) = e^{-y} \], \( 0 < x < 1, \ y > 0 \)

and

\[ P(Y > X) = \iint_{\text{shaded region}} f(x,y) \, dy \, dx = \int_0^\infty \left[ \int_x^\infty e^{-y} \, dy \right] dx = \sum \frac{P\{Y > X \mid X = x\} \, P\{X = x\}}{dx} \]

\[ = \int_0^1 (1 - e^{-y}) \, dy = \int_0^1 e^{-x} \, dx = -e^{-x} \bigg|_0^1 = (1 - e^{-1}) = 0.632 \]

On campus scores:

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<th>6</th>
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<th>4</th>
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<td>7</td>
<td>9</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

\( n = 13 \)

\( \bar{X} = 55.54 \)

\( \sigma = 18.00 \)

70 \( \uparrow \) A

50 \( \uparrow \) B

30 \( \uparrow \) C