Lecture 1: Basic Matrix Operations

1. Column vector (a variable)

\[ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \]

2. Row vector (an observation) \( X = [x_1, \ldots, x_p] \)

3. \( cX, X + Y \), inner product \( X'Y, XY' \)

4. Length of a vector \( ||X|| \); Angle between 2 vectors; Linearly dependent vectors

5. \( X_{nxp} \) data matrix: n observations and p variables;

6. Matrix transpose, \( cX, X + Y \)

7. \( n = p \) square matrix, identity matrix \( I_n \)

8. Matrix multiplication. In general \( AB \neq BA \)

9. Symmetric matrix \( A' = A \) (must be square matrix too, correlation or covariance matrix)

10. Inverse of a square matrix \( AA^{-1} = A^{-1}A = I_n \)

11. Orthogonal matrix \( QQ' = Q'Q = I_n \)

12. Trace of a matrix \( A: tr(A) = \sum a_{ii}; tr(AB) = tr(BA) \).

13. Eigenvalues and eigenvectors of a square matrix (PCA, FA) \( AX = \lambda X \), where \( \lambda \) is an eigenvalue and \( X \) is an eigenvector.

14. A symmetric matrix is positive definite if all its eigenvalues are positive.
15. Quadratic form of a symmetric matrix $A$ is $Q(X) = X^TAX$. $Q(X) \geq 0$ for non-negative definite matrix $A$


$$A = \lambda_1 e_1 e_1' + \ldots + \lambda_n e_n e_n' = P \Lambda P'$$

$$tr(A) = \sum \lambda_i$$

$Q(X) = (X'P)\Lambda(P'X) \geq 0$.

16. We can then define the square root of a non-negative definite matrix $A$

17. Singular value decomposition of $A_{m \times k}$, $A = U \Lambda V'$. $UU' = U'U = I_m$, $V'V = VV' = I_k$,

$$= \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ \vdots \\ 0 & \ldots & 0 & \lambda_k \\ 0 & \ldots & 0 & 0 \\ \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix}$$

18. $AA' = U \Lambda^2 U'$, relationship between singular value decomposition and eigen decomposition

19. Random vectors and random matrices. In this class, we focus on mean vector, variance-covariance matrix, and correlation matrix.

$$E(AXB) = AE(X)B, E(X + Y) = E(X) + E(Y)$$

$$E(cX) = cE(X), V(cX) = c^2V(X)$$