Lecture 4: Discrete Random Variables and Probability Distributions

Definition: $(E, \mathcal{S})$: Experiment and Sample Space.
$\mathcal{R}$: all real numbers.
Random variable: X: $\mathcal{S} \rightarrow \mathcal{R}$.
: $X(s) = x$

Examples:
Toss an unfair coin.
Toss an unfair coin until head.
Toss two fair 4-sided dice.
Randomly select a student on campus.
1. gender
2. undergraduate/graduate
3. which year
4. height

Discrete Random Variable vs Continuous Random Variable
Bernoulli Random Variable

Probability Distributions for d.r.v
Toss two fair dice (continued):

$$\mathcal{S} = \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{matrix}$$

Random variable Y: the sum of the outcomes

possible values	2	3	4	5	6	7	8
probabilities							

Probability Distribution or Probability Mass Function
For each possible value x, $p(x)$: the prob. of observing x when the experiment is performed

Bernoulli r.v.

$$p(x) = \begin{cases} 1-\alpha & \text{if } x = 0 \\ \alpha & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

·Bernoulli distribution
·α called parameter
·Family of Bernoulli distributions

Toss a coin until head (continued):
X: the number of tosses

Let $P(\text{H}) = p$,
$P(X = 1) = P(\text{H}) = p$
$P(X = 2) = P(\text{TH}) = P(\text{T})P(\text{H}) = (1 - p)p$
$P(X = 3) = P(\text{TTH}) = P(\text{T})P(\text{T})P(\text{H}) = (1 - p)^2 p$
In general, if x is an positive integer,
$P(X = x) =$

Tire Example
X: the number of tires on a randomly selected car that are underinflated. Its probability mass function is given as follows,

x	0	1	2	3	4
$p(x)$	0.4	0.1		0.1	0.3

Compute $P(2 \leq X \leq 4)$ and $P(X \neq 0)$.

Another Representation of Probability Distribution
Tire Example continued
Let's calculate $P(X \leq x)$ for any given x
If $x < 0$, $P(X \leq x) = 0$
If $x = 0$, $P(X \leq 0) = P(X = 0) = 0.4$
If $0 < x < 1$, $P(X \leq x) = P(X = 0) = 0.4$
If $x = 1$, $P(X \leq x) = P(X = 0) + P(X = 1) = 0.4 + 0.1 = 0.5$
If $1 < x < 2$, $P(X \leq x) = 0.5$
if $2 \leq x < 3$, $P(X \leq x) = 0.6$
if $3 \leq x < 4$, $P(X \leq x) = 0.7$
if $x \geq 4$, $P(X \leq x) = 1$

Hence, $P(X \leq x)$ is defined for any x, it is called cumulative distribution function, denoted by $F(x)$.
Step function

Definition The cumulative distribution function (cdf) $F(x)$ of a discrete rv X with pmf $p(x)$ is defined for every number x by

$$F(x) = P(X \leq x) = \sum_{y:y \leq x} p(y)$$

For any number x, $F(x)$ is the probability that the observed value of X will be at most x.

Calculate Probability Using cdf $F(x)$
For any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = F(b) - F(a-)$$

where $a-$ represents the largest possible value of X that is strictly less than a.
·pmf and cdf are equivalent.

Measure Location and Dispersion of Probability Distributions

Expected value (or mean value, or population mean)

Suppose D is the collection of all possible values of X, and $p(x)$ is the pmf.

$$E(X) = \mu_x = \sum_{x \in D} x \cdot p(x)$$

Tire Example continued

Expected Value of a Function of X

Let $h(X)$ be any function depending on X, then

$$E(h(X)) = \mu_{h(X)} = \sum_{D} h(x) \cdot p(x)$$

Proposition

$$E(aX + b) = aE(X) + b$$

or

$$\mu_{aX+b} = a \cdot \mu_X + b$$

Tire Example continued

Suppose the time used to inflate the tires is $0.5x^2$, where x is the number of underinflated tire.

Variance of X

Let X have pmf $p(x)$ and expected value μ. Then the variance of X, denoted by $V(X)$, or σ_X^2, or just σ^2, is

$$V(X) = \sum_{D} (x - \mu)^2 \cdot p(x) = E((X - \mu)^2)$$

The standard deviation (SD) of X is

$$\sigma_X = \sqrt{\sigma_X^2}$$

Example

Proposition

1.

$$V(h(X)) = \sigma_{h(X)}^2 = \sigma_D (h(X) - E(h(X)))^2 \cdot p(x)$$

2

$$V(aX + b) = \sigma_{aX+b}^2 = a^2 \cdot \sigma_X^2 \text{ and } \sigma_{aX+b} = \mid a \mid \cdot \sigma_X$$

3

$$V(X) = E(X^2) - (E(X))^2$$

Useful Probability Distributions

Binomial Probability Distribution

1. A sequence of n trials
2. The trials are identical, each with two outcomes, denoted by S or F.
3. The trials are independent.
4. $P(S)$=constant, denoted by p

Binomial Experiment

Example: Toss a coin 10 times, X: the number of heads

Approximate Binomial Experiment

Example: Select 10 students on campus, X: the number of female student

Definition: Given a binomial experiment consisting of n trials (exact or approximate), let

$$X = \text{the number of } S\text{'s among the } n \text{ trials.}$$

X is a binomial random variable.
Possible values?
pmf $b(x; n, p)$?

Example: n=4

0	1	2	3	4
$b(0;4,p)$	$b(1;4,p)$	$b(2;4,p)$	$b(3;4,p)$	$b(4;4,p)$

$b(0;4,p) = P(X = 0) = P(FFFF) = P(F)P(F)P(F)P(F) = (1-p)^4$

$b(1;4,p) = P(X = 1) = P(1S \text{ and } 3F's) = P(SFFF)+P(FSFF)+P(FFSF)+P(FFFS) = (1-p)^3p+(1-p)^3p+(1-p)^3p+(1-p)^3p = 4(1-p)^3p$

$b(2;4,p) = P(X = 2) = P(2S's \text{ and } 2F's) = P(SSFF)+P(SFFS)+P(SFSF)+P(FSSF)+P(FSFS)+P(FFSS) = (1-p)^2p^2 + \cdots + (1-p)^2p^2 = 6(1-p)^2p^2$

Similarly,
$b(3;4,p) = 4(1-p)p^3$

$b(4;4,p) = p^4$

General Formula

$X \sim \text{Bin}(n,p)$
pmf:

x	$p(x)$
0	$\binom{n}{0}(1-p)^n p^0$
1	$\binom{n}{1}(1-p)^{n-1} p^1$
2	$\binom{n}{2}(1-p)^{n-2} p^2$
$\vdots$	$\vdots$
i	$\binom{n}{i}(1-p)^{n-i} p^i$
$\vdots$	$\vdots$
n	$\binom{n}{n}(1-p)^0 p^n$

cdf of binomial distribution

$$P(X \le x) = B(x; n, p) = \sum_{y=0}^{x} b(y; n, p)$$

For $n = 5, 10, 15, 20, 25$, and $p = 0.01, 0.05, \ldots, 0.95, 0.99$, the probabilities are given in Appendix Table A.1

Mean and Variance

$$E(X) = np$$
$$V(X) = n(1-p)p$$

Examples 3.48

Poisson Probability Distribution

X: the number of events in a specific time period (or in a specific region)

Examples:

the number of phone calls at an office

the number of accidents at an intersection

the number of certain animals found in a sqare mile area

...

X is a Poisson r.v., its pmf (Poisson distribution) is

$$p(x; \lambda) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Cumulative Distribution function (cdf)

$$F(x;\lambda) = \sum_{y=0}^{y=x} \frac{\lambda^y e^{-\lambda}}{y!}$$

cdf of Poisson distributions are given in Table A.2

x	0	1	2	3	4	5	6	7	8	...
$p(x;\lambda)$	$\frac{\lambda^0 e^{-\lambda}}{0!}$	$\frac{\lambda^1 e^{-\lambda}}{1!}$	$\frac{\lambda^2 e^{-\lambda}}{2!}$	$\frac{\lambda^3 e^{-\lambda}}{3!}$	$\frac{\lambda^4 e^{-\lambda}}{4!}$	$\frac{\lambda^5 e^{-\lambda}}{5!}$	$\frac{\lambda^6 e^{-\lambda}}{6!}$	$\frac{\lambda^7 e^{-\lambda}}{7!}$	$\frac{\lambda^8 e^{-\lambda}}{8!}$	...

Proposition

$$\sum_{x=0}^{x=+\infty} p(x;\lambda) = \frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \frac{\lambda^4 e^{-\lambda}}{4!} + \cdots = 1$$

$E(X) = \Sigma_{x=0}^{+\infty} x \cdot p(x;\lambda)$

$$= 0 \cdot \frac{\lambda^0 e^{-\lambda}}{0!} + 1 \cdot \frac{\lambda^1 e^{-\lambda}}{1!} + 2 \cdot \frac{\lambda^2 e^{-\lambda}}{2!} + 3 \cdot \frac{\lambda^3 e^{-\lambda}}{3!} + 4 \cdot \frac{\lambda^4 e^{-\lambda}}{4!} + \cdots = \lambda$$

$E(X^2) = \Sigma_{x=0}^{+\infty} x^2 \cdot p(x;\lambda)$

$$= 0^2 \cdot \frac{\lambda^0 e^{-\lambda}}{0!} + 1^2 \cdot \frac{\lambda^1 e^{-\lambda}}{1!} + 2^2 \cdot \frac{\lambda^2 e^{-\lambda}}{2!} + 3^2 \cdot \frac{\lambda^3 e^{-\lambda}}{3!} + 4^2 \cdot \frac{\lambda^4 e^{-\lambda}}{4!} + \cdots = \lambda^2 + \lambda$$

What are the mean and variance of X?

Connection with Binomail Distributions

Suppose that in the binomial pmf $b(x;n,p)$, we let $n \to \infty$ and $p \to 0$ in such a way that np approaches a value $\lambda > 0$. Then

$b(x; n, p) \to p(x; \lambda)$

In any binomail experiment in which n is large and p is small,

$$b(x; n, p) \approx p(x; \lambda) \text{ where } \lambda = np,$$

when

$$n \geq 100, p \leq 0.01 \text{ and } np \leq 20$$

Examples 3.81 and 3.82