Using Fourier Methods for Estimating Central Subspace and Central Mean Subspace

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Outline

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Introduction

- Response $Y \in \mathbb{R}$ and predictor $X \in \mathbb{R}^p$

$$Y = g(X, \varepsilon)$$

  where $X \perp\!\!\!\perp \varepsilon$, and $E[\varepsilon] = 0$.

- High-dimensional nonparametric regression
- Curse of dimensionality

Structural Nonparametric Approach

Assume some structures on regression function in order to mitigate the curse of dimensionality.

- Projection pursuit regression (Friedman and Stuetzle 1981)

$$Y = g_1(\beta_1^\tau X) + g_2(\beta_2^\tau X) + \cdots + g_M(\beta_M^\tau X) + \varepsilon$$

- Single index model

$$Y = g(\beta^\tau X) + \varepsilon$$

- Multiple index model $(1 < k < p)$

$$Y = g(\beta_1^\tau X, \beta_2^\tau X, \ldots, \beta_k^\tau X) + \varepsilon$$

Regression Graphics Approach

Identify a subspace of X that can *completely* capture the information of X about Y *without estimating the regression function* of Y on X

$\Downarrow$

Visualization

$\Downarrow$

Parametric or Nonparametric Regression of Y on X in the subspace

Three Equivalent Models

regression (Li 1991)	$Y = g(B^\tau X, \varepsilon)$
conditional independence (Cook 1994, 1998)	$Y \perp\!\!\!\perp X \mid B^\tau X$
conditional distribution	$F(Y \mid X) = F(Y \mid B^\tau X)$

where $B = (\beta_1, \ldots, \beta_k)$ is a $p \times k$ matrix, $k < p$,
$g(\cdot)$ is an unknown function, $X \perp\!\!\!\perp \varepsilon$, $E[\varepsilon] = 0$
$F(\cdot \mid \cdot)$ is conditional distribution.

$\mathcal{S}(B)$, space spanned by the columns of B, is called a *dimension reduction subspace* (DRS).

Central Subspace

- DRS is not unique. For any two spaces $\mathcal{S} \subseteq \mathcal{S}'$,

$$\mathcal{S} \text{ is a DRS } \Rightarrow \mathcal{S}' \text{ is a DRS}$$

- *Central subspace* (CS), denoted by $\mathcal{S}_{Y|X}$, is a DRS such that

$$\mathcal{S}_{Y|X} = \bigcap_{\text{all DRS}} \mathcal{S}$$

- $\mathcal{S}_{Y|X}$ is unique and the minimal DRS if it exists.
- Let $Z = \Sigma^{-1/2}(X - E[X])$, where $\Sigma = cov(X)$, then

$$\mathcal{S}_{Y|X} = \Sigma^{-1/2}\mathcal{S}_{Y|Z}$$

- In the following, we assume $E[X] = 0$ and $cov[X] = I_p$.

Example

Suppose $X = (X_1, \ldots, X_5)^\tau \in \mathbb{R}^5$. Consider model

$$\begin{aligned} Y &= X_1 + (X_1 + X_3)^2 + \varepsilon\, X_4 \\ &= g(\beta_1^\tau X, \beta_2^\tau X) + \varepsilon\, h(\beta_3^\tau X) \end{aligned}$$

where $\varepsilon \perp\!\!\!\perp X$ and $E[\varepsilon] = 0$,

$$\begin{aligned} \beta_1 &= (1, 0, 0, 0, 0)^\tau \\ \beta_2 &= (1, 0, 1, 0, 0)^\tau \\ \beta_3 &= (0, 0, 0, 1, 0)^\tau \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{S}_{Y|X} &= span\{\beta_1, \beta_2, \beta_3\} \\ \mathcal{S}_{E[Y|X]} &= span\{\beta_1, \beta_2\} \end{aligned}$$

Central Mean Subspace

regression (Li 1992)	$E[Y \mid X] = h(B^\tau X)$
conditional independence (Cook and Li 2002)	$Y \perp\!\!\!\perp E[Y \mid X] \mid B^\tau X$

where B is a $p \times k$, $k < p$, $h(\cdot)$ is an unknown function.

$\mathcal{S}(B)$ is a *mean dimension reduction subspace* (MDRS).

Central mean subspace (CMS), denoted by $\mathcal{S}_{E[Y|X]}$, is a MDRS such that

$$\mathcal{S}_{E[Y|X]} = \bigcap_{\text{all MDRS}} \mathcal{S}$$

Note that $\mathcal{S}_{E[Y|X]} \subseteq \mathcal{S}_{Y|X}$.

A Common Approach for Estimating CS/CMS

- Population version:
  - Find a matrix M depending on X and Y, such that

$$\mathcal{S}(M) \subseteq \mathcal{S}_{Y|X} \quad (\text{or } \mathcal{S}_{E[Y|X]})$$

- Sample version:

  Given a sample (x_i, y_i), $i = 1, 2, \dots, n$,

  1. find an estimate $\hat{M}$ of M.
  2. perform spectral decomposition of $\hat{M}$.
  3. estimate CS (or CMS) by the space spanned by the eigenvectors corresponding to the k largest eigenvalues.

Existing Methods for Estimating CS

- Sliced inverse regression (SIR; Li 1991). Under linear condition,

$$E[X \mid Y] \in \mathcal{S}_{Y|X}, \qquad M_{\mathrm{SIR}} = cov[E[X \mid Y]].$$

$$\text{linear condition:} \quad E[X \mid B^{\tau} X] = P_B X$$

- Sliced average variance estimate (SAVE; Cook and Weisberg 1991). Under linear condition and constant variance condition,

$$M_{\mathrm{SAVE}} = E[(I_p - cov[X \mid Y])^2].$$

$$\text{constant variance condition:} \quad cov[X \mid B^{\tau} X] = I_p - P_B.$$

Existing Methods for Estimating CMS

- Principal Hessian direction (pHd; Li 1992). Under linear condition and constant variance condition,

$$M_{\text{pHd}} = E[(Y - E[Y])XX^{\tau}].$$

  Note that $M_{\text{pHd}} = E\big(\frac{\partial^2}{\partial X \partial^{\tau} X} E[Y \mid X]\big)$ when X is normal.

- Iterative Hessian transformation (IHT; Cook and Li 2002). Under linear condition and constant variance condition,

$$M_{\text{IHT}} = (\beta, M_{\text{pHd}}\beta, M_{\text{pHd}}^2\beta, \ldots, M_{\text{pHd}}^{p-1}\beta)$$

  where $\beta = E[YX]$.

Some Methods from Structural Nonparametric Approach

Need to estimate $\frac{\partial}{\partial X}E[Y \mid X]$.

$$\frac{\partial}{\partial X}E[Y \mid X] = B\frac{\partial}{\partial(B^\tau X)}E[Y \mid B^\tau X] \in \mathcal{S}_{E[Y|X]}$$

Hristache, Juditsky, Polzehl and Spokoiny (2001)

Local linear approximation of regression function at point X_0,

$$E[Y \mid X] \approx a + d^\tau(X - X_0).$$

Xia, Tong, Li, and Zhu (2002)

Local linear approximation of regression function at point X_0,

$$E[Y \mid X] \approx a + b^\tau B^\tau(X - X_0).$$

Comparison between Two Approaches

	regression graphics	structural nonparam.
estimate regression function	No	Yes
distributional assumption	Yes	No
explicit estimate	Yes	No
need iterative algorithm	No	Yes
targeted subspace	CS/CMS	CMS

Limitation of Existing Methods

Limitation of existing methods (Regression Graphics Approach)

- Need distributional assumptions on X
- No theoretical result on when the entire CS/CMS can be recovered

Next: population versions of FMM and FMC.

Assume all the density functions exist and are differentiable. The model is

$$f_{Y|X}(y \mid x) = f_{Y|B^\tau X}(y \mid B^\tau x) = h(y; B^\tau x)$$

where f is the conditional density function.

Estimate CMS

Heuristics: Find some vectors that belong to CMS, and let them span the whole CMS.

- $m(X) = E[Y \mid X]$ is a function of $U = B^\tau X$, then

$$\frac{\partial}{\partial X} m(X) = B \frac{\partial}{\partial U} m(U) \in \mathcal{S}_{E[Y|X]}$$

- For any $\omega \in \mathbb{R}^p$,

$$\begin{aligned} \psi(\omega) &= \int \exp\{\imath\, \omega^\tau X\} \left(\tfrac{\partial}{\partial X} m(X)\right) f_X(X)\, dX \\ &= -E_{(X,Y)}\left[Y(\imath\omega + G(X)) \exp\{\imath\, \omega^\tau X\}\right] \quad \in \mathcal{S}_{E[Y|X]} \end{aligned}$$

where f_X is the density function of X, and $G(X) = \frac{\partial}{\partial X} \log f_X(X)$.

FMM

Because $\psi(\omega) \in \mathcal{S}_{E[Y|X]}$, then

$$\mathcal{S}(\psi(\omega)\bar{\psi}(\omega)^\tau) \subseteq \mathcal{S}_{E[Y|X]}, \quad \text{for any } \omega \in \mathbb{R}^p$$

Proposition 1. *Define matrix*

$$M_{FMM} = \int \psi(\omega)\bar{\psi}(\omega)^\tau K(\omega)\, d\omega$$

where $K(\omega)$ is a positive weight function on $\mathbb{R}^p$. Then M_{FMM} is non-negative definite, and

$$\mathcal{S}(M_{FMM}) = \mathcal{S}_{E[Y|X]}$$

FMM When $K(\omega)$ is Gaussian Function

- Suppose $K(\omega) = (2\pi\sigma_\omega^2)^{-p/2} \exp\{-\|\omega\|^2/2\sigma_\omega^2\}$, then

$$M_{\mathrm{FMM}} = E\left[Y_1 Y_2\, e^{-\frac{\sigma_\omega^2}{2}\|X_{12}\|^2}\left[\sigma_\omega^2 I_p + (G(X_1) - \sigma_w^2 X_{12})(G(X_2) + \sigma_w^2 X_{12})^\tau\right]\right]$$

  where $X_{12} = X_1 - X_2$.

- σ_ω^2 is different from bandwidth in kernel estimation.
  - Proposition 1 is valid for any σ_ω^2.
- Other weight function can also be used.

Connection Between FMM and ADE

Consider single index model

$$m(X) = E[Y \mid X] = g(\beta^\tau X)$$

The average derivative estimate (ADE; Härdle and Stoker 1989) is

$$E[\tfrac{\partial}{\partial X} m(X)] = -E[YG(X)] \propto \beta$$

- $\psi(\omega)$ is a generalization of ADE.

$$\psi(\omega) = E[\exp\{\imath\, \omega^\tau X\} \left(\tfrac{\partial}{\partial X} m(X)\right)]$$

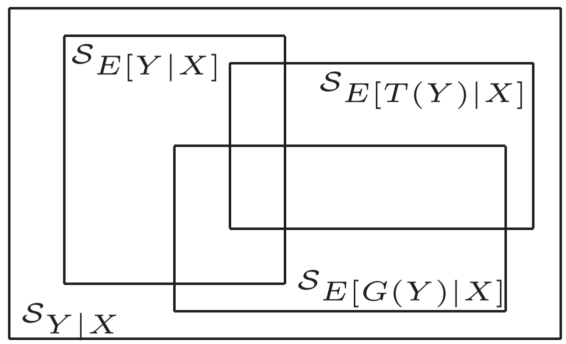
- When $K(\omega)$ is point mass on origin ($\sigma_w^2 = 0$),

$$M_{\text{FMM}} = E\big[YG(X)\big]E\big[YG(X)\big]^\tau$$

In a model with multiple directions, *FMM can estimate multiple directions when $\sigma_\omega^2 \neq 0$, while ADE can only estimate single direction.*

Heuristics on Estimating CS

Relationship between the central subspace and the central mean subspace. $T(Y)$ and $G(Y)$ are two transformations of Y.



It is possible to recover $\mathcal{S}_{Y|X}$ by all possible central mean subspaces.

$$\mathcal{S}_{Y|X} = \sum_{\text{all possible } T} \mathcal{S}_{E[T(Y)|X]}$$

Represent CS by CMSs

- Define transformation,

$$T_t(Y) = \exp\{\imath\, tY\} = \cos(tY) + \imath \sin(tY), \quad t \in \mathbb{R}.$$

- $E[T_t(Y) \mid X]$ is the Fourier transform (characteristic function) of $f_{Y|X}$.

$$m_t(X) = E[T_t(Y) \mid X] = \int \exp\{\imath\, tY\}\, f_{Y|X}(Y \mid X)\, dY.$$

Proposition 2. *CS can be represented as the sum of a family of CMSs.*

$$\mathcal{S}_{Y|X} = \sum_{t\in\mathbb{R}} \mathcal{S}_{E[T_t(Y)|X]} = \sum_{t\in\mathbb{R}} \mathcal{S}_{E[\cos(tY)|X]} + \sum_{t\in\mathbb{R}} \mathcal{S}_{E[\sin(tY)|X]}$$

FMC

Define

$$\begin{aligned}\phi(t,\omega) &= \int \exp\{\imath\,\omega^\tau X\}\left(\tfrac{\partial}{\partial X}\, m_t(X)\right) f_X(X)\, dX \\ &= -E_{(X,Y)}\left[(\imath\omega + \tfrac{\partial}{\partial X}\log f_X(X))\,\exp\{\imath tY + \imath\,\omega^\tau X\}\right] \in \mathcal{S}_{E[Y|X]}\end{aligned}$$

It is obtained by substituting Y by $\exp\{\imath tY\}$ in $\psi(\omega)$.

Proposition 3. *Define matrix*

$$M_{FMC} = \iint \phi(t,\omega)\bar{\phi}(t,\omega)^\tau K(\omega)k(t)\, d\omega dt$$

where $K(\omega)$ and $k(t)$ are positive weight functions. Then M_{FMC} is non-negative definite, and

$$\mathcal{S}(M_{FMC}) = \mathcal{S}_{Y|X}$$

FMC when $K(\omega)$ and $k(t)$ are Gaussian Functions

- When $k(t) = (2\pi\sigma_t^2)^{-1/2}\exp\{-t^2/2\sigma_t^2\}$, and $K(\omega)$ as before,

$$M_{\text{FMC}} = E\left[a_{12}\left[\sigma_\omega^2 I_p + (G(X_1) - \sigma_w^2 X_{12})(G(X_2) + \sigma_w^2 X_{12})^\tau\right]\right]$$

  where $a_{12} = \exp\{-\sigma_t^2(Y_1 - Y_2)^2/2 - \sigma_\omega^2\|X_1 - X_2\|^2/2\}$.

- σ_ω^2 and σ_t^2 are different from bandwidth in kernel estimation.
  - Proposition 3 is valid for any σ_ω^2 and σ_t^2.
- Other weight function can also be used.

Connection between FMC and SIR

- When $K(\omega)$ is point mass on origin ($\sigma_\omega^2 = 0$),

$$M_{\mathrm{FMC}} = \int E[e^{\imath tY} E[G(X) \mid Y]] E[e^{-\imath tY} E[G(X) \mid Y]]^\tau \, k(t) \, dt$$

  Therefore,

$$\begin{aligned}\mathcal{S}(M_{\mathrm{FMC}}) &= span\{E[e^{\imath tY} E[G(X) \mid Y]], t \in \mathbb{R}\} \\ &= span\{E[G(X) \mid Y], Y \in supp(Y)\}\end{aligned}$$

- When X is normal, then $G(X) = -X$,

$$\mathcal{S}(M_{\mathrm{FMC}}) = span\{E[X \mid Y], Y \in supp(Y)\} = \mathcal{S}(M_{\mathrm{SIR}})$$

- *FMC degenerates to SIR when $\sigma_\omega^2 = 0$ and X is normal.*

Summary of Population Versions

In population version, FMC/FMM can

- work for X with almost arbitrary distribution
- recover the *entire* central subspace (central mean subspace)

Next: sample version:

Given a sample (x_i, y_i), $i = 1, 2, \ldots, n$,

- estimate $\hat{M}_{\text{FMC}}$
- asymptotic property

The discussion of FMM is similar and is omitted.

Estimation of $M_{\mathbf{FMC}}$

Note that

$$M_{\text{FMC}} = E\left[a_{12}\left[\sigma_\omega^2 I_p + (G(X_1) - \sigma_w^2 X_{12})(G(X_2) + \sigma_w^2 X_{12})^\tau\right]\right]$$

where $a_{12} = \exp\{-\sigma_t^2 (Y_1 - Y_2)^2/2 - \sigma_\omega^2 \|X_1 - X_2\|^2/2\}$. It can be estimated by sample average, and the only unknown component is

$$G(x_i) = \frac{\partial}{\partial X} \log f_X(x_i) = \frac{\frac{\partial}{\partial X} f_X(x_i)}{f_X(x_i)}$$

How to estimate $G(x_i)$?

- If the density function is known, $G(x_i)$ can be directly calculated.
- If X belongs to a family of distributions indexed by parameter θ, $G(x_i)$ is a function of θ. It can be estimated by substituting θ by its MLE.
- If there is no information about the density function, kernel density estimator can be used to estimate $G(x_i)$.

When X Follows Normal Distribution

- Suppose X is normally distributed with $E[X] = 0$ and $cov[X] = I_p$.

$$G(x_i) = -x_i$$

- We can have an estimates of M_{FMC}.

$$\boxed{\hat{M}_{\text{FMC}} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left[\sigma_\omega^2 I_p + (x_i + \sigma_w^2 x_{ij})(x_j - \sigma_w^2 x_{ij})^\tau \right]}$$

where $a_{ij} = \exp\{-\sigma_t^2 y_{ij}^2/2 - \sigma_\omega^2 x_{ij}^\tau x_{ij}/2\}$, $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$.

Asymptotic Results

By Hoeffding decomposition,

$$\hat{M}_{FMC} = M_{FMC} + \frac{1}{n}\sum_{i=1}^{n}\big(A(x_i, y_i) + A(x_i, y_i)^\tau - 2M_{\text{FMC}}\big) + o_p(n^{-1/2})$$

where

$$A(y_i, x_i) = E\big[a_{12}\left[\sigma_\omega^2 I_p + (x_1 + \sigma_w^2 x_{12})(x_2 - \sigma_w^2 x_{12})^\tau \mid (x_1, y_1) = (x_i, y_i)\right]$$

Proposition 4. *When $n \to \infty$,*

$$\sqrt{n}\,(vec(\hat{M}_{FMC})) - vec(M_{FMC})) \xrightarrow{\mathcal{L}} N(0, \Sigma)$$

where Σ is a positive definite matrix.

General Cases

- Estimate $G(x_i)$ by plugging in kernel estimate

$$\hat{G}(x_i) = \frac{\frac{\partial}{\partial X}\hat{f}_h(x_i)}{\hat{f}_h(x_i)}$$

where

$$\hat{f}_h(x_i) = \frac{1}{nh^p}\sum_{\ell=1}^{n} W\left(\frac{x_i - x_\ell}{h}\right)$$

$$\frac{\partial}{\partial X}\hat{f}_h(x_i) = \frac{1}{nh^{p+1}}\sum_{\ell=1}^{n} W'\left(\frac{x_i - x_\ell}{h}\right)$$

and $W(\cdot)$ is a kernel function, h is the bandwidth.

Estimate in General Cases

We have estimates

$$\hat{M}_{\text{FMC}} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left[\sigma_\omega^2 I_p + (\hat{G}(x_i) - \sigma_w^2 x_{ij})(\hat{G}(x_j) + \sigma_w^2 x_{ij})^\tau \right] \hat{I}_i \hat{I}_j$$

where $a_{ij} = \exp\{-\sigma_t^2 y_{ij}^2/2 - \sigma_\omega^2 x_{ij}^\tau x_{ij}/2\}$, $x_{ij} = x_i - x_j$, and $y_{ij} = y_i - y_j$, $\hat{I}_i = I_{[\hat{f}_h(x_i) > b]}$, $I_{[\cdot]}$ is an indicator function, and b is a threshold.

The technique of using $I_{[\cdot]}$ is called *trimming*. Its purpose is to rule out the cases that the estimated densities are extremely small.

Asymptotic Results

Proposition 5. *Under some regularity conditions, if* $f_X(x)$ *has partial derivatives up to order* $r \geq p+2$, *and*

(1) $n \to \infty$, $h \to 0$, $b \to 0$ *and* $b^{-1}h \to 0$;

(2) for some $\varepsilon > 0$, $b^4 n^{1-\varepsilon} h^{2p+2} \to \infty$;

(3) $nh^{2r-2} \to 0$

then

$$\sqrt{n}\,(vec(\hat{M}_{FMC}) - vec(M_{FMC})) \xrightarrow{\mathcal{L}} N(0, \Sigma)$$

where Σ *is a positive definite matrix.*

Sketch of Proof

- When b is small enough, the effect of trimming is negligible.
- Linearization.

$$\hat{G}(x_i) = \frac{\frac{\partial}{\partial X}\hat{f}_h(x_i)}{\hat{f}(x_i)} = G(x_i) + \frac{\frac{\partial}{\partial X}\hat{f}_h(x_i)}{f(x_i)} - \frac{\hat{f}_h(x_i)}{f(x_i)}G(x_i) + o_p(n^{-1/2})$$

  Then there is no kernel estimator in the denominators.

- Substitute $\hat{G}(x_i)$ by its approximation in $\hat{M}$, and write it as a U-statistic.
- Asymptotic normality is obtained by Hoeffding decomposition and central limit theorem.

Intuition for $\sqrt{n}$ Convergence Rate

- Analog to semi-parametric model.
  - central subspace (central mean subspace) is essentially the parametric part in a semi-parametric model.
- Under some conditions, $\sqrt{n}$ convergence rate is possible for the estimate of an integral functional of density and its derivatives

$$\int T(X, f(X), f'(X), \dots, f^{(v)}(X))\, dX$$

  See Bickel and Ritov (1988), Birge and Massart (1995).
- Although individual $\hat{G}(x_i)$ cannot achieve $\sqrt{n}$ convergence rate, their average can converge more quickly. Same phenomenon was observed in Härdle and Stoker (1989).

Generic Estimation Procedure

Suppose we have observations (x_i, y_i), $i = 1, \ldots, n$.

1. Specify parameters: k, σ_ω^2, σ_t^2, and h, if applicable.
2. Standardize data by $\tilde{x}_i = \hat{\Sigma}^{-1/2}(x_i - \bar{x})$ and $\tilde{y}_i = (y_i - \bar{y})/s_y$
3. Calculate an estimate $\hat{M}$ of M_{FMC} (or M_{FMM}) using data $(\tilde{x}_i, \tilde{y}_i)$.
4. Perform spectral decomposition of $\hat{M}$. The eigenvalues are $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_p \geq 0$, and their corresponding eigenvectors are $\hat{\gamma}_1, \ldots \hat{\gamma}_p$.
5. Estimate $\mathcal{S}_{Y|X}$ (or $\mathcal{S}_{E[Y|X]}$) by $\hat{\mathcal{S}} = span\{\hat{\Sigma}^{-1/2}\hat{\gamma}_1, \ldots, \hat{\Sigma}^{-1/2}\hat{\gamma}_k\}$.

Discrepancy Measure

- If A and B are two matrices, $P_A = A(A^\tau A)^- A^\tau$ and $P_B = B(B^\tau B)^- B^\tau$ are projection matrices, then

$$r^2 = \frac{1}{k} trP_A P_B$$

where $k = rank(A) = rank(B)$, and r is called the trace correlation.

- Define

$$D(\hat{\mathcal{S}}, \mathcal{S}) = 1 - r$$

- $0 \leq D(\hat{\mathcal{S}}, \mathcal{S}) \leq 1, \quad D(\hat{\mathcal{S}}, \mathcal{S}) = 0$, if $\hat{\mathcal{S}} = \mathcal{S}, \quad D(\hat{\mathcal{S}}, \mathcal{S}) = 1$, if $\hat{\mathcal{S}} \perp \mathcal{S}$.

Choose Tuning Parameters

- k: assume known.
- $\sigma_\omega^2, \sigma_t^2$: recommend $\sigma_\omega^2 = 0.1$ and $\sigma_t^2 = 1.0$.
- h: use optimal bandwidth given by kernel density estimator.

or

- Choose the parameters that minimize $D(\hat{\mathcal{S}}, \mathcal{S})$.
- If $\mathcal{S}$ is unknown, bootstrap procedure can be used (Ye and Weiss 2003).

Example 1

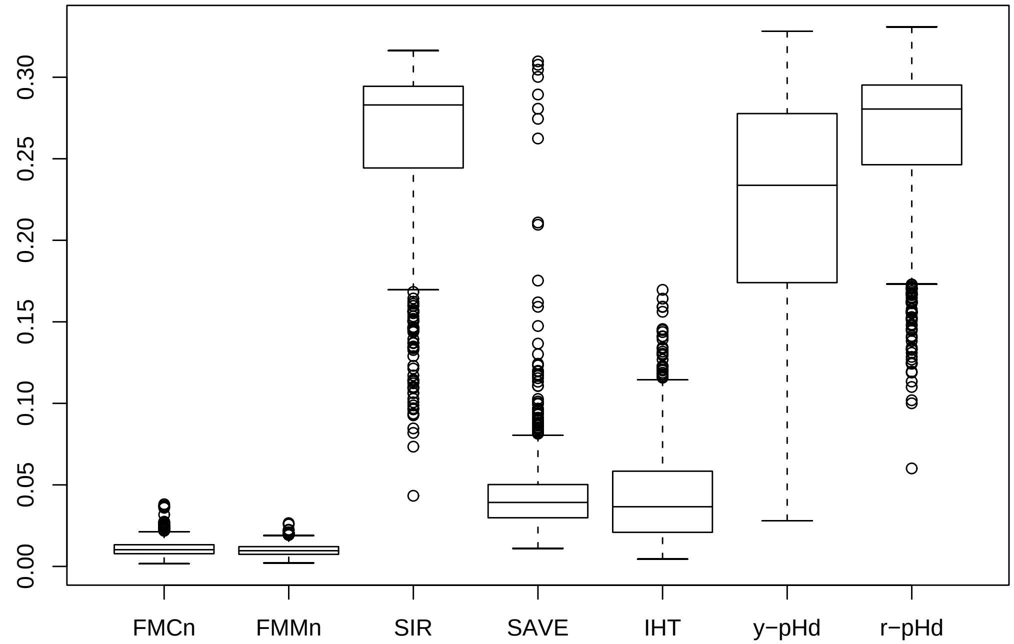
Consider model

$$Y = 2(\beta_1^\tau X) + (\beta_2^\tau X)^2 + \varepsilon$$

where $X \sim N(0, I_{10})$. $\varepsilon \perp\!\!\!\perp X$ and $\varepsilon \sim N(0,1)$. Sample size is 500.

$$\beta_1 = (1,1,1,0,0,0,0,0,0,0)^\tau$$
$$\beta_2 = (0,0,0,0,0,0,0,1,1,1)^\tau$$

1000 replicates are used to construct boxplots.



FMCn: $\sigma_\omega^2 = 0.1$, $\sigma_t^2 = 1.0$. FMMn: $\sigma_\omega^2 = 0.1$.

SIR, SAVE: $H = 5$.

Example 2: Heteroscedastic Model

Consider model

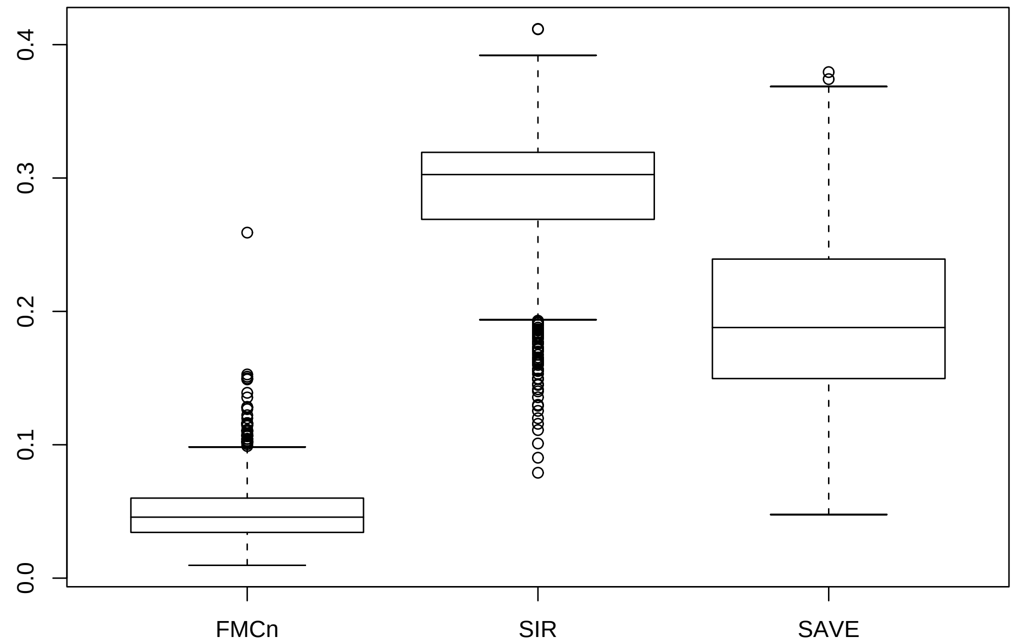
$$Y = (\beta_1^\tau X) + 4(\beta_2^\tau X)\,\varepsilon$$

where $X \sim N(0, I_{10})$. $\varepsilon \perp\!\!\!\perp X$ and $\varepsilon \sim N(0,1)$. Sample size is 500.

$$\beta_1 = (1,1,1,0,0,0,0,0,0,0)^\tau$$

$$\beta_2 = (0,0,0,0,0,0,0,1,1,1)^\tau$$

1000 replicates are used to construct boxplots.



FMCn: $\sigma_\omega^2 = 0.1$, $\sigma_t^2 = 1.0$. SIR, SAVE: $H = 5$.

Example 3: Y is Discreet

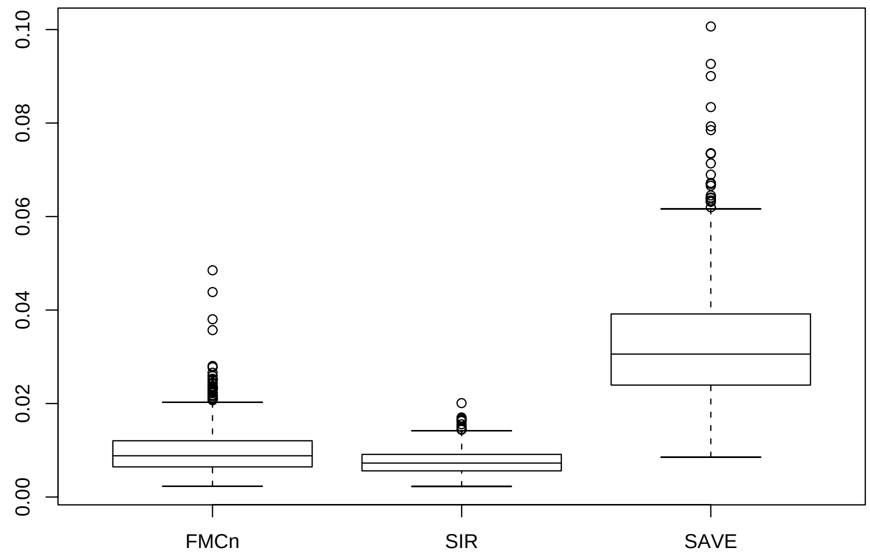
Consider model

$$Y = I_{[\beta_1^\tau X + 0.2\varepsilon_1 > 1]} + 2I_{[\beta_2^\tau X + 0.2\varepsilon_2 > 0]}$$

where $X \sim N(0, I_{10})$. $\varepsilon \perp\!\!\!\perp X$ and $\varepsilon \sim N(0,1)$. $I_{[\cdot]}$ is indicator function, and $Y = 0, 1, 2, 3$. Sample size is 500.

$$\beta_1 = (1,1,1,0,0,0,0,0,0,0)^\tau$$
$$\beta_2 = (0,0,0,0,0,0,0,1,1,1)^\tau$$

1000 replicates are used to construct boxplots.



FMCn: $\sigma_\omega^2 = 0.1$, $\sigma_t^2 = 1.0$. SIR, SAVE: $H = 5$.

Example 4: Mixture Normal

Consider model

$$Y = \frac{\beta_1^\tau X}{3 + (2 + \beta_2^\tau X)^2} + 0.1\varepsilon$$

where $X \sim 0.5N(-d\beta_3, I_{10}) + 0.5N(d\beta_3, I_{10})$, $\varepsilon \sim N(0,1)$ and $\varepsilon \perp\!\!\!\perp X$.
Sample size is 500.

$$\beta_1 = (1,1,1,0,0,0,0,0,0,0)^\tau$$
$$\beta_2 = (0,0,0,0,0,0,0,1,1,1)^\tau$$
$$\beta_3 = (0,0,0,0,1,1,0,0,0,0)^\tau$$

Let $d = 0, 1, 2, \ldots, 10$.

1000 replicates are used to compare the performance of different methods.

Mean and standard deviation of 1000 replicates for each case

	mean of $D(\hat{\mathcal{S}}, \mathcal{S})$'s				std. dev. of $D(\hat{\mathcal{S}}, \mathcal{S})$'s			
d	0	3	7	10	0	3	7	10
SIR (s)	0.025	0.040	0.079	0.094	0.012	0.042	0.077	0.082
SAVE (a)	0.147	0.589	0.656	0.643	0.048	0.308	0.316	0.318
IHT (i)	0.068	0.150	0.240	0.253	0.028	0.071	0.065	0.061
y-pHd (y)	0.083	0.274	0.375	0.394	0.039	0.125	0.142	0.143
r-pHd (r)	0.025	0.104	0.263	0.290	0.010	0.087	0.122	0.126
FMC-n (c)	0.020	0.018	0.018	0.018	0.009	0.009	0.009	0.008
FMM-n (m)	0.031	0.028	0.028	0.028	0.015	0.014	0.013	0.015
FMC-k (k)	0.088	0.129	0.185	0.191	0.063	0.083	0.082	0.082
FMM-k (q)	0.022	0.020	0.020	0.020	0.013	0.012	0.014	0.013

SIR, SAVE: $H = 5$. FMCn: $\sigma_\omega^2 = 0.1$, $\sigma_t^2 = 1.0$. FMMn: $\sigma_\omega^2 = 0.1$.

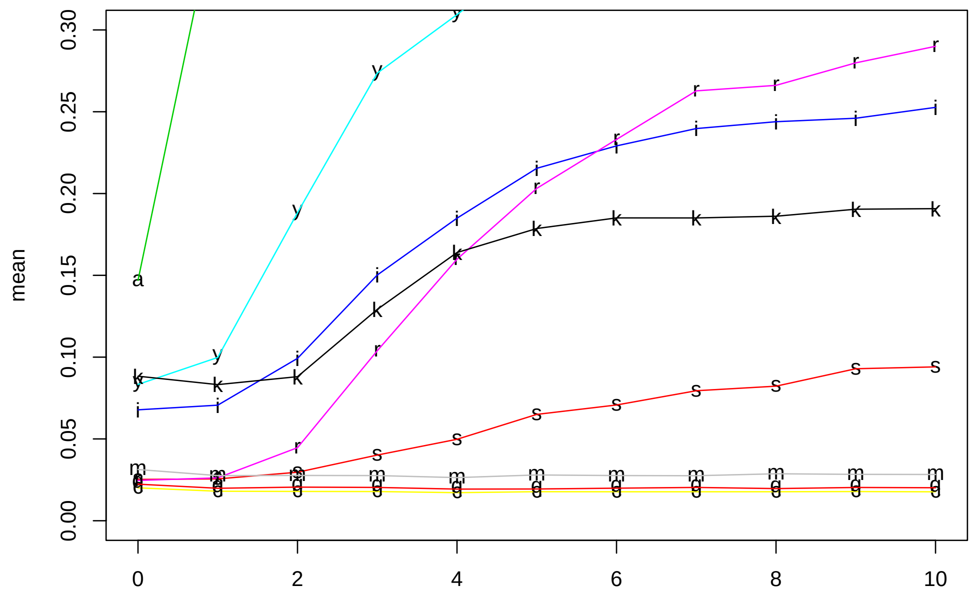
FMCk: $\sigma_\omega^2 = 0.1$, $\sigma_t^2 = 1.0$, $h = 1.0$. FMMk: $\sigma_\omega^2 = 0.1$, $h = 1.0$.

The best three:

FMC-n (c),

FMM-k (q),

FMM-n (m)



The best three:

FMC-n (c),

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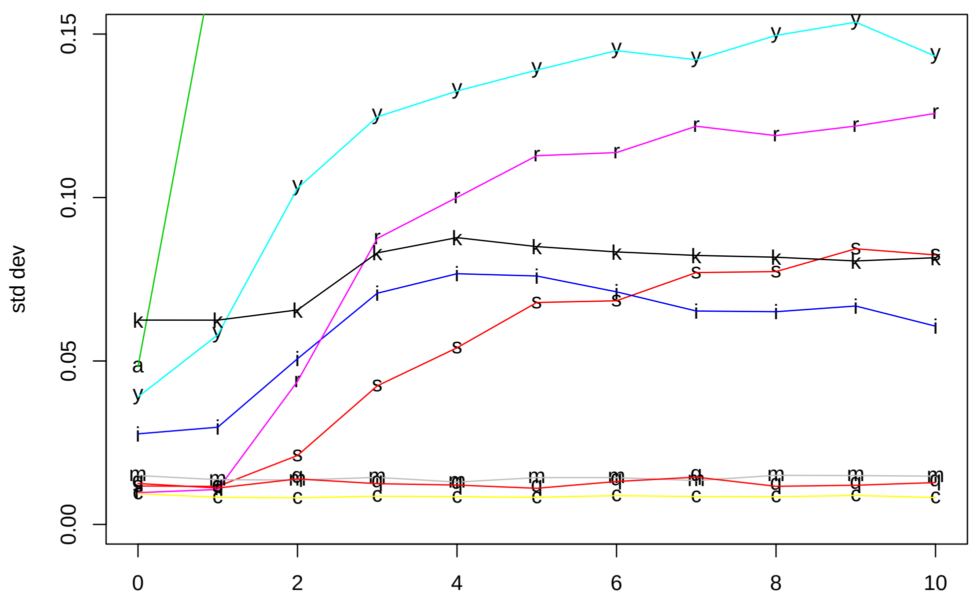

Conclusion

- FMC (FMM) can recover the *entire* central subspace (central mean subspace).
- Represent central subspace as the sum of a family of central mean subspaces.
- Implement FMC/FMM under normal distribution and general case.
- Establish the connection of FMC/FMM with other existing methods.

Further Research Direction

- Further study statistical properties of FMM/FMC, including the optimal selection of k, h, σ_ω^2 and σ_t^2.
- Categorical or ordinal predictor variables.
- Dependent data and time series data.
- Robustness.
  - when the model is not true?
  - when the distributional assumption is not true?
- Theoretical comparison with the existing methods.