Lecture 3: Probability

Experiment: action or process that generates observations.
Sample Space $\mathcal{S}$: the set of all possible outcomes.
Examples:

1. Toss a coin:
2. Toss a coin twice:
3. Toss a coin till getting a head:
4. Throw a six-sided die:
5. Throw a six-sided die twice:
6. Travel from Lafayette to Chicago (time):

Event: subset of outcomes in $\mathcal{S}$:

Simple Event vs. Compound Event

Examples:

2. $\{\text{H}, \text{T}\}$:
   There is at least on head:
3. {H, TH, TTH}:
   The number of throws is more than 4:
4. $\{1, 2, 3\}$: $\{2, 4, 6\}$:
   The outcome is an odd number:
   The outcome is bigger than 3:
5. $\{(1, 3), (2, 2), (3, 1)\}$:
   The outcomes are the same from the throws:

Set Theory

·Union: $A \cup B$

·Intersection: $A \cap B$

·Complement: A'

Mutually exclusive/disjoint events:

Venn Diagrams

Axioms

$$(\mathcal{S}, A, P(A))$$

1. For any event A, $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$.
3. a. if $A_1, A_2, \ldots, A_k$ is a finite collection of mutually exclusive events, then $P(A_1 \cup A_2 \cup \cdots A_k) = \Sigma_{i=1}^{i=k} P(A_i)$.

   b. if $A_1, A_2, A_3, \ldots$ is an infinite collection of mutually exclusive events, then $P(A_1 \cup A_2 \cup \cdots) = \Sigma_{i=1}^{i=\infty} P(A_i)$.

Probability Assignment for coin-tossing examples:

1.
2.
3.

Interpretation:

Objective Interpretation vs. Subjective Interpretation

Relative Frequency:

Limiting Relative Frequency:

Let's do an experiment:

Basic Properties:

1. $P(A) + P(A') = 1$
2. If $A \cap B$ is empty, $P(A \cap B) = 0$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example 2.14

Systematic Approach:

Equally Likely Outcomes:

Example and Counterexample:

If True: probability assignment = Counting

$$P(A) = \frac{N(A)}{N}$$

Counting Techniques

Product Rule:

· An experiment with 2-steps, n_1 and $n2$:

The number of outcomes:

Example: 1. toss a coin then toss a die.

2.

Tree Diagram:

Ordered Pair (2-tuple):

General Product Rule:

An experiment with k steps, n_1, n_2, ... and n_k:

The number of outcomes:

k-tuple:

Example: toss a coin, then toss a die, then toss a coin.

Permutations

Model: Choose k objects (to form k-tuple) from a collection of n objects without replacement.

Experiment: choose one by one (k steps).

The number of outcomes:

Example:

m factorial:

Combinations

Model: choose k objects (unordered set) from a collection of n objects without replacement

Experiment: select k objects at once.

The number of outcomes:

Example: $k = 2$ and $n = 4$

Example 2.22

A bridge hand consists of any 13 cards selected from a 52-card deck without regard to order.

$A = \{$the hand consists entirely of spades and clubs with both suits represente

$B = \{$the hand consists of exactly two suits$\}$

Example 2.23

A rental car service facility has 10 foreign cars and 15 domestic cars. Only 6 of them can be serviced due to lack of labor.

$A = \{$3 of the selected cars are domestic and the other 3 are foreign$\}$

$B = \{$at least 4 of the selected cars are domestic$\}$

Conditional Probability

Definition (using Venn Diagram)

$$P(A \mid B) = \frac{\#(A \cap B)}{\#B} = \frac{\#(A \cap B)/\#S}{\#B/\#S} = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rule:

$$P(A \bigcap B) = P(B)P(A \mid B)$$

$$P(A_1 \bigcap A_2 \bigcap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_2, A_1)$$

Example

	engineering	non-engineering
female	3	7
male	6	4

A=have an engineering backgroud, B= a female student

Tree Diagram

knots: various events; branches: labeled with conditional probability
Example:

Example 2.29
$A_i = \{\text{brand } i \text{ is purchased}\}$, $i = 1, 2$, and 3.
$P(A_1)$=0.50, $P(A_2)$=0.30, $P(A_3)$=0.20
$B = \{needsrepair\}$, $B' = \{doesn'tneedrepair\}$
$P(B \mid A_1)$=0.25, $P(B \mid A_2)$=0.20, $P(B \mid A_3)$=0.10
To calculate:
1. $P(A_1 \cap B)$; 2. $P(B)$; 3. $P(A_1 \mid B)$, $P(A_2 \mid B)$ and $P(A_3 \mid B)$.

Bayes' Theorem

The Law of Total Probability
Let A_1, A_2, ..., A_k be mutually exclusive and exhaustive. Then for any event B,

$$P(B) = P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k)$$

Proof with $k = 3$ using Venn Diagram

Bayes' Theorem
Let A_1, A_2, ..., A_k be mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, \ldots, k$. Then for any other event B for which $P(B) > 0$

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)}$$
$$= \frac{P(B \mid A_j)P(A_j)}{P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k)}$$

Example 2.30
$A = \{\text{individual has the disease}\}$, A'=$\{\text{individual does not have the disease}\}$
$B = \{\text{positive test result}\}$, $B' = \{\text{negative test result}\}$
Known probabilities:
$P(A) = 0.001$, $P(A') = 0.999$, $P(B \mid A) = 0.99$, $P(B \mid A')$=0.02
To calculate $P(A \mid B)$

Independence

Two Equivalent Definitions
1. Two events A and B are independent if $P(A \mid B) = P(A)$
2. Two events A and B are independent if $P(A \cap B) = P(A)P(B)$

How about A' and B, A and B', A' and B'....

Example: Toss a fair die, $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ and $C = \{1, 2, 3, 4\}$:

Mutually Exclusive Events

Example: The proportions of blood phenotypes are as follows:

A	B	AB	O
.42	.10	.04	.44

Two people are randomly selected.

1. P(both phenotypes are O)?
2. P(both phenotypes are the same)?

General Definition

Events $A_1, A_2, \ldots, A_n$ are mutually independent if for every k ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, i_2, \ldots, i_k$,

$$P(A_{i_1} \bigcap A_{i_2} \bigcap \cdots \bigcap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

Example 2.35