Stat 516
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Plan Session 1
1. Outcome spaces
2. Events
3. Probability as limiting relative frequency
4. Empirical law of averages

Example 1
Roll black die. Let $X$ = result.
Roll red die. Let $Y$ = result.

Outcome Space
Def. An outcome space for a random experiment is a list of possible outcomes so that exactly one must occur.

Example: Other outcome spaces
$\Omega_B = \{1, 2, 3, 4, 5, 6\}$
$\Omega_C = \{1, 2, 3, 4, 5, 6\}$
$\Omega_D = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
where $\Omega_D$ outcomes give value of $T > 6$.
NOT $\Omega_D' = \{1, 2, 3, 4, 5, 6\}$

Def. An event is a subset of the outcome space in use.
Examples:
- $B$: event black die is 1 = $\{X = 1\}$
- $R_3$: $\{Y = 3\}$
- $T = \{T = 7\}$
- $L = \{T < 7\}$
- $M = \{T > 7\}$

Primary Definition
The probability of an event $A$ is the fraction of the time that it happens in the long run = limiting relative frequency in $\infty$ many identical, independent repetitions of the experiment.

Empirical Law of Averages
Relative frequencies stabilize (converge to limiting values) as the number of experiment repetition gets larger and larger.

Example 2
Flip a fair coin. $\Omega = \{H, T\}$
$P_n(H) = \frac{\text{# Heads in } n \text{ Flips}}{n}$

0.5

'Typical' behavior of $P_n(H)$.
Example. Assume 36 outcomes in \( \Omega \) are equally likely (occur equally often in long run).

\[
P(B_i) = \\
P(B_1 \cup B_4) = \\
P(B_1 \cap B_4) = \\
P(B_1 \cup R_1) = \\
P(B_1 \cap R_1) = \\
\]

Q. When do we have
\[P(A \cup B) = P(A) + P(B)\]?

\[
\begin{array}{c|cccc}
\text{Event} & 1 & 2 & 3 & 4 \\
\hline
A & 2 & 2 & 1 & 1 \\
B & 3 & 4 & 3 & 2 \\
A \cup B & 5 & 6 & 4 & 3 \\
\end{array}
\]

always

General Formula:
\[P(A \cup B) = P(A) + P(B)\]

Text, p. 5

**IF** all outcomes in a finite outcome space \( \Omega \) are equally likely, then

\[P(A) = \frac{\#A}{\#\Omega} = \frac{\text{#outcomes in } A}{\text{#outcomes in } \Omega}\]

for any event \( A \subseteq \Omega \).

Pascal-Fermat Problem

Roll a die 4 times.

Win if you get at least one Ace.

Lose if no aces.

\[A_i \text{ event of ace on } i^{th} \text{ roll, etc.}\]

\[P(W) = P(A_1, A_2, A_3, A_4)\]
\[= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6}\]

P(at least 1 ace in 7 rolls)

\[= P(A_1 \cup A_2 \cup \ldots \cup A_7)\]
\[= \frac{1}{6} + \frac{1}{6} + \ldots + \frac{1}{6} = \frac{7}{6}\]
Correct Sol'n of Pascal-Fermat
\[ \mathcal{L} = \{ 1111, 1112, 1113, ... , 6666 \} \]
\[ \mathcal{L} \text{ has } 6^4 \text{ equally likely outcomes} \]

\[ P(W) = \frac{\# W}{\# \mathcal{L}} = \]
\[ \mathcal{L} = \{ 2222, 2223, ... , 6666 \} \]

\[ \# \mathcal{L} = \]

\[ P(L) = \frac{5^4}{6^4} = 0.482 \]

\[ P(W) = 1 - P(L) = 0.518. \]

---

Variant
Roll pair of dice 214 times.
Win if at least one double-ace.
Review of Session 1
1. Outcome space $\Omega$, events
2. Probability = limiting relative frequency
   (or = opinion)
3. Empirical Law of Averages
4. $\Omega$ with outcomes equally likely:
   
   \[ P(A) = \frac{\#A}{\#\Omega} \]

Rules of Proportion & Probability [p. 2]
Non-negative: $P(B) \geq 0$
Addition*: If $B_1 \cap B_2 = \emptyset$,

\[ P(B_1 \cup B_2) = P(B_1) + P(B_2) \]

Total one:

\[ P(\Omega) = 1 \]

Plan Session 2
1. Rules of Proportion and Probability
2. Conditional Probability

Additivity implies:
If $B_1, B_2, B_3, B_4$ partition $B$,

\[ P(B) = P(B_1) + P(B_2) + P(B_3) + P(B_4) \]

Additivity implies:
If $B_1, B_2, B_3, B_4$ partition $B$,
More consequences: pages 21-22

Complement rule
\[ P(A^c) = 1 - P(A) \]

Inclusion-Exclusion
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

Example 100 women in \( \Delta \Delta \Delta \)
40 have taken Bio.
25 \( \subset \) Chem.
17 \( \subset \) both.

\% who took at least one of Bio.+Chem.
\[ P(B \cup C) = P(B) + P(C) - P(B \cap C) \]
\[ = .40 + .25 - .17 = .48 \]

\% in neither
\[ = P(B^c \cap C^c) \]
\[ = P(\overline{B \cup C}) \] (DeMorgan)
\[ = 1 - P(B \cup C) = 1 - .48 = .52 \]

Pick one \( \Delta \Delta \Delta \) at random.*
Given that she took Chem.
what is probability she took Bio.?
\[ P(B \mid C) = \text{cond. prob. of } B \text{ given } C \]
\[ = \frac{17}{25} = \frac{\#(B \cap C)}{\#C} = \frac{P(B \cap C)}{P(C)} \]

\% in neither
\[ = \frac{17}{25} \]
\[ = \frac{.17}{.52} \]
\[ P(B^c \mid C) = \]
\[ P(C \mid B) = \]

\[ \begin{array}{ccc}
B & C \\
0.23 & 0.17 & 0.08 \\
0.52 & & \\
\end{array} \]

Def: \[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{(assuming \( P(B) \neq 0 \))} \]

Freq Interpretation:
\[ P(A \mid B) = \text{long-run relative frequency of } A, \text{ among those trials where } B \text{ occurred.} \]

Consequence: Multiplication Rule
\[ P(A \cap B) = P(B) \cdot P(A \mid B) \]

Example cont:
\[ P(C \cap B) = P(C) \cdot P(B \mid C) \]
\[ = \frac{25}{100} \cdot \frac{12}{25} = \frac{12}{100} \]

Tree diagrams:
\[ \text{Q1: Had Chem?} \]
\[ \text{Q2: Had Bio?} \]

\[ \begin{array}{c}
C \\
B \\
\end{array} \]

Often:
- start with some conditional, some unconditional probabilities
- determine prob. distribution on \( \Omega \) from these

Ex. Have 3 coins
- 2 "fair" in has prob. \( \frac{2}{3} \) of heads
Pick coin at random and toss.
\[ \text{P(H)? P(F \mid H)?} \]
Summary
1. Probability axioms and consequences

2. Conditional probability $P(A|B)$
   - definition
   - interpretation
   - mult. rule for
   - in Tree diagrams

Rule of Average Cond. Probs. $P(A)$
If $B_1, B_2, B_3$ partition $\Omega$,
$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$
Last time (Session 2)
1. Prob. axioms & consequences
2. conditional probability

This time (Session 3)
1. rule of average cond. prob.'s
2. Bayes' rule
3. independence of 2 events

Bayes' Rule
If $B_1, B_2, B_3$ partition $\Omega$,

$$P(B_i | A) = \frac{P(A | B_i) P(B_i)}{P(A | B_1) P(B_1) + P(A | B_2) P(B_2) + P(A | B_3) P(B_3)}$$

Ex. AIDS test:
If have HIV, prob. of + test is .95
If no HIV, prob. of + test is .02
1% of population has HIV.
Q1: What % of pop. tests +?
Q2: If test is +, prob. of HIV = ?

$P(D) = .01$
$P(D') = .99$

$P(+) | D = .95$
$P(+) | D' = .02$
Q1. Rule of Avg. Cond. Prob.'s
\[ P(+) = P(+) \cdot P(D) + P(+) \cdot P(D^c) \cdot P(D^c) \]
\[ = 0.0095 + 0.0198 = 0.0293 \]

Q2. Bayes rule
\[ P(D|+) = \frac{P(D \cdot +)}{P(+)} = \frac{0.0095}{0.0293} = 0.324 \]

(Independence). Flip coin, then toss die
- Coin result should not affect die probabilities, whether or not coin, die is fair
For black die - red die, which pairs are independent?
(i) B₁ and R₁?
(ii) B₁ and T₄?
(iii) B₁ and T₅?
(iv) B₁ and B₂?
(v) T₄ and T₅?

Even if coin and die are unfair, we expect die prob's to be unaffected by coin result, and so that prob's of 12 outcomes have above structure.

\[
P(A|H) = P(A|H^c) \iff P(A|H) = P(A)
\]

\[
\iff \frac{P(A \cap H)}{P(H)} = P(A) \iff P(A \cap H) = P(A)P(H)
\]

\[
\iff P(H|A) = P(H) \quad \text{definition of \ } A, H \text{ indep}
\]

\[
\iff P(H^c|A) = P(H^c)
\]

\[
\iff P(H|A^c) = P(H)
\]

\[
\iff P(H^c|A^c) = P(H^c)
\]
For black die-red die, which pairs are independent?

(i) $B_1$ and $R_1$?
(ii) $B_1$ and $T_4$?
(iii) $B_1$ and $T_7$?
(iv) $B_1$ and $B_6$?
(v) $T_4$ and $T_7$?

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Summary:
1. Calculations using trees
   - rule of avg. cond. probs
   - Bayes rule
2. Independence
Last time (Session 3)
- Tree diagrams
- Rule of Avg., cond. prob.'s
- Bayes rule
- Independence of 2 events

This time (Session 4)
- Multiplication rules
- For counting
- For cond. prob.'s
- Independence, >2 events

# subsets of \( \{A, B, C, D, E, F, G\} \)?

If \( \# \Sigma = n \),
\# subsets \( \mathcal{E} \) = \( 2^n \)

Multiplication Rule for Counting p50?
For \( k \) successive choices, with
\( n_j \) choices at stage \( j \) \( (1 \leq j \leq k) \)
The total number of possible successive choices is
\( n_1 \cdot n_2 \cdot \ldots \cdot n_k \)

Pascal-Fatigue: Roll die 4 times
Win if \( \geq 1 \) ace
\( \Omega = \{1111, 1112, 1113, \ldots, 6666\} \)
\( \mathcal{L} = \{2222, 2223, 2224, \ldots, 6666\} \)
\( P(W) = 1 - P(L) = 1 - \frac{\# L}{\# \Omega} = 1 - \frac{6^4}{6^4} \)

= 0.512

by counting Mult. Rule,
since outcomes equally likely.
Multiplication Rule for 3 events

\[ P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2) \]

Likewise for \( n \) events.

Independency of \( A_1, A_2, A_3 \) put

\[ P(A_3|\text{event involving other } A_i's) = P(A_3) \]

Consequence (Alt. de'min)

\[ P(A_1^c \cap A_2^c \cap A_3^c) = P(A_1^c)P(A_2^c)P(A_3^c) \]

for any choice of \( c \)’s.

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Pascal-Fractal Trees

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\[ \text{Pascal-Fractal, again} \]

\[ P(\text{L}) = P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c) \]

\[ = P(A_1^c)P(A_2^c)P(A_3^c)P(A_4^c) \]

(assuming rolls independent)

\[ = \left(\frac{5}{6}\right)^4 \]

---

Ex. Roll die 4 times

\[ P(\text{get 4 different numbers}) \]

\[ = P(D_1D_2D_3D_4) \]

(\( D_i \) event that \( i \)-th roll different from previous rolls

\[ = P(D_1)P(D_2|D_1)P(D_3|D_2D_1)P(D_4|D_3D_2D_1) \]

\[ = \]
**Geometric Distribution**  p. 58-59
Roll die until 1st ace

\[ \Omega = \{a, na, nna, nnna, \ldots\} \]

value of \( x \# \) rolls

\[ \text{prob} = \frac{1}{6} \left( \frac{5}{6} \right)^{x-1} \]

\[
P(X > 3) = \]

\[
P(X > 4 | X > 3) = \]

---

**Birthday Problem**  p. 62

23 "random" people

P(3 shared B-day) = P(all diff.) = \( P(D_1D_2 \ldots D_{23}) \)

\[
= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{343}{365}
\]

\[
= 0.494 \quad \text{so} \\
\]

P(3 shared B-day) \approx 0.506
$C_i = \text{event component $i$ conducts elec.}$

$F = \text{event current flows thru circuit}$

Assume: $\{ C_i \text{'s independent, indicated prob.'s}$

(a) $F = \quad \#1 \quad \#2$

$P(F) =$

(b) $F = \quad \#1 \quad \#2 \quad \#3$

$P(F) =$

$P(C_i | F) =$

$P(C_i | F^c) =$
(a) $F =$

```
1 2 3
4 5 6
```

```
7 8 9
```
Last time (Session 4)
- Multiplication rule
- For counting
- For cond. prob.
- Independence for >2 events
- Geometric random variables

This time (Session 5)
- Finish Elec. Circuits
- Permutations \(_{26}^{4} \) 
- Combinations \(_{26}^{4} \)
- Binomial r.v.'s

Recall Mult. Rule for Counting p509
For \( k \) successive choices, with
\( n_j \) choices at stage \( j \), \( 1 \leq j \leq k \),
the total number of possible successive choices is
\[ n_1 n_2 \cdots n_k \]

# 4-letter "words"?
\[
\begin{array}{cccc}
\times & \times & \times & \times \\
\end{array}
\]

# 4-letter "words" with no repeated letters?
\[
\begin{array}{cccc}
\times & \times & \times & \times \\
\end{array}
\]

= \# "permutations" of size 4 from 26 letters.

5.3

Permutations and Combinations
(Line-ups and Committees)
30 people
Q1. \# ways to make a line-up of 4?
(# permutations of size 4)
\[
\begin{array}{cccc}
\times & \times & \times & \times \\
\end{array}
\]
\[
= \frac{30!}{26!}
\]

5.4
Q2. \# possible committees of size 4?
\(\binom{30}{4}\)
# line-ups = \# committees \times ________
so
\[
\frac{30!}{26!} = \# committees \times ________
\]
\[
\frac{30!}{26!} = \# committees \times \frac{5!}{1!}
\]
$\frac{30!}{26!} = \# committees \times \frac{5!}{1!}$

How many unordered 5-card hands?

How many are spade flushes?

How many are flushes?

5.7

\[
P\{\text{Ace-high straight}\}
= P\{\text{Hand has 10, J, Q, K, A}\}
= \_
\]

\[
P\{\text{straight}\}
= \_
\times \text{above}
\]
How many routes from A to B?
(always going east or north)

5.9

How many arrangements of 
\text{eeennnnn}?

Binomial Distribution

Roll a die 4 times.
\[ \Omega = \{\text{aaaa, aaan, ..., nnnn}\} \]
(or \[ \Omega = \{1111, 1112, ..., 6666\} \])

\[ X = \# \text{aces in 4 rolls} \]

\[ P\{X = 0\} = P\{0 \text{ aces}\} \]

\[ = \]

5.11

\[ P\{X = 1\} = P\{1 \text{ ace}\} \]

\[ = P\{\text{annn, nann, nnan, nna}\} \]

\[ = \]
\[ P\{X = 2\} = P\{2 \text{ aces}\} = P\{\text{aann, anan, ..., nnaa}\} = \]

\[ P\{X = 3\} = P\{3 \text{ aces}\} = P\{\text{aana, anaa, anaa, naaa}\} = \]

\[ P\{X = 4\} = P\{4 \text{ aces}\} = P\{\text{aaaa}\} = \]

---

5.13

5.15

If \( X = \# \text{"successes"} \) in \( n \) independent S-F trials, where \( p = \text{prob. of success in each trial} \), then \( X \sim \text{Binomial}\left(n, p\right) \),

\[ P\{X = k\} = \binom{n}{k} p^k q^{n-k} \]

where \( k = 0, 1, 2, \ldots, n \) and \( q = 1 - p \).
\[ X = \# \text{Successes} \]

\[ \text{in } n \text{ independent trials (i.e., } S-F) \]

is Binomial \((n, p)\) r.v.

\[ P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \]

\[ k = 0, 1, \ldots, n \]

\[ a, \quad p^1 = b \]

\[ n = \{ b b b b b b b \ldots \} \text{ or } \{ w w w w w \} \]

\[ P(X = 0) = P \{ \text{Sequence (a), \& \, (b) \} \} \]

\[ \text{Equally likely?} \]

\[ \text{Random sample (a) with repl. (b) without repl.} \]

\[ 3 \text{ black} \]

\[ 4 \text{ balls, one-by-one} \]

\[ \text{Third Time (Session 5)} \]

\[ \text{Binomial r.v.'s} \]

\[ \text{Normal distn. (for approximating Bernoulli prob's)} \]

\[ \text{This time (Session 6)} \]

\[ \text{More Binomial's} \]

\[ \text{Normal distn's.} \]

\[ \text{for approximating Bernoulli prob's} \]
\[ P(X=1) = P \{ bbbw, bbwb, bwbb, wbbb \} = \binom{4}{1} P \{ bbbw \} \text{ in (a) or (b)} = \binom{4}{4}(.7)^3 = .0756 \text{ in (a)} = \binom{4}{4} \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} \cdot \frac{3}{4} = .0333 \text{ in (b)} \]

Likewise,
\[ P(X=2) = \binom{4}{2} P \{ bbww \} \]
\[ P(X=3) = \binom{4}{3} P \{ bwww \} \]
\[ P(X=4) = \binom{4}{4} P \{ wwww \} \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{X} & 0 & 1 & 2 & 3 & 4 \\
\hline
\text{(a)} \ w/c & .008 & .076 & .265 & .412 & .240 \\
\text{(b)} \ w/o & 0 & .033 & .300 & .500 & .166 \\
\hline
\end{array}
\]

In (b)
\[ P(X=k) = \binom{4}{k} \left( \frac{3}{10} \right)^{k-1} \left( \frac{7}{10} \right)^{4-k} \]

Draw balls until 3rd black draw.
\[ T = \# \text{ draws needed. \ P(T=5)} = \]

\[
\begin{array}{|c|c|}
\hline
\text{7 white} & \text{3 black} \\
\hline
\end{array}
\]
If \( X \sim \text{Binomial}(n=13, p=\frac{1}{2}) \), then \( Y = 13 - X \) is \( \text{Binomial}(n=\_\_, p=\_\_) \) and \( P\{X=4\} = P\{Y=\_\_\} \).

Ratios of consecutive probabilities:

For \( X \sim \text{Binomial}(n, p) \); \( 0 < k < n \),

\[
P\{X=k\} = \frac{n!}{k!(n-k)!} p^k q^{n-k}
\]

\[
P\{X=k-1\} = \frac{n!}{(k-1)!(n-k+1)!} p^{k-1} q^{n-k+1}
\]

\[
= \left[ \frac{n-k+1}{k} \right] \frac{p}{q}
\]

\( \downarrow \) as \( k \uparrow \)

For \( X \sim \text{Binomial}(n, p) \),

\( P\{X=k\} \uparrow \) in \( k \) for \( k < np \)

\( \downarrow \) in \( k \) for \( k > np \)

So maximum attained by one of the integers to either side of \( np \).

\( p \approx .6 \)

\( \cdot \)

Page 87
The normal distribution is a continuous probability distribution that is symmetric about the mean, showing that data near the mean are more frequent in occurrence than data far from the mean. It is often used to model natural phenomena such as the distribution of heights or IQ scores. The normal distribution is defined by two parameters: the mean (μ) and the standard deviation (σ).

The probability density function of the normal distribution is given by:

$$ f(x; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} $$

Where:
- $x$ is the random variable
- $\mu$ is the mean of the distribution
- $\sigma$ is the standard deviation of the distribution
- $\pi$ is the mathematical constant approximately equal to 3.14159
- $e$ is the base of the natural logarithm, approximately equal to 2.71828

The standard normal distribution is a normal distribution with mean 0 and standard deviation 1. The area under the standard normal curve is exactly 1, and the total area under the normal curve is all the way from negative to positive infinity.
Figure 2. A histogram for heights of women compared to the normal curve. The area under the histogram between 60 inches and 66 inches (the percentage of women within one SD of average with respect to height) is about equal to the area between -1 and +1 under the curve—95%.

Key Fact:
Any normal distribution is standard normal on the standard units scale.

\[ z = \frac{x - \mu}{\sigma} \] that \( x \) is above \( \mu \).