Assignment 1
Due Next Thursday 5pm

1. Let $A$ be an $n \times n$ nonsingular square matrix, $x$ be a $n$ dimensional vector and $c$ be a constant.
   (a) Show that
   \[
   \det \begin{pmatrix} A & x \\ x' & c \end{pmatrix} = |A|(c - x'A^{-1}x)
   \]
   (b) Show that $|A + xx'| = |A|(1 + x'A^{-1}x)$.

2. Let $A$ be an $m \times n$ matrix.
   (a) Show that $\text{rank}(AA') = \text{rank}(A'A) = \text{rank}(A) = \text{rank}(A')$.
   (b) Show that $\mathcal{R}(AA') = \mathcal{R}(A)$.

3. Find the eigenvalues and eigenvectors of the matrix given below by hand.
   \[
   \begin{pmatrix}
   4 & 2 & 0 & 4 \\
   0 & 2 & -1 & 0 \\
   0 & 0 & 3 & 3 \\
   0 & 4 & 0 & 7
   \end{pmatrix}
   \]

4. Show that a symmetric matrix $A$ is of rank 1 if and only if $A = aa'$ where $a$ is a nonzero vector.

5. Let $A$ be a positive definite matrix.
   (a) Show that
   \[
   \max_{x \neq 0} \frac{(b'x)^2}{x'Ax} = b'A^{-1}b
   \]
   where $b$ is a given vector.
   (b) $(x'Ay)^2 \leq (x'Ax)(y'Ay)$ for any $x$ and $y$.
   (c) $(x'y)^2 \leq (x'Ax)(y'A^{-1}y)$ for any $x$ and $y$.

6. Let $A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be an $n \times n$ matrix and $X$ be a $r \times r$ sub-matrix. Suppose $\text{rank}(A) = r < n$ and $X$ is nonsingular.
   (a) Show that $W = ZX^{-1}Y$.
   (b) Show that $\begin{pmatrix} X^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is a generalized inverse of $A$. 