Outline on Chapter 6

Experiments with Two Crossed Treatment Factors

6.1 Introduction

Two factors are crossed if all combinations of the two factors are used in the experiment, though it is possible that there are missing values for some of the combinations. One advantage of the factorial design is that fewer sample data may be needed. It becomes possible when the means are assumed to have some additive structure.

Example (Battery Experiment) Two treatment factors: Duty (two levels: alkaline and heavy duty, coded as 1 and 2, respectively) and Brand (two levels: name brand and store brand, coded as 1 and 2, respectively). The two factors are crossed. The four treatment combinations were coded as 1, 2, 3, and 4, and one-way model was used in previous chapters to analyze the effects of the treatments.

If we code the treatment combinations as 11, 12, 21 and 22, the same one-way model can be equivalently written as

\[ Y_{ijt} = \mu_{ij} + \epsilon_{ijt}, i = 1,2, j = 1,2, t = 1,\ldots, n \]

where \( \epsilon_{ijt} \) are i.i.d \( \mathcal{N}(0, \sigma^2) \). This is also called a cell-mean model.

In this chapter, we investigate the contributions that each of the factors individually makes to the response. We could either have a 2-way complete model or a 2-way main effects model (if it is assured that this is no interaction).

6.2. Models and Factorial Effects

6.2.1 The Meaning of Interaction

If the change in the level of one factor have the same effect on the responses from the other factor, there is *no interaction* between the two factors. Otherwise, there is an interaction.

When there is no interaction, we see approximately parallel lines in interaction plots.

When there is no interaction, we can write

\[ \mu_{ij} = \mu + \alpha_i + \beta_j, i = 1,\ldots, a, j = 1,\ldots, b \]

where \( \sum_i \alpha_i = 0, \sum_j \beta_j = 0 \).

When there is interaction, we can write
\[ \mu_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b \]

where \( \sum_i \alpha_i = 0 \) and \( \sum_j \beta_j = 0 \).

Example of No Interationd (left plot above)
Example of Interaction (right plot above)

Decomposition: \( \alpha_i = \mu_i - \bar{\mu}, \beta_j = \mu_j - \bar{\mu}, \)
\( (\alpha\beta)_{ij} = \mu_{ij} + \bar{\mu} - \bar{\mu}_i - \bar{\mu}_j. \)

When there is no interaction, \( (\alpha\beta)_{ij} = 0 \) for all \( i \) and \( j \).

### 6.2.2 Models for Two Treatment Factors

The two way complete model (or the two way ANOVA model):
\[
Y_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijt}, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b, \quad t = 1, \ldots, r_{ij}.
\]

where \( \alpha_i \) is the effect on the response due to the \( i \)th treatment of factor A; \( \beta_j \) is the effect on the response due to the \( j \)th treatment of factor B; \( (\alpha\beta)_{ij} \) is the extra effect due to the \( i \)th level of factor A and the \( j \)th level of factor B.

When there is a reasonable certainty that the two factors do not interact, the two-way main effects model is applied: \( Y_{ijt} = \mu + \alpha_i + \beta_j + \epsilon_{ijt}, i = 1, \ldots, a, \quad j = 1, \ldots, b \).

### 6.2.3 ANOVA for the Two-Way Complete Model

The three standard hypotheses that are usually examined with a two-way complete model:
I. There is no interaction, or equivalently interaction plots show parallel lines, or symbolically, \( H_0^{AB}: (\alpha \beta)_{ij} - (\alpha \beta)_{ij} = (\alpha \beta)_{sj} - (\alpha \beta)_{sq} \) for all \( i \neq j \). or equivalently: \( (\alpha \beta)_{ij} = 0, \) for all \( i \) and \( j \).

II. The levels of \( A \) have the same average effect (averaged over the levels of \( B \)) on the response, i.e., \( H_0^A: \alpha_i^* = \alpha_2^* = \ldots = \alpha_a^* \), where \( \alpha_i^* = \alpha_i + (\alpha \beta)_i \).

III. The levels of \( B \) have the same average effect (averaged over the levels of \( A \)) on the response, i.e., \( H_0^B: \beta_j^* = \beta_2^* = \ldots = \beta_b^* \), where \( \beta_j^* = \beta_j + (\alpha \beta)_j \).

Each of these three null hypotheses results in a reduced model, and the reduced model will have an increased SSE. The differences between the SSEs of the reduced models and the complete model, denoted by ssAB, ssA and ssB respectively, is attributed to the interaction or a factor, resulting in F-tests for the hypotheses.

The **Type III sums of squares** are the values of ssA, ssB and ssAB. They are used for the three hypotheses whether or not the sample sizes (replicates) are equal.

The **Type I sums of squares** are the sequential sums of squares, and are used when the sample sizes are equal. In this case, the two types of sums of squares are identical, and the total sum of squares is decomposed into

\[
\text{sstot} = \text{ssA} + \text{ssB} + \text{ssAB} + \text{ssE}.
\]

The Two-way ANOVA table of page 156.

6.3. Estimation and Contrasts

6.3.1 Least Squares Estimators

The least squares estimator for the treatment mean \( \mu_{ij} \) is \( \bar{y}_{ij} \), as in a one-way model.

6.3.2. Contrasts and Inferences for Contrasts

Interaction Contrast:
Main Effects Contrasts: A 100(1- \( \alpha \ ))% confidence interval for a single contrast is of the form

\[
\text{estimate} \pm t_{\alpha/2, df} \ (\text{std error of estimate}), \ \text{with df=n-ab}.
\]
For multiple contrasts, the $100(1- \alpha )\%$ simultaneous confidence intervals are of the form

$$\text{estimate} \pm w \times (\text{std error of estimate}),$$

where $w$ varies with each method.

For example, for comparing main effects of factor A, use the $w$ values provided in the following table:

<table>
<thead>
<tr>
<th>Method</th>
<th>Bonferroni</th>
<th>Scheffe</th>
<th>Tukey</th>
<th>Dunnett</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>$t_{n-ab,a/(2m)}$</td>
<td>$(a-1) F_{a-1,n-ab,a}$</td>
<td>$\frac{q_{a,n-ab,a}}{\sqrt{2}}$</td>
<td>needs the multivariate t-distribution</td>
</tr>
<tr>
<td>Table</td>
<td>A.4, p704</td>
<td>F value in A.6,p706</td>
<td>q value in A.8, p718</td>
<td>A.9, p720</td>
</tr>
</tbody>
</table>