A Review of Matrix Algebra

Basic notation and concepts

vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

matrices

$$A_{n \times k} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} = (a_{ij})$$

Basic Concepts

vectors:

linear independent, length, angle, inner product, perpendicular, projection, Gram-Schimdt procedure, etc.

matrices:

determinant, rank of a matrix, nonsingular, inverse, orthogonal, symmetric, positive definite

Definition: Let $A = (a_{ij})$ be a $k \times k$ matrix. The trace of the matrix A, written $tr(A)$, is the sum of the diagonal elements. That is $tr(A) = \sum_{i=1}^{k} a_{ii}$.

Results: Let $A_{k \times k}$ and $B_{k \times k}$ be two matrices. Let c be a scalar.

(a) $tr(A) = tr(B)$

(b) $tr(A \pm B) = tr(A) \pm tr(B)$

(c) $tr(AB) = tr(BA)$

(d) $tr(B^{-1}AB) = tr(A)$

(e) $tr(AA') = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij}^2$

Definition Let A be a $k \times k$ matrix. Let I be the $k \times k$ identity matrix.

(a) $f(\lambda) =\mid A - \lambda I \mid$ is call the characteristic polynomial of A.

(b) $\lambda_1, \lambda_2, \cdots, \lambda_k$, satisfying $f(\lambda) = 0$, are called the eigenvalues of A.

Definition Let A be a $k \times k$ matrix and λ an eigenvalue of A. If $\vec{x}$ is a nonzero vector ($\vec{x} \neq 0$) such that $A\vec{x} = \lambda\vec{x}$, then $\vec{x}$ is said to be an eigenvector of A associated with the eigenvalue λ. Let $\vec{e} = \frac{\vec{x}}{||\vec{x}||}$, then $(\lambda, \vec{e}$ is called a pair of eigenvalue-eigenvector of A.

Definition A quadratic function $Q(x)$ in $(x_1, x_2, \ldots, x_k)$ is

$$Q(x) = \vec{x}' A \vec{x},$$

where $\vec{x}' = (x_1, x_2, \ldots, x_k)$ and A is a $k \times k$ symmetric matrix.

Results

(1) Eigenvalues of real symmetric matrix are real and so are their corresponding eigenvectors.

(2) A symmetric positive definite matrix has real positive eigenvalues

Spectral Decomposition:

Let A be a $k \times k$ symmetric matrix. Then A can be expressed in terms of its k eigenvalue-eigenvector pairs (λ_i, e_i) as

$$A = \sum_{i=1}^{k} \lambda_i e_i e_i'$$

If A is positive definite, then

$$A^{-1} = \sum_{i=1}^{k} \frac{1}{\lambda_i} e_i e_i'$$

and

$$A^{1/2} = \sum_{i=1}^{k} \sqrt{\lambda_i} e_i e_i'$$

Singular Value Decomposition

Let A be an $n \times k$ matrix, then there exists an $n \times n$ orthogonal matrix U and an $k \times k$ matrix V such that

$$A = U \Lambda V$$

where the $n \times k$ matrix Λ has (i, i) entry $\lambda_i \geq 0$, for $i = 1, 2, \ldots, \min(n, k)$ and the other entries are zero. The positive constants λ_i are called the singular values of A.

Assume that the rank of A is r. There exist r positive constants $\lambda_1, \lambda_2, \ldots, \lambda_r$, r orthogonal $n \times 1$ unit vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_r}$ and r orthogonal $k \times 1$ unit vector $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}$ such that

$$A = \sum_{i=1}^{k} \lambda_i \vec{u_i} \vec{v_i}' = U_r \Lambda_r V_r$$

where $U_r = (\vec{u_1}, \vec{u_2}, \ldots, \vec{u_r})$, $V_r = (\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r})$, and Λ_r is an $r \times r$ diagonal matrix with diagonal entries λ_i.

Rayleigh-Ritz Theorem

Let $B_{k \times k}$ be a positive definite matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and associated normalized eigenvectors $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_k}$. Then,

$$\max_{\vec{x} \neq 0} \frac{\vec{x}' B \vec{x}}{\vec{x}' \vec{x}} = \lambda_1, (\text{attained when } \vec{x} = \vec{e_1})$$

$$\min_{\vec{x} \neq 0} \frac{\vec{x}' B \vec{x}}{\vec{x}' \vec{x}} = \lambda_k, (\text{attained when } \vec{x} = \vec{e_k})$$

Moreover

$$\max_{\vec{x} \perp \vec{e_1}, \ldots, \vec{e_i}} \frac{\vec{x}' B \vec{x}}{\vec{x}' \vec{x}} = \lambda_i + 1, (\text{attained when } \vec{x} = e_{\vec{i+1}},\ i = 2, \ldots, k-1)$$