### Last time (Session 6)
**Binomial distribution**
**Normal curves**

### This time (Session 7)
**Normal curve calculations**
**Normal approximation for binomial probabilities**

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#### Appendix Table

Table shows values of \( \Phi(z) \) for \( z \) from 0 to 3.08 by steps of .1. Example: to find \( \Phi(1.23) \), look in row 1.2 and column .0. To find \( \Phi(1.2 + .03) = \Phi(1.23) = .5098 \). Use \( \Phi(-z) = 1 - \Phi(z) \) for negative \( z \).

<table>
<thead>
<tr>
<th>( z )</th>
<th>( .00 )</th>
<th>( .01 )</th>
<th>( .02 )</th>
<th>( .03 )</th>
<th>( .04 )</th>
<th>( .05 )</th>
<th>( .06 )</th>
<th>( .07 )</th>
<th>( .08 )</th>
<th>( .09 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.039</td>
<td>0.078</td>
<td>0.118</td>
<td>0.159</td>
<td>0.199</td>
<td>0.238</td>
<td>0.277</td>
<td>0.315</td>
<td>0.354</td>
</tr>
<tr>
<td>0.1</td>
<td>0.040</td>
<td>0.080</td>
<td>0.120</td>
<td>0.160</td>
<td>0.200</td>
<td>0.240</td>
<td>0.280</td>
<td>0.320</td>
<td>0.360</td>
<td>0.400</td>
</tr>
<tr>
<td>0.2</td>
<td>0.079</td>
<td>0.158</td>
<td>0.237</td>
<td>0.316</td>
<td>0.395</td>
<td>0.474</td>
<td>0.553</td>
<td>0.632</td>
<td>0.711</td>
<td>0.790</td>
</tr>
<tr>
<td>0.3</td>
<td>0.128</td>
<td>0.227</td>
<td>0.326</td>
<td>0.425</td>
<td>0.524</td>
<td>0.623</td>
<td>0.722</td>
<td>0.821</td>
<td>0.920</td>
<td>1.019</td>
</tr>
<tr>
<td>0.4</td>
<td>0.187</td>
<td>0.306</td>
<td>0.415</td>
<td>0.524</td>
<td>0.633</td>
<td>0.742</td>
<td>0.851</td>
<td>0.960</td>
<td>1.069</td>
<td>1.178</td>
</tr>
<tr>
<td>0.5</td>
<td>0.256</td>
<td>0.395</td>
<td>0.504</td>
<td>0.613</td>
<td>0.722</td>
<td>0.831</td>
<td>0.940</td>
<td>1.049</td>
<td>1.158</td>
<td>1.267</td>
</tr>
<tr>
<td>0.6</td>
<td>0.335</td>
<td>0.504</td>
<td>0.613</td>
<td>0.722</td>
<td>0.831</td>
<td>0.940</td>
<td>1.049</td>
<td>1.158</td>
<td>1.267</td>
<td>1.376</td>
</tr>
<tr>
<td>0.7</td>
<td>0.424</td>
<td>0.613</td>
<td>0.722</td>
<td>0.831</td>
<td>0.940</td>
<td>1.049</td>
<td>1.158</td>
<td>1.267</td>
<td>1.376</td>
<td>1.485</td>
</tr>
<tr>
<td>0.8</td>
<td>0.523</td>
<td>0.722</td>
<td>0.831</td>
<td>0.940</td>
<td>1.049</td>
<td>1.158</td>
<td>1.267</td>
<td>1.376</td>
<td>1.485</td>
<td>1.595</td>
</tr>
<tr>
<td>0.9</td>
<td>0.632</td>
<td>0.831</td>
<td>0.940</td>
<td>1.049</td>
<td>1.158</td>
<td>1.267</td>
<td>1.376</td>
<td>1.485</td>
<td>1.595</td>
<td>1.705</td>
</tr>
</tbody>
</table>

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#### 68-95-99.7 Rule

For any normal distribution:

- \( P\{\mu - \sigma < X < \mu + \sigma\} \approx 68\% \)
- \( P\{\mu - 2\sigma < X < \mu + 2\sigma\} \approx 95\% \)
- \( P\{\mu - 3\sigma < X < \mu + 3\sigma\} \approx 99.7\% \)
Normal Calculation Example

If \( X \sim \mathcal{N}(\mu = 63.5, \sigma = 2.5) \),

\[
P\{61 < X < 67\} = P\left\{ \frac{61 - 63.5}{2.5} < \frac{X - 63.5}{2.5} < \frac{67 - 63.5}{2.5} \right\}
\]

\[
= P\{-1 < Z < 1.4\}
\]

\[
0.9244 
\]

\[
\begin{array}{c}
-1 \quad 0 \quad 1.4
\end{array}
\]

Normal Approx. to Binomial (p99)

Histogram for \( X \sim \text{Binomial}(n,p) \)

\( X \sim \mathcal{N}(\mu = np, \sigma = \sqrt{npq}) \) curve

\[
P\{a < X < b\} \approx \Phi\left( \frac{b + \frac{1}{2} - \mu}{\sigma} \right) - \Phi\left( \frac{a - \frac{1}{2} - \mu}{\sigma} \right)
\]

\text{Fatwa: Use only when } \sqrt{npq} \geq 3.
Binomial \((n = 100, p = 0.1)\); \(np = 10, \sqrt{npq} = 3\)

\[
P\{11 \leq X \leq 15\} = P\{10.5 < X < 15.5\}
= P\left\{\frac{10.5 - 10}{3} < \frac{X - 10}{\sqrt{3}} < \frac{15.5 - 10}{3}\right\}
\approx P\{0.16 < Z < 1.83\} = 0.4004
\]

**Exact:** 0.3770

**Error:** 0.0234

```
\frac{|0.3770 - 0.4004|}{0.4004} = 0.032
```

For \(X \sim \text{Binomial}(n = 100, p = 0.2)\)

\[
P\{45 < X \leq 55\} = P\{45.5 < X < 55.5\}
= P\left\{\frac{45.5 - 50}{5} < \frac{X - 50}{5} < \frac{55.5 - 50}{5}\right\}
\approx P\{-0.9 < Z < 1.1\} = 0.1802
\]

**Exact:** 0.1802

```
\frac{|0.1802 - 0.1802|}{0.1802} = 0
```
Steps for Normal Approx to Binomial

1. Write equivalent event involving half-integers. (= Continuity correction)

2. Standardize: \( \frac{X - np}{\sqrt{npq}} \)
   (= Transform to \( z \)-scale)

3. Use (std) normal table; since \( \frac{X - np}{\sqrt{npq}} \approx N(0, 1) \).

Graph of \( \sqrt{pq} \) versus \( p \), 0 ≤ \( p \) ≤ 1

Maximum at \( p = \frac{1}{2} \), value = \( \frac{1}{2} \).

Result: \( \sigma = \sqrt{npq} \) for Binomial \( (n, p) \)

Square Root Law

For \( X \sim \text{Binomial}(n, p) \), \( n \) not too small,

(a) \( X = \# \text{successes} \) has \( \geq 95\% \) chance of falling between \( np \pm \sqrt{n} \)
   (since \( \sqrt{n} \geq 2 \sqrt{npq} = 2 \sigma \), whatever \( p \)).

(b) \( \hat{p} = \frac{X}{n} = \text{proportion of successes} \)
   has \( \geq 95\% \) chance of falling between \( p \pm \frac{1}{\sqrt{n}} \).
Confidence Intervals
(For estimating p, n not too small.)

The random interval \( \hat{p} \pm \frac{1}{\sqrt{n}} \)
has a \( > 95\% \) chance that it will cover the true value of \( p \).

\( n = 100 \) : \( \hat{p} \pm .10 \)
\( n = 400 \) : \( \hat{p} \pm .05 \)

Remarks
1. Worst possible error in normal approximation
\( W(n, p) \approx \frac{1 - 2p}{10 \sqrt{npq}} \)
\( = \frac{p - \frac{1}{2}}{5 \sqrt{npq}} \)

(So Fatwa guarantees \( W(n, p) \leq \frac{1}{30} = .033 \))

2. Pitman has formula for skewness correction.

\( p \) 106
Last time (Session 7)
Normal curve calculations
Normal approx. for Binomial Prob.'s

This time (Session 8)
Square root law (LLN)
Confidence intervals
Poisson approx.
for small-p Binomials
Start sampling (?)

4.1

Poisson Approximation p119
For small $p$, (large $n$)
in Binomial $(n, p)$ dist.,
P[$k$ successes] $\approx e^{-\mu} \frac{\mu^k}{k!}$

$\mu = np, k = 0, 1, \ldots, n$

4.2

Ex. 500 people each throw $3$ dice.
# triple-aces $\sim$ Binomial $(n=500, p=\frac{1}{6})$
P(4 "successes")
$= \binom{500}{4} \left(\frac{1}{6}\right)^{4} \left(\frac{5}{6}\right)^{496}$
$= 0.1183^{2x}$

$[\mu = np = \frac{500}{216} = 2.315]$
$\approx e^{-\mu} \frac{\mu^4}{4!} = 0.1182$, approx.

5.3

Ex. 3,000 people each throw $4$ dice.
# quadruple-aces $\sim$ Binom. $(n=3,000, p=\frac{1}{1296})$
P(4 "successes")
$= \binom{3,000}{4} \left(\frac{1}{1296}\right)^{4} \left(\frac{1295}{1296}\right)^{2996}$
$= 0.11820$

$[\mu = np = \frac{3,000}{1296} = 2.315]$
$\approx e^{-\mu} \frac{\mu^4}{4!} = 0.11818$, approximately

5.4
Proof For $k = 3$. [Think: $n = 500$, $p = \frac{1}{200}$, $\mu = 2.5$]

$$P \{ 3 \text{ successes} \} = \frac{n!}{(n-3)!3!} P^3 (1-P)^{n-3}$$

$$= \frac{n(n-1)(n-2)}{n \cdot n \cdot n} (\frac{np}{3!})^3 \left( \frac{1}{n} \right)^{n-3} \approx 1 = \frac{n^3}{3!} \propto e^{-\mu} \approx 1$$

*Recall:

$$\left(1 - \frac{\mu}{n}\right)^n \rightarrow e^{-\mu} \text{ as } n \rightarrow \infty.$$
2. rev 13 p133
50 seeds/packet, seeds indep.
P(Seed bad) = .01
Packet replaced if > 3 bad
Q: P(replace > 50 out of 4000 packets)?
#bad/packet ~ Bin (n=50, p=.01)
P(#3 bad) = 1 - P(0, 1, or 2 bad)
\[
\approx -e^{-\mu} + \mu e^{-\mu} - \frac{\mu^2}{2} e^{-\mu}
\]
(\mu = np = \frac{50}{50} = .01439)

8.9

Sampling
Box with 10 black, 30 white cards
Sample 8 cards
(a) with replacement \( X = \# \text{ black} \)
(b) without \( X = \text{ Binomial (n=8, p=\frac{1}{4})} \)
\[
P(X=0) = \left( \frac{3}{4} \right)^8 = .1001
\]
\[
P(X=2) = \binom{8}{2} \left( \frac{1}{4} \right)^2 \left( \frac{3}{4} \right)^6
\]
\[
= .3115
\]

8.11

# packets replaced ~ Bin (n=4000, p=.0144)
\( \mu = np = 57.55 \quad \sigma = \sqrt{npq} = 7.53 \)

P(>50 out of 4000 replaced) = P(X>50.5)
\[
= P \left( \frac{X-\mu}{\sigma} > \frac{50.5-\mu}{\sigma} \right)
\]
\[
= P \left( Z > -0.936 \right)
\]
\[
= P (Z < .936)
\]
\[= .8251 \]
(\( = P(Z < .94) = .8240 \text{ OK} \))

8.10

(b) \( X \) Hypergeometric
Use \( \text{H}_c = \binom{10}{2} \) possible combinations
\[
P(X=0) = \frac{\# \text{ comb. with no black}}{\text{total # comb.}}
\]
\[
= \frac{\binom{90}{30}}{\binom{100}{30}} = .0761 \quad \text{P(wwwww)}
\]
\[
P(X=2) = \frac{\binom{10}{2} \binom{90}{28}}{\binom{100}{30}} = .3474
\]
\[
= \left[ \frac{10}{3} \right] \text{P(bbwwwwww)}
\]
8.12
2. What if:
   1000 black, 3000 white in box,
sample 8 (a) with repl.; (b) w/o
   X = # black?

(a) ___________________

(b) ___________________

9.13

Chapter 2 Review
1. Binomial (n, p) dist.
2. Normal approx. (npq \geq 3)
3. Poisson approx. (p \leq 0.05)
4. Hypergeometric vs. Binomial

Sampling
w/o replacement
vs
w/ replacement

Today (Session 9)
1. Sampling w/o replacement
2. Poker
3. Random variables
   - distributions
   - joint distributions
   - conditional dist.'s
   - independence

Box with 100 black balls
Sample of 100 balls
w/o replacement
w/o replacement

St Py of Hypergeometric (p.241)

Box with 
B black, N-B white cards
Sample n w/ replacement

\[ X = \# \text{ black sampled} \]

\[ \text{mean} (X) = np \quad \{ p = \frac{B}{N} \} \]

\[ \text{SD}(X) = \sqrt{n \frac{N-n}{N-1} \frac{npq}{N-p}} \quad q = 1 - p \]

Finite population correction factor

Problem 2.5.12 Poker Hands

(a) Straight flush \((3, 4, 5, 6, 7, \text{ all H})\)

(b) 4-of-a-kind \((\text{AAAAA})\)

(c) Full house \((\text{AAAXX})\)

(d) Flash, not straight flush \((\text{all H})\)

(e) Straight, not straight flush \((3, 4, 5, 6, 7)\)

(f) 3-of-a-kind \((\text{AAAQQ})\)

(g) 2 pairs \((\text{AAKKQ})\)

(h) Pair \((\text{AAKQS})\)

(i) None of the above

\[(a) \# = 40, \quad P = \frac{40}{\binom{52}{5}} = 0.000154\]

\[(b) \# = 13 \times 4 \times 3, \quad P = 0.000240\]

\[(c) \# = 13 \times 12 \times (\frac{4}{5}) \times (\frac{4}{5}) = 0.00144\]

\[(d) \# = 5 \times (\frac{52}{5}) - 40, \quad P = 0.00197\]

\[(e) \# = 10 \times 5^5 - 40, \quad P = 0.00392\]

\[(f) \# = 13 \times 4 \times (\frac{12}{5}) \times 4 \times 4, \quad P = 0.0211\]

\[(g) \# = (\frac{5}{5}) \times (\frac{12}{5}) \times (\frac{12}{5}) \times 4 \times 4, \quad P = 0.0475\]

\[(h) \# = 13 \times (\frac{4}{5}) \times (\frac{12}{5}) \times 4^3, \quad P = 0.423\]

\[(i) \quad P = 1 - \frac{5}{52} \quad P = 0.501, \quad \text{since a \& h disjoint}\]
**DePn's of Random Variable**

1. A random variable is a number* determined by the outcome of a random experiment.  
   [p. 139](#)

2. A random variable is a function \( X: \Omega \rightarrow \mathbb{R} \) which assigns a number \( X(\omega) \) to each outcome \( \omega \in \Omega \).  
   [p. 143](#)

**Example**

Teams A+B play *best-3-of-5* series.  
\( X = \) # games won by A  
\( Y = \) # games won by B  
\( T = X + Y = \) # games played  
\( \Omega = \{aaa, bbb, baaa, \ldots, aabbb\} \)  
\( X: 3 \ 0 \ 3 \ldots \)  
\( Y: 0 \ 3 \ 1 \ldots \)  
\( T: 3 \ 3 \ 4 \ldots \)

**Assume:** Games indep.  
\[ P(A \text{ wins}) = \frac{1}{3}, \text{each game} \]

**Generally:** The dist. of \( X \) is the assignment of probabilities \( P(X = b) \) to sets of numbers \( B \).  
[p. 140, p. 144](#)

**Here:** (Since \( X \) has finite # of values)  
The dist. of \( X \) is the set of possible values, together with their probabilities.  
[p. 141](#)
### Distribution of $X$

<table>
<thead>
<tr>
<th>Value of $X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X=x)$</td>
<td>$\frac{3}{81}$</td>
<td>$\frac{6}{81}$</td>
<td>$\frac{8}{81}$</td>
<td>$\frac{4}{81}$</td>
</tr>
</tbody>
</table>

$P(X=0) = P\{bbbb\} = \left(\frac{1}{3}\right)^3 = \frac{3}{81}$

$P(X=1) = P\{abbb, babb, bbbb\} = 3 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^3 = \frac{4}{81}$, etc.

### Dist. of $T$

<table>
<thead>
<tr>
<th>Value of $T$</th>
<th>$P{T=t}$</th>
</tr>
</thead>
</table>

#### $U = \min(X, Y)$

<table>
<thead>
<tr>
<th>$U$ values</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{3}{81}$</td>
<td>$\frac{6}{81}$</td>
<td>$\frac{8}{81}$</td>
<td>$\frac{4}{81}$</td>
</tr>
</tbody>
</table>

#### $V = \max(X, Y)$

<table>
<thead>
<tr>
<th>$V$ values</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{3}{81}$</td>
<td>$\frac{6}{81}$</td>
<td>$\frac{8}{81}$</td>
<td>$\frac{4}{81}$</td>
</tr>
</tbody>
</table>

(marginal) dist. of $X$
Conditional dist. of $X$, given $Y = 3$

<table>
<thead>
<tr>
<th>Value of $X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X=x</td>
<td>Y=3)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

$P(X=0 | Y=3) = \frac{P(X=0, Y=3)}{P(Y=3)} = \frac{3/81}{17/81}$

Q. Cond. dist. of $X$, given $Y = 1$?

Equivalent conditions for independence of $X$ and $Y$

1. $P(X=x, Y=y) = P(X=x)P(Y=y)$, all $x, y$
2. $P(X\in A, Y\in B) = P(X\in A)P(Y\in B)$, all $A, B$
3. Cond. dist. of $Y$, given $X=x$, same for all $x$.
4. Cond. dist. of $X$, given $Y=y$, same for all $y$.

Independence Example

<table>
<thead>
<tr>
<th>X values</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y values</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Expectation (= expected value = mean) of r.v. 
\[ E(X) = \sum_{\text{possible values } x} x \cdot P(X=x) \]

\[ = \sum_{\text{outcomes } w} X(w) \cdot P\{w\} \]

Example. Roll die twice
\( X_1 = \# \text{aces on first roll} \) independent
\( X_2 = \# \text{aces on second roll} \) Bernoulli \( \left( \frac{1}{6} \right) \)
\( T = X_1 + X_2 = \text{total } \# \text{aces} \)

\begin{align*}
\begin{array}{c|cc}
\text{values } x \text{ of } X_i & 0 & 1 \\
\text{prob.'s } P\{X_i = x\} & \frac{5}{6} & \frac{1}{6} \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c|ccc}
\text{values of } T & 0 & 1 & 2 \\
\text{prob's } P\{T = t\} & \frac{25}{36} & \frac{10}{36} & \frac{1}{36} \\
\end{array}
\end{align*}

\[ E(T) = \left( \frac{25}{36} \right) \left( \frac{5}{6} \right) + \left( \frac{10}{36} \right) \left( \frac{1}{6} \right) + \left( \frac{1}{36} \right) \left( \frac{1}{6} \right) = \frac{12}{36} = \frac{1}{6} \]

OR \( \Omega = \{ nn, na, an, aa \} \)

\begin{align*}
\begin{array}{c|cccc}
P\{w\} & \frac{25}{36} & \frac{5}{36} & \frac{5}{36} & \frac{1}{36} \\
T(w) & 0 & 1 & 1 & 2 \\
X_1(w) & 0 & 0 & 1 & 1 \\
X_2(w) & 0 & 1 & 0 & 1 \\
\end{array}
\end{align*}

\[ E(T) = \sum_{w} T(w) \cdot P\{w\} \]
\[ = \frac{12}{36} = \frac{1}{6}, \text{ and } \]
\[ T(w) = X_1(w) + X_2(w) \quad \forall w. \]

Example, cont. Note that
\[ E(T) = \sum_{w} T(w) \cdot P\{w\} \]
\[ = \sum_{w} \left[ X_1(w) + X_2(w) \right] \cdot P\{w\} \]
\[ = \sum_{w} X_1(w) \cdot P\{w\} + \sum_{w} X_2(w) \cdot P\{w\} \]
\[ = E(X_1) + E(X_2) \]
Addition Rule for Expectations
If X and Y are r.v.'s in same setting (on same outcome space Ω), then
\[ E(X+Y) = E(X) + E(Y) \]
\[
\frac{\sum_{\omega} [X(\omega) + Y(\omega)] P(\omega)}{\sum_{\omega} P(\omega)} = \frac{\sum_{\omega} X(\omega) P(\omega) + \sum_{\omega} Y(\omega) P(\omega)}{\sum_{\omega} P(\omega)}
\]
\[ = E(X) + E(Y) \]

Remark: An indicator r.v. \( I_A \) takes only the values
1 (when event A occurs)
0 (when event A doesn't occur).

\[ E(I_A) = 1 \cdot P(I_A=1) + 0 \cdot P(I_A=0) = P(A). \]

Examples
\[ X_1 \text{ above } I_A, \]
\[ X_2 \text{ } = I_{A_2}, \]
where
\[ A_i \text{ = event of Ace on } i^{th} \text{ roll.} \]

Constants
\[ E(7X) = 7 E(X) \]
\[ \frac{\sum_{\omega} 7X(\omega) P(\omega)}{\sum_{\omega} P(\omega)} = 7 \sum_{\omega} X(\omega) P(\omega) = 7 E(X). \]

Likewise for any other constant \( c \).
Combining w. Addition Prop.
gives LINEARITY?
\[ E(ax + by + c) = aE(X) + bE(Y) + c \]
where \( a, b, c \) constants.

Binomial and Hypergeometric Means
Box with 13 black, 17 white balls
Sample 4 balls, (a) with repl.
(b) w/o repl.
\[ T = \# \text{ black draws} \]
\[ = I_{B_1} + I_{B_2} + I_{B_3} + I_{B_4} \text{ (scenario)} \]
\[ E(T) = E(I_{B_1}) + \ldots + E(I_{B_4}) \]
\[ = \frac{13}{30} + \frac{13}{30} + \frac{13}{30} + \frac{13}{30} \]
\[ = \frac{4 \times 13}{30} = \binom{4}{3} \]
3.2.14. Elevator with 10 floors
12 people, indep., floors equally likely for @
\[ X = \# \text{ stops} \quad \text{EX} = ? \]
\[ X = I_1 + I_2 + \ldots + I_{10} \]
\[ I_j = \# \text{ stops on floor } j \]
\[ E I_j = \]

**Interpretations**

\( X = \# \text{ from rolling fair die} \)

\[ \text{EX} = 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{3}{6} + \ldots + 6 \cdot \frac{6}{6} = 3 \frac{1}{2} \]

\( \text{Interpret: } \text{Balance pt. of prob Histogr.} \)

\[ \frac{1}{6} \uparrow \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

**Interp. 2**
Long-run average for many independent repetitions of Expt.

For 6 million numbers (Refer to p. 20.12)
\[ x_1, x_2, \ldots, x_{6,000,000}, \text{each } x_i \in \{1, \ldots, 6\} \]

\[ \text{Average} = \frac{x_1 + x_2 + \ldots + x_{6,000,000}}{6,000,000} \]

\[ = \frac{n_1 + n_2 + n_3 + \ldots + n_6 \cdot 6}{6,000,000} \]

\[ = \frac{f_1 + f_2 + \ldots + f_6}{6} \]

\( n_6 = \#6's \text{ in list} \)

\( f_6 = \text{proportion of 6's in list} \)

**Tail Sum Formula for EX**

\[ \text{If } X \text{ has values } 0, 1, 2, \ldots \]

\[ \text{then } \text{EX} = \sum_{j=1}^{\infty} P(X \geq j) \]

**Pitman Pf:**

\[ X = I_{A_1} + I_{A_2} + I_{A_3} + \ldots \]

where \( A_j \text{ = event that } X \geq j, j = 1, 2, \ldots \)

**Challenge:** Find analog formula for general \( X \).
Ex. Roll die until 1st Ace
\(X = \#\) rolls \(\sim \text{Geometric}(p = \frac{1}{6})\) r.v.
\[EX = \sum_{x=1}^{\infty} x \cdot P(X=x) = \sum_{x=1}^{\infty} x \cdot \left(\frac{5}{6}\right)^{x-1} \left(\frac{1}{6}\right) = ?\]
OR
\[EX = \sum_{j=1}^{\infty} P(X=j) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{j} \frac{1}{j^2}\right) = \sum_{j=1}^{\infty} \left(\frac{1}{j^2}\right) + \left(\frac{1}{j}\right) + \ldots\]
\[= \frac{1}{1} + \frac{\frac{1}{2}}{1} + \frac{\frac{1}{3}}{1} + \ldots\]
= sum of all \(P(X=j)\)

or note that \(\frac{1}{n}\) times sum

Last time (Session 10)
Joint, conditional distributions
Expectation formulae, add rule
Means of indicators, bin., hyper.

This time (Session 11) More Expectation
Interpretations tail sum formulae
Means of geometric, negative binomial
Markov inequality
\(E_g(X), E_g(X,Y)\)
Mult. rule \((X,Y \text{ indep.})\)
If $X$ has values 0, 1, 2, 3, 4, 5 and $EX=1$, then how large can $P\{X=4\}$ be?

<table>
<thead>
<tr>
<th>Values of $X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P{X=x}$</td>
<td>$p_0$</td>
<td>$p_1$</td>
<td>$p_2$</td>
<td>$p_3$</td>
<td>$p_4$</td>
<td>$p_5$</td>
</tr>
</tbody>
</table>

How large can $P\{X=4\}$ be?

Markov Inequality p174

If $X \geq 0$ (always), then

$$P(X \geq a) \leq \frac{EX}{a} \text{ for every } a > 0.$$

Proof for $a = 4$:

$$EX = \sum_{x=0}^{\infty} x \cdot P\{X=x\} \geq 4 \cdot \sum_{x=0}^{\infty} x \cdot P\{X=x\}$$

$$\geq \sum_{x=0}^{\infty} 4 \cdot P\{X=x\} = 4 \cdot P\{X=4\}$$

If, $X$ are $\textit{aces}$ in 5 cards—Hypergeo.

$$EY = E(I_1 + I_2 + I_3 + I_4 + I_5) = 5 \cdot \frac{4}{52} = \frac{20}{52}$$

Markov:

True Value

$$P(Y \geq 1) \leq \frac{EY}{1} = \frac{20}{52} = 0.385$$

$$P(Y \geq 2) \leq \frac{EY}{2} = \frac{19}{52} = 0.365$$

$$P(Y \geq 3) \leq \frac{EY}{3} = \frac{12}{52} = 0.231$$

$$P(Y \geq 4) \leq \frac{EY}{4} = \frac{9}{52} = 0.173$$
Expectation of function of $X$, or of $(X,Y)$

$$E(g(X)) = \sum_x g(x)P\{X=x\}$$

These are special cases of

$$E(T) = \sum_w T(w)P\{w\}$$

Important Counterexamples

1. $E(X^2) \neq (EX)^2$ (except when...?)

\[
\begin{array}{c|c|c}
\text{values} & \text{P}\{X=x\} & E(X^2) \\
\hline
\text{x} & -1 & 1 \\
\hline
\end{array}
\]

2. $E(XY) \neq (EX)(EY)$ usually (but = if...?)

Take $X$ as in (1), $Y = X$, i.e.

\[
\begin{array}{c|c|c|c}
\text{x-values} & \text{y-values} & E(XY) & (EX)(EY) \\
\hline
-1 & -1 & D \\
\hline
1 & 1 & D \\
\hline
\end{array}
\]

Means are a measure of location. Standard deviation measures spread.

Example: Wechsler IQ's, ages 20-34

All: $N(\text{mean 110, std.dev. 25})$

A.F.'s: $N(\text{mean 122, std.dev. 7})$

*In 1970.*
Definitions

\[ \text{Var}(X) = E[(X-\mu)^2] \]
\[ = \sum_x (x-\mu)^2 P\{X=x\} \]
\[ = \sum_\omega (\omega-\mu)^2 P\{\omega\} \]

\[ \text{SD}(X) = \sqrt{\text{Var}(X)} \]

Var of indicator

\[
\begin{array}{c|cc}
\text{value } x & 0 & 1 \\
\text{P}(X=x) & 1-p & p \\
\end{array}
\]
\[ \mu = \Sigma X = p \]
\[ \text{var } X = E[(X-p)^2] = \sum_x (x-p)^2 P\{X=x\} \]
\[ = (0-p)^2 (1-p) + (1-p)^2 p \]
\[ = p(1-p)(p+1-p) / \sqrt{\text{SD}(X)} \]
\[ = p(1-p) = pq \]

Interpretation: Avg. sq. error of prediction.
If you use \( \mu \) to predict the value of \( X \)
and pay a penalty = squared error,
then \( \text{Var}(X) \) = mean squared error

Remark \( \mu \) = best guess w.r.t. mean

\[
\begin{align*}
E[(X-b)^2] &= E[(X-\mu + \mu - b)^2] \\
&= E[(X-\mu)^2 + 2(X-\mu)(\mu - b) + (\mu - b)^2] \\
&= \text{var}(X) + (\mu - b)^2 + 0 \\
&\text{since } E[c(X-\mu)] = c E(X-\mu) = 0
\end{align*}
\]

Computational Formula.

For \( \text{Var } p(1-p) \)

\[ \text{Var}(X) = E(X^2) - (EX)^2 = \sum_x x^2 P\{X=x\} - \left[ \sum_x x P\{X=x\} \right]^2 \]

\[ \text{Var}(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] \]
\[ = E(X^2) - 2\mu E(X) + \mu^2 \]
\[ = E(X^2) - 2\mu^2 + \mu^2 \]
\text{since } E(X) = \mu.
Indicators, cont.
For indicator $X$,

$$X^2 = X_1, \text{ so } E(X^2) = EX = p.$$  

$$\text{var } X = E(X^2) - (EX)^2$$  

$$= p - p^2 = p(1-p) = pe$$

Example Heights of men 18-24 in US have mean $\approx 70$ inches
$SD \approx 3$ inches
(and are approx. normal)
Heights in cm ($y = 2.54x$) have mean $\approx 2.54 \times 70 = 177.8$ cm
$SD \approx 2.54 \times 3 = 7.62$ cm

Scaling and Shifting \hspace{1cm} p188

For constants $a, b$

$$SD(aX + b) = |a| SD(X) \quad \text{(Prove)}$$

(and $\text{var}(aX + b) = a^2 \text{var}(X)$)

Special case (Standardization)\hspace{1cm} p188
If $\mu = EX, \sigma = SD(X)$,

$$X^* = \frac{X - \mu}{\sigma} \quad \text{"X in std. units"}$$

then

$$EX^* = 0, SD(X^*) = 1$$

Last time (Session 11)
Expectation: interpretations, tail sum formula, Markov inequality.

This time (Session 12)
Expectation: $Eg(X), Eg(Y)$
Mult. rule $(X$ and $Y$ indep.)
$SD$ and variance
Def's
scaling & shifting
Chebychev inequ.
Add. rule for var
Simple (Math.) example of scaling

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<td>$P{X = x}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
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</table>

$E[X] = 1$
$Var X = E[(X-1)^2] = (0-1)^2 \cdot \frac{1}{2} + (2-1)^2 \cdot \frac{1}{2} = 1$
$SD X = \sqrt{1} = 1$

$Y = 3X$

<table>
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<tr>
<th>Values of $Y$</th>
<th>0</th>
<th>$\frac{3}{2}$</th>
<th>$\frac{3}{2}$</th>
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<td>$P{Y = y}$</td>
<td>$\frac{1}{2}$</td>
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so $SD(Y) = 3$

100 Exam Scores
mean = 60
SD = 10
$\sqrt{\frac{\sum (x_i - 60)^2}{100}}$

At most _____ scores \leq SD or \geq 70.
At most _____ scores \leq 40 or \geq 80.
At most _____ scores \leq 30 or \geq 90.

What if score dist.
approximately normal?

Note: $Var X = \sum (x - \mu)^2 P\{X = x\} \geq 0$
Q Is $Var X = 0$ possible? How?

Recall $Var X = E(X^2) - (E(X))^2$
Thus, $E(X^2) = (E(X))^2$
if and only if

Chebychev: For $X^* = \frac{X - \mu}{\sigma}$, $k > 0$
$P\{|X - \mu| \geq k\sigma\} = P\{|X^*| \geq k\} \leq \frac{1}{k^2}$
Words: The prob. that $X$ is $k$ or more SD's from its mean $\mu$ is at most $\frac{1}{k^2}$.

OR: The prob. that $X$ is more than $k$ SD's from $\mu$ is $< \frac{1}{k^2}$.

PE: Set $Y = (X^*)^2$.
$Y \geq 0$, always, and $EY = 1$.
P\{1 \leq Y \leq k\} \leq P\{Y \geq k^2\} \leq \frac{EY = \frac{1}{k^2}}{k^2}$
same events

Markov: 12.6
Addition Rule for Variances p.143

If $X$ and $Y$ are independent, then
\[ \text{var}(X+Y) = \text{var}(X) + \text{var}(Y) \]

If $X_1, X_2, \ldots, X_n$ are independent, then
\[ \text{var}(X_1 + \ldots + X_n) = \text{var}(X_1) + \ldots + \text{var}(X_n) \]

Last Time (Session 12)
- SD and Var
- Shifting and scaling
- addition rule (indep. r.v.s)
- Chebyshev

This Time (Session 13)
- CLT, with and w/o Continuation
- $\mu$ rule, moments
- Craps principle
- Recall: If $|q_1| < 1$, $p = 1 - q$
- $1 + q + q^2 + q^3 + \ldots = \frac{1}{1 - q} = \frac{1}{p}$
- so $\sum_{k=1}^{\infty} q^k p = 1$

Proof of Add. Rule p.144

\[ S = X + Y, \quad \mu_S = \mu_X + \mu_Y \]
\[ \text{var} S = \text{var}(S - \mu_S)^2 \]
\[ = \text{var} [(X - \mu_X) + (Y - \mu_Y)] \]
\[ = \text{var}(X - \mu_X)^2 + \text{var}(Y - \mu_Y)^2 + 2 \text{cov}(X - \mu_X, Y - \mu_Y) \]
\[ = \text{var} X + \text{var} Y + 0 \]

since $X, Y$ independent implies $X - \mu_X$ and $Y - \mu_Y$ independent.
\[ \text{and} \ E(X - \mu_X) = E(Y - \mu_Y) = 0 \]

Remember:

\[ E(X + Y) = E(X) + E(Y) \]
\[ E(aX) = aE(X) \]
\[ \text{var}(X + Y) = \text{var}(X) + \text{var}(Y) \]

\[ i \equiv X \text{ and } Y \text{ independent} \]
\[ \text{var}(aX) = a^2 \text{var}(X) \]

13.2
Square Root Law

\[ \overline{X_n} = \frac{S_n}{n} = \frac{X_1 + \ldots + X_n}{n} \]

Note:
Standardized \( S_n = \text{Standardized} \overline{X_n} \)

\[ \frac{S_n - \mu}{\sigma} = \frac{\overline{X_n} - \mu}{\sigma/\sqrt{n}} \]

(Divide numerator and denominator of LHS by \( n \) to get RHS.)

Central Limit Theorem (+) p194

If \( X_1, X_2, \ldots, X_n \) independent, same dist., mean \( \mu \), SD \( \sigma \) < \( \infty \), then for large \( n \),

\[ P\left\{ a \leq \frac{S_n - \mu}{\sigma/\sqrt{n}} \leq b \right\} \approx \Phi(b) - \Phi(a) \]

for any numbers \( a \leq b \).

If \( S_n \) has (consecutive) integer values, then approx. better with continuity correction.
Recall \( E g(X) = \sum_x g(x) P[X = x] \)

so for \( X = \) die roll,

\[
E(X^2) = \sum_{x=1}^{6} x^2 P[X = x] = 1.5 + 2.5 + \ldots + 6.5 = \frac{91}{6}
\]

Q1. \( P \{ 320 < X_{100} \leq 370 \} \)

\[
= P \left\{ \frac{329.5 - 350}{17.1} < \frac{X_{100} - 350}{17.1} < \frac{370.5 - 350}{17.1} \right\}
\]

\[
\approx P \{ -1.20 < Z < +1.20 \} = 0.7698
\]

Q2. \( P \{ 3.2 \leq \bar{X} \leq 3.7 \} \)

\[
\approx \frac{0.7698}{6} \quad (w/o \ cont. \ corr. \ 0.7584)
\]

Ex. For \( W = \) wt. of random man, 18-24,

\( E(W) = 162 \) \( \sigma = 30 \) pounds

If you get a random sample of 25 such men, what is \( P \left\{ \bar{W}_{25} > 170 \right\} \)?

\[
= P \left\{ \frac{\bar{W}_{25} - 162}{30/\sqrt{25}} > \frac{170 - 162}{30/\sqrt{25}} \right\} = P \{ Z > 1.33 \}
\]

\[
= 0.0912 \quad \text{Note: Cont. Corr. NOT appropriate!}
\]
Density (≈ Normal Curve)
For Avg. Wt. of random sample of 25 men

Density For
W = weight of 1 man

100 162

(See Chapter 4, especially page 2.68.)

Post-Section 3.4 Wrinkles:
Infinite sum rule:
If events \( A_1, A_2, A_3, \ldots \) disjoint, then
\[
P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)
\]

Moments:
\[EX = \sum_{x} x P\{X=x\}\]
provided \( \sum_{x} |x| P\{X=x\} < \infty \)

provided at least one
\[\sum_{x>0} \text{finite} \quad \text{and} \quad \sum_{x<0} \text{finite}\]

Remark: More general CLT
If \( X_1, \ldots, X_n \) (approx.) independent,*
and every \( X_k \sim \mu_k \) negligible
relative to \( SD(S_n) = \sqrt{\frac{\sum_{k=1}^{n} \sigma_k^2}{n}} \),
then \( S_n \) is approx. normal,
(*perhaps diff. dist.)*

\[
\text{mean } \sum_{k=1}^{n} \mu_k
\]

\[
\text{SD } \sqrt{\sum_{k=1}^{n} \sigma_k^2}
\]

St. Petersburg Paradox
Start with \#1.
Flip coin until first Heads, then stop.
Every coin flip doubles your money!
\( X = \# \) dollars you end up with.


$$
E = \sum_{x} x P\{X=x\} = 1 + 1 + 1 + 1 + \ldots = \infty
$$
Variation: Toss die, then coin as above.
Die odd: Buffet pays Gates X dollars.
Die even: Gates pays Buffet X dollars.
W = Gates' net winnings = \( \pm X \)
values of W \( \ldots -4 \quad -2 \quad +2 \quad +4 \quad \ldots \)
probabilities \( \frac{-1}{6} \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{6} \quad \ldots \)

\[ E(W) = \sum_{w} w P(W=w) \]
\[ = \frac{1}{2} \cdot \frac{-1}{6} + \frac{1}{2} \cdot \frac{-2}{3} + \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{-2}{3} + \ldots \]
\[ = 7 \quad \text{Undefined.} \]

Verification using Infinite Rule
\[ N = \# \text{rolls to get 7 or 8} \]
\[ \sim \text{Geometric} (p = \frac{11}{36}), \text{so} \]
\[ P(N=k) = \left( \frac{25}{36} \right)^{k-1} \cdot \frac{11}{36}, \quad k=1,2,\ldots \]

\[ E7 = \text{event you stop with 7.} \]
\[ P(E7) = \sum_{k=1}^{\infty} P(N=k \text{ and } E7) \]
\[ = \sum_{k=1}^{\infty} \frac{11}{36} \left( \frac{25}{36} \right)^{k-1} \cdot \frac{11}{36} \]
\[ = \frac{\frac{11}{36}}{1 - \frac{25}{36}} \]
\[ = \frac{11}{11} \cdot \frac{36}{11} \]
\[ = 1 \]

Craps Principle
\[ T = \text{total when 2 dice rolled.} \]
\[ \text{Roll until } T = 7 \text{ or } T = 8 \quad (W.C.F.) \]
\[ P(\text{end with 7}) \]
\[ = P(T=7) \]
\[ = \frac{6}{36} \]
\[ = \frac{6}{6+5} = \frac{6}{11} \]

Remark
\[ \text{Since } P(N=k \text{ and } E7) \]
\[ = \left( \frac{25}{36} \right)^{k-1} \cdot \frac{11}{36} \]
\[ = P(N=k) \cdot P(E7) \quad \text{for all values } k \text{ of } N, \]
\[ \text{the r.v. } N \text{ and the event } E7 \]
\[ \text{are independent.} \]
General Craps Principle p. 210
If in each A vs. B game,
\[ P(A) = P(A \text{ wins}), \quad P(B) = P(B \text{ wins}), \quad P(D) = P(\text{draw}) \]
\[ P(A) + P(B) + P(D) = 1 \]
games indep., play until win then
\[ P(\text{A wins overall}) = \frac{P(A)}{P(A) + P(B)} \]
\[ = P(\text{A wins 1st game | not draw}) \]
\[ \frac{13}{x=2} \]
\[ \sum_{x=4}^{11} P(X = x) P(\text{Win} | X = x) \]
\[ = \frac{3}{36} \left( \frac{3}{9} \right) + \frac{4}{36} \left( \frac{5}{10} \right) + \frac{5}{36} \left( \frac{5}{11} \right) + \frac{6}{36} \left( \frac{1}{7} \right) + \frac{5}{36} \left( \frac{1}{8} \right) + \frac{3}{36} \left( \frac{1}{9} \right) + \frac{3}{36} \left( \frac{1}{10} \right) + \frac{2}{36} \left( \frac{1}{11} \right) \]
\[ x = 4 \quad x = 5 \quad x = 6 \quad x = 7 \quad x = 8 \]
\[ = 0.4929 \]

3.4.8 Craps p. 218
Roll 2 die. \( X_0 = \text{result} \) is "point"
Win if \( X_0 = 7 \) or 11
Lose if \( X_0 = 2 \) or 3 or 12
Else, keep rolling until total = \{"point" (Win) or 7 (Lose)\}

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & & & & & \\
2 & 4 & & & & \\
3 & & 4 & & & \\
4 & & & 7 & & \\
5 & & & & 7 & \\
6 & & & & & 7 \\
\end{array} \]

So
\[ P(X_0 = 4) = \frac{3}{36}, \quad \text{and} \]
\[ P(\text{Win} | X_0 = 4) = \frac{3}{9} \]
by Craps Principle.

by Craps Principle.
Example: Sudden death Ping Pong, A vs. B
\[ P(A\text{ wins}) = p, \text{ each point points indep.} \]
First player ahead by 2 wins match.
\[ P(A \text{ wins match}) = \sum_{k=1}^{\infty} P(A \text{ wins after } 2k \text{ pts.}) \]
\[ = P(A \text{ in } 2) + P(A \text{ in } 4) + P(A \text{ in } 6) + \ldots \]
\[ = p^2 + (2pq)p^2 + (2pq)^2 p^2 + \ldots \]
\[ = \frac{p^2}{1 - 2pq} = \frac{p^2}{p^2 + q^2} \text{ when } pq \neq 0. \]

3.217 Box has 3 red, 10 white balls
Draw w/o repl. until get all 3 red.
\[ D = \# \text{ draws} \]
\[ P(D \leq 9) = \frac{\# \text{ comb. with } 3 \text{ red}, 6 \text{ white}}{\# \text{ comb. of size 9}} \]
\[ = \frac{\binom{3}{3} \binom{10}{6}}{\binom{13}{9}} = \frac{42}{143} \approx 0.2937. \]

Last time: Session 13
Normal approx. w/o cont corr.
Crops principle

This time: Session 14
Crops principle
Examples:
probability by counting
expectation using addition rule
(normal approx.
(add. rule for var)
Geometric mean, var
# trials to get \( r \) successes

(a) Or, as outcomes (equally likely)
use the \( \binom{13}{3} \) possible color sequences from drawing all 13 balls,
one-by-one w/o replacement,
e.g.
\[
1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13
\]
\[ P(D \leq 9) = \frac{\binom{9}{3}}{\binom{13}{9}} \]
(b) \[ P(D = 9) = P(D \leq 9) - P(D \leq 8) \]
\[ = \frac{3}{13} \binom{10}{9} - \frac{3}{13} \binom{10}{8} \] using first method in (a)
\[ = \frac{8}{13} = 0.6154 \] 2nd method in (a)

(c) \[ E \bar{D} ? \]
Number white balls 1, 2, ..., 10
\[ D = 3 + I_1 + I_2 + ... + I_{10} \]
\[ I_8 = \begin{cases} 1 & \text{if white ball 8 drawn before last red} \\ 0 & \text{else} \end{cases} \]
\[ E I_8 = \frac{3}{4} \]
\[ E \bar{D} = 3 + 10 \times \frac{3}{4} = 10.25 \]

3.2.16 Deal cards until 1st ace.
\[ X = \# \text{cards dealt. } EX = ? \]
Number non-aces 1, 2, ..., 48
\[ X = 1 + I_1 + I_2 + ... + I_{48} \]
\[ I_8 = \begin{cases} 1 & \text{if card 8 drawn before 1st ace} \\ 0 & \text{if not} \end{cases} \]
\[ E I_8 = \frac{1}{5} \]
\[ EX = 1 + 48 \times \frac{1}{5} = 10.6 \]
\[ \text{G. } P(2\text{nd Ace on 20th card}) = ? \]
Ex. Al and Bob each roll die 100 times. 
\( S = \# \text{ Al aces} \), \( T = \# \text{ Bob aces} \) 
\( P(S = T) = ? \) 
\[
\text{exact} = \sum_{k=0}^{100} P(S = T = k) = \sum_{k=0}^{100} \binom{100}{k} \left( \frac{1}{2} \right)^k \left( \frac{1}{2} \right)^{100-k}
\]

Same as 
\( P(S - T = 0) \), 
\( E(S - T) = 0 \) 
\( \text{var}(S - T) = \text{var}(S) + \text{var}(-T) \) 
\( = 200pq = \frac{1000}{36} \) (normal)

Let \( X_k = \# \text{ aces on Al's } k\text{th roll} \) 
\( Y_k = \# \text{ aces on Bob's } k\text{th roll} \) 
\( \Delta_k = X_k - Y_k \) 

\[
\begin{array}{c|c|c}
\Delta_k \text{ values} & -1 & +1 \\
\hline
\text{probs.} & \frac{5}{36} & \frac{5}{36} & \frac{5}{36} \\
\end{array}
\]

\( \mathbb{E} \Delta_k = 0 \) 
\( \text{var} (\Delta_k) = \frac{10}{36} \) from dist. table, or 
\( \text{var}(X_k) + \text{var}(-Y_k) \)

\( S - T = \sum_{k=1}^{100} X_k - \sum_{k=1}^{100} Y_k = \sum_{k=1}^{100} \Delta_k \)

so 
\( \text{L'T} \Rightarrow S - T \text{ approx. } N(\mu = 0, \sigma^2 = \frac{1000}{36}) \)

and since \( S - T \) has consec. int. values, 
\( P(S - T = 0) = P(-.5 < S - T < .5) \) 
\( = P\left\{ \frac{-5}{1000} < \frac{S - T}{\sqrt{1000/36}} < \frac{5}{1000/36} \right\} \)
\( \approx P(|Z| < 0.95) = 0.7577 \)

approx. v. good (\( \Delta_k \) is symmetric)

Recall: If \( W = \# \text{ independent Bernoulli}(p) \) trials needed to get first success, 
then \( W \sim \text{Geometric}(p) \) 
\( \text{P}(W = k) = q^{k-1} p \quad ; k = 1, 2, 3, \ldots \)

\( \mathbb{E} W = \sum_{j=1}^{\infty} j q^{j-1} = \frac{q}{1-q} = \frac{1}{p} \)

Fact: \( \text{var}(W) = \frac{q^2}{p^2} \) 
\( \text{SD}(W) = \frac{\sqrt{q}}{p} \)

Example: \( W = \# \text{ die rolls needed to get Ace} \)
Bernoulli $(p)$ trials to get $r$ successes

Roll die until 3rd ace, $T_3 = \# rolls$

$P(T=5) = P\{\text{naaa, naaa, naaaa} \ |
\text{anna, annaa, anana, aanna}\}$

$= \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right) \cdot \frac{1}{6}$

$P(T_3 = k) = P\{2 \text{ aces in } 1^{st} \text{ } k-1 \text{ rolls, ace on } k^{th}\}$

$= \left(\frac{1}{2}\right)^2 \frac{1}{6} \left(\frac{5}{6}\right)^{k-3}$, $k = 3, 4, 5, ...$

Let $T_{15} = \# \text{ die rolls to get } 15 \text{ aces}$

$N_{100} = \# \text{ aces in } 100 \text{ rolls}$

$\sim \text{ Binomial } (n=100, p = \frac{1}{6})$

Then

$\{T_{15} \leq 100\} = \{N_{100} \geq 15\}$

$\{T_{15} > 100\} = \{N_{100} < 15\}$

---

$T_3 = W_1 + W_2 + W_3$ so $E(T_3) = 3 \cdot \frac{1}{2} = 18$

$\text{ indep. Geom} \left(\frac{1}{6}\right)$

$\text{ var}(T_3) = 3 \cdot \frac{9}{2} = 9.0$

$T_3 - 3 = \# \text{ non-aces } \& \text{ get } 3 \text{ aces}$

$\sim \text{ Negative Binomial } (r = 3, p = \frac{1}{6})$

$P(T_3 - 3 = n) = P\{T_3 = n+3\} = \left(\frac{n+2}{2}\right) \left(\frac{1}{6}\right)^{n} \left(\frac{5}{6}\right)^{n}$, $n = 0, 1, 2, ...$

$E(T_3 - 3) = 15$

$\text{ var}(T_3 - 3) = 9.0$

---

"Collector's Problem"

$X = \# \text{ die rolls to see all 6 numbers}$

$EX = ?$ \hspace{1cm} $\text{ var } X = ?$

$X = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6$

$Y_k = \# \text{ rolls to see } k^{th} \text{ new number,}$

(after seeing $k-1^{th}$ new number)

$e.g.,$ if see

445452512343551326

$Y_1 = 1 \hspace{1cm} Y_2 = 2 \hspace{1cm} Y_3 = 3 \hspace{1cm} Y_4 = 1$

$Y_5 = 5 \hspace{1cm} Y_6 = 8$
\[ \begin{align*}
Y_1 & \sim \text{Geom}(p_1 = 1) \\
Y_2 & \sim \text{Geom}(p_2 = \frac{5}{6}) \\
Y_3 & \sim \text{Geom}(p_3 = \frac{4}{6}) \\
Y_4 & \sim \text{Geom}(p_4 = \frac{3}{6}) \\
Y_5 & \sim \text{Geom}(p_5 = \frac{2}{6}) \\
Y_6 & \sim \text{Geom}(p_6 = \frac{1}{6}) \\
\end{align*} \]

\begin{align*}
\text{independent,} & \quad \text{so} \quad \text{var} \ X = \frac{1}{k} \text{var} \ Y_k \\
\text{var} \ X = \frac{1}{k} \sum \text{var} \ Y_k & = \frac{1}{k} \left( \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{6} \right) \\
& = 7.7 \text{ var } Y
\end{align*}

\[ \begin{align*}
\text{Poisson Distribution} & \quad N \sim \text{Poisson} (\mu), \mu > 0, \text{ means} \\
P\{N = k\} & = e^{-\mu} \frac{\mu^k}{k!}, k = 0, 1, 2, \ldots \\
\text{For small } p, & \quad \frac{\text{Bin}(n,p)}{n p q} \approx \text{Poisson}(\mu = n p) \\
\text{mean} & \quad np \quad \mu \\
\text{var} & \quad np q \quad \mu \\
\text{SD} & \quad \sqrt{np q} \quad \sqrt{\mu} \\
\text{Proof: p 223} & \quad \text{p 223}
\end{align*} \]

**Last time (Session 14)**
Craps principle calculation examples

**This time (Session 15)**
Geometric r.v.'s
# trials to get r successes
Poisson r.v.'s
Poisson scatters

Recall (Problem 3.1.10)
\[ \text{Independent?} \]
\[ \begin{align*}
\text{Bin} \left( n = 72, p = \frac{1}{3} \right) + \text{Bin} \left( n = 54, p = \frac{1}{3} \right) & = \text{Bin} \left( n = 126, p = \frac{1}{3} \right) \\
\text{Pois} \left( \mu_1 = 2 \right) + \text{Pois} \left( \mu_2 = 1.5 \right) & \approx \text{Pois} \left( \mu = 3.5 \right) \\
\rightarrow \text{Indep.} & \quad \text{Indep} \\
\text{So, sums of independent Poissons are Poisson.} & \quad \text{are Poisson.}
\end{align*} \]
Proof: $N_1 \sim \text{Poisson}(\mu_1 = 2)$ \(\text{Indep.}\) $N_2 \sim \text{Poisson}(\mu_2 = 1.5)$

\[
P(N_1 + N_2 = 5) = \sum_{j=0}^{5} P(N_1 = j) P(N_2 = 5 - j)
\]

\[
= \sum_{j=0}^{5} \left[ \frac{e^{-2} 2^{j}}{j!} \right] \left[ \frac{e^{-1.5} (1.5)^{5-j}}{(5-j)!} \right]
\]

\[
= \frac{e^{-3.5} \left(\frac{3.5}{5}\right)^5}{5!} \times 2
\]

Example: Lump of carbon.

$N = \# \text{Carbon } 14 \text{ atoms} = 5 \text{ zillion}$

$p = \text{prob. of decay in next minute} = \frac{1}{\text{zillion}}$ for each atom

$N_1 = \# \text{decays in next minute} \sim \text{Poisson}(\mu = )$

Lump of Strontium.

$N_2 = \# \text{decays in next minute} \sim \text{Poisson}(\mu = 3)$.

Then

$N_1 + N_2 \sim \text{Poisson}(\mu = )$

Recall earlier claim (Session 8)

If $N = X_1 + X_2 + \ldots + X_n$

independent, $X_k \sim \text{Bernoulli}(p_k)$

then

$N \sim \text{Poisson}(\mu = \sum_{1}^{n} p_k)$
Decay ex. cont.
Given that total # decays in one min. is 4, what is the (cond.) prob. that exactly 3 were $^{14}$C decays (and one was $^{14}$N)?

\[ P(N_1=3 \mid N_1+N_2=4) = \frac{P(N_1=3, N_2=1)}{P(N_1+N_2=4)} = \frac{\binom{4}{3} \frac{3^3}{3!} \frac{1}{1!}}{e^{-2} \frac{2^4}{4!}} \]

\[ = \frac{\left( \frac{3^3}{3!} \right) \left( \frac{1}{1!} \right)}{e^{-2} \frac{2^4}{4!}} \]

\[ = \frac{(1)}{\left( \frac{2^4}{4!} \right)} \frac{N_1+N_2=x}{N_1 \sim \text{Binomial}(n, p = \frac{\lambda_1}{\lambda_1+\lambda_2})} \]

Poisson Scatter
- Random pattern of "hits" in space (or time)
- Examples
  1. T: Times when phone calls come into switch board. (Section 4.2)
  2. T: Points at which raindrops hit sidewalk over 1 second interval.

Poisson Scatter Theorem
If (1) No multiple hits (at 1 point)
(2) Independence of hits on nonoverlapping regions
(3) Equal (small) areas have equal prob. of being hit

Then there is a $\lambda > 0$ so that

\[ \# \text{Hits in region } B \sim \text{Poisson} \left( \mu = \lambda \cdot \text{size}(B) \right) \]
Rain falls at rate 2 drops per square foot per second. 

\[ P\left\{ \text{all } 9 \text{ square feet in a square yard} \right. \geq 1 \text{ hit at least once in next second} \} \]

\[ = \left[ 1 - P(N = 0) \right]^9 = (1 - e^{-2})^9 = 0.27017 \]

\( \text{since conditions (1), (2), (3) of Poisson Scattering are satisfied here.} \]

More generally, if (1), (2) and (3) hold, then

\[ \text{if } B \text{ is a region, } \lambda = \frac{\text{# of hits in } B}{\text{size}(B)} \]

\[ \text{# of hits in } A \sim \text{Poisson}(\lambda \text{ size}(A)) \]

\[ P(1 \text{ hit in } A) = e^{-\lambda \text{ size}(A)} \]

\[ \approx \lambda \text{ size}(A) \]

Rain fall, ex. cont.

\( \lambda = 2 \) drops per square foot (per second) means 

\( \text{(Global): } 20,000 \pm \text{a couple hundred drops on a } 10,000 \text{ sq. ft. region in 1 sec} \)

\[ \left( \text{SD} = \sqrt{2 \times 100} = \sqrt{2000} \right) \]

\( \text{(Local): } \text{A single square inch has prob. } = \lambda^2 (1/2)^2 = \frac{1}{12} \text{ of being hit in 1 sec} \)

\[ \left( \frac{1 - e^{-\frac{1}{12}}}{1 - e^{-1/2}} = \frac{1}{12} \cdot 0.0001 \right) \]

\[ P(k > 1 \text{ hit}) = 1 - e^{-\frac{1}{12}} - \frac{1}{12} e^{-\frac{1}{12}} \]

\[ = 0.0001 \]

\[ \text{Interpretations of Rate } \lambda \]

\[ \text{Global (LLN): } \lambda = \frac{\text{# of hits in very large region } B}{\text{size}(B)} \]

\[ \text{Local: For small region } A, \# \text{ of hits } \sim \text{Poisson}(\lambda \text{ size}(A)) \]

\[ P(1 \text{ hit in } A) = e^{-\lambda \text{ size}(A)} \]

\[ \approx \lambda \text{ size}(A) \]
Example of "General Poisson Scatter"

\[ \lambda(t) = \text{rate at which real car accidents occur at time of week } t \text{ in Tippecanoe Co.} \]

\[ \text{\# in } [t_1, t_2] \sim \text{Pois}(\mu = \int_{t_1}^{t_2} \lambda(t) \, dt) \]