1. Given any variables $X_1, \ldots, X_n$, the best linear prediction for a random variable $X$ is the linear combination $\sum_{i=1}^{n} a_i X_i$ that minimizes the mean-squared error (assuming the means of all variables are 0),

$$E(X - \sum_{i=1}^{n} a_i X_i)^2 = Var(X) - 2 \sum_{i=1}^{n} a_i Cov(X, X_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j Cov(X_i, X_j)$$

Show that the partial derivative of the MSE with respect to $a_i$ is

$$-2C(X, X_i) + 2 \sum_{j=1}^{n} Cov(X_i, X_j) a_j. \quad (1)$$

These $n$ equations can be written in matrix notation

$$Ra = b \quad (2)$$

where $R$ is an $n \times n$ matrix whose $(i, j)$th element is $Cov(X_i, X_j)$, i.e.,

$$R = (Cov(X_i, X_j)), \text{ and}$$
$$b = (Cov(X, X_1), \ldots, Cov(X, X_n))^\prime.$$
$$a = (a_1, \ldots, a_n)^\prime.$$

2. Let us apply the result to a stationary sequence $Y_t$ with mean 0. Write the best linear predictor for $Y_{n+1}$ given $Y_n, \ldots, Y_1$ as

$$\hat{Y}_{n+1} = \sum_{i=1}^{n} \phi_{n,i} Y_{n+1-i}.$$

Write

$$\phi_n = (\phi_{n,1}, \ldots, \phi_{n,n})^\prime \text{ note this is a vector}$$
$$R_n = (\gamma(i-j))_{n \times n}$$
$$k_n = (\gamma(1), \ldots, \gamma(n))^\prime.$$

By (2),

$$\phi_n = R_n^{-1} k_n. \quad (3)$$

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You can use this to obtain the partial autocorrelation function (PACF) $\phi_{n,n}$, which is the last element of $\phi_n$.

3. The Durbin-Levinson Algorithm. There is an iterative way to get the PACF. It goes as follows.

$$\phi_{11} = \gamma(1)/\gamma(0); \quad v_1 = \gamma(0)(1 - \phi_{11}^2).$$

For $n \geq 1$

$$\phi_{n+1,n+1} = (\gamma(n + 1) - \sum_{j=1}^{n} \phi_{n,j}\gamma(n + 1 - j))/v_n,$$

$$\phi_{n+1,j} = \phi_{n,j} - \phi_{n+1,n+1}\phi_{n,n-j+1}, \quad j = 1, 2, \ldots, n,$$

or

$$\begin{pmatrix} \phi_{n+1,1} \\ \vdots \\ \phi_{n+1,n} \end{pmatrix} = \begin{pmatrix} \phi_{n,1} \\ \vdots \\ \phi_{n,n} \end{pmatrix} - \phi_{n+1,n+1} \begin{pmatrix} \phi_{n,n} \\ \vdots \\ \phi_{n,1} \end{pmatrix}, \quad \text{and}$$

$$v_{n+1} = v_n(1 - \phi_{n+1,n+1}^2)$$

Proof is simple by using block matrix. Write $b_n = (\gamma(n), \ldots, \gamma(1))'$. Then

$$R_{n+1} = \begin{pmatrix} R_n & b_n \\ b_n' & \gamma(0) \end{pmatrix}.$$ 

The inverse of this block matrix is given by

$$R_{n+1}^{-1} = \begin{pmatrix} R_n^{-1} + c_n R_n^{-1} b_n b_n' R_n^{-1} & -c_n R_n^{-1} b_n \\ -c_n b_n' R_n^{-1} & c_n \end{pmatrix}$$

where $c_n = (\gamma(0) - b_n' R_n^{-1} b_n)^{-1}$.

By (3), we have

$$\phi_{n+1} = R_{n+1}^{-1} k_{n+1}. \quad (4)$$

The proof is completed by applying the inverse given in the previous page and the fact that

$$c_n = 1/v_n, \quad n \geq 1. \quad (5)$$

The last equation can be shown by induction.
1. Prove equation (1).

2. Prove equation (5) and use equation (4) to complete the proof of the Durbin-Levinson algorithm. Hint: You just need to show that $c_1 = 1/v_1$ and

$$1/c_{n+1} = (1 - \phi_{nn}^2)/c_n$$

for $n > 1$.

3. Write a computer program to calculate the PACF of an MA($q$) model up to the lag of 50 by applying equation (3). You computer program should take the MA coefficients $\theta_1, \cdots, \theta_q$ as the only input and your output should be the 50 PACFs.

4. Do the same thing in the previous problem by applying the Durbin-Levinson algorithm. Report the number of lines in your code.