Session 12
HW examples

Start continuous r.v.'s
definition densities
cdf's (+ survival functions)
expectation
(infinitesimals)

\[ 3.4.12 \quad W_1 \sim \text{Geom}(p_1) \quad \text{indep.} \]
\[ W_2 \sim \text{Geom}(p_2) \]

(a) \( P(W_1 = W_2) \)

\[ 3.4.10 \quad X = \# \text{Bernoulli}(p) \text{ trials (indep.)} \]
\[ \text{to get} \geq 1 \text{ success,} \geq 1 \text{ failure} \]

(a) Dist. of \( X \)

(b) \( \mathbb{E}X = \)

(c) \( \text{var} X = \)
(c) $P(W_1 > W_2)$

(e) Dist. of $Y = \max(W_1, W_2)$

(d) Dist. of $X = \min(W_1, W_2)$

Geiger counter beeps at average rate of once per minute.

For $0 \leq a < b$

$N[a, b] = \#$ beeps in time int. $(a, b]$
\[ P\{\text{exactly 4 beeps in 1st 3 minutes}\} \]
\[ P\{\text{no beeps in 1st 3 minutes}\} \]

\[ P\{\text{2 beeps in 1st min., 3 beeps in 2nd min.}\} \]

Let \( T_r \) = time of \( r \)-th beep, so \( T_1 \) = time of 1st beep.

\[ P\{T_1 > 3\} = \]
For any $t > 0$

\[ P\{T_1 > t\} = \]

\[ P\{T_1 \leq t\} = \]

Let $X = r$ if 1st beep during $r$th minute

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ X = 1 \quad X = 2 \quad X = 3 \quad X = 4 \]

Dist. of $X$?
For $0 \leq a < b$

$$P\{a < T \leq b\} = \int_a^b \, dt$$

$P\{T_2 > 3\} =$

$P\{\text{1st beep during 61st second}\} =$

For any $t \geq D$

$$P\{T_2 > t\} =$$
For any $t > 0$, 
\[ P\{T_2 > t\} = \int_t^\infty \] 

Ex. 2, cont. $R =$ distance to center 
\[ f(r) \, dr = P(R = r) \, dr = \frac{\text{ring area}}{\pi} = \frac{2\pi r \, dr}{\pi} \]
\[
(12.2)
\]
\[
(12.21)
\]
\[
(19.2)
\]
\[
(21.2)
\]
\[
(12.23)
\]

$P(2^{nd} \text{ beep during 61st second})$

\[ E[R] = \int_{-\infty}^{\infty} r \, f(r) \, dr = \int_0^1 2r^2 \, dr = \left[ \frac{2}{3} r^3 \right]_0^1 = \frac{2}{3} \]

\[ E[R^2] = \int_{-\infty}^{\infty} r^2 \, f(r) \, dr = \int_0^1 2r^3 \, dr = \left[ \frac{2}{4} r^4 \right]_0^1 = \frac{1}{2} \]

\[ \text{var } R = \frac{1}{2} - \left( \frac{2}{3} \right)^2 = \frac{1}{18} \]

\[ P(R \leq \frac{2}{3}) = \int_0^{2/3} 2r \, dr = r^2 \bigg|_0^{2/3} = \frac{4}{9} \]
Formula for $P\{R \leq b\}$:
For $0 \leq b \leq 1$,
$$P\{R \leq b\} = \int_0^b 2r\,dr = r^2\bigg|_0^b = b^2$$
So $b^2 = \text{area of circle of radius } b = \frac{\pi b^2}{\pi}$

So $\text{cdf } F_R(b) \triangleq P\{R \leq b\}$
$$= \begin{cases} 
0 & \text{if } b \leq 0 \\
 b^2 & \text{if } 0 \leq b \leq 1 \\
 1 & \text{if } b \geq 1 
\end{cases}$$

1 For every $b \in \mathbb{R}$,
$$P\{R \leq b\} = \text{area under } f \text{ to left of } b$$
$$= \int_{-\infty}^b f_R(r)\,dr = F_R(b) = \text{height of } f_R \text{ at } b$$

2 For every $a < b$,
$$P\{a < R \leq b\} = \text{area under } f \text{ over } (a, b]$$
$$= \int_a^b f_R(r)\,dr = F_R(b) - F_R(a)$$
$$= \text{increase in } F_R \text{ over } (a, b]$$

Compare 1 and 2 on previous page with relationship between standard normal density $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and standard normal cdf $\Phi$ (Appendix 5).

3 For every $r$ (except $r = 0$, $r = b$)
$$f(r) = \text{slope of } f_R \text{ at } r = \frac{d}{dr} \left. f_R(r) \right|_{b=r}$$
$$= P\{R \leq r + dr\} - P\{R \leq r\}$$
$$= P\{r < R \leq r + dr\} = f(r)\,dr$$
Now divide by $dr$.

Median of $R$ is (any) number $m$ with $P\{R < m\} \leq \frac{1}{2} \leq P\{R \leq m\}$. Here, $m$ solves $P\{R \leq m\} = \frac{1}{2}$
$$\text{so } m^2 = \frac{1}{2}, \quad m = \frac{1}{\sqrt{2}} = .707.$$
3. A circle with radius \( m = \frac{1}{2} \) has half the area of a unit circle.

\[ ER = \frac{2}{3} = 0.667 \]

\[ \text{Median}(R) = \frac{1}{\sqrt{2}} = 0.707 \]

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**Total area under \( f_x \)?**

**EX =**

**median of \( X = \)**

**Slope of \( f_x \) at \(-1\)?** At \( 0\)? At \( 1\)?

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Recall \( U \sim \mathcal{U}[0, 1] \) has \( \text{Var}(U) = \frac{1}{12} \).

Note \( 2U \sim \mathcal{U}[0, 2] \)

\[ V = 2U - 1 \sim \mathcal{U}[-1, 1] \]

and

\[ \text{Var}(V) = \text{Var}(2U - 1) = 2^2 \text{Var}(U) = \frac{4}{12} = \frac{1}{3} \]

Q. Is the variance of \( X \) above \( \frac{1}{3} \)?

\[ \frac{1}{3} \]
Independence of r.v.s

General Definition: \( X \) and \( Y \) indep.
if \( P\{X \in A, Y \in B\} = P\{X \in A\} P\{Y \in B\} \)
for all sets of numbers \( A, B \).

[Recall: For \( X, Y \) discrete, used]
\( P\{X=x, Y=y\} = P\{X=x\} P\{Y=y\}, \text{ all } x, y \).

Still true that
\( E(XY) = E(X)E(Y) \)
\( \text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) \).

Are \( X \) and \( Y \) indep.? 

But
\( E(XY) = \)

Are \( W \) and \( R \) indep.?