(1) Exponential Regression: Let $X_1, \ldots, X_n$ be independent exponential variables, and let $t_1, \ldots, t_n$ be a sequence of constants. The failure rate for $X_i$ is related to $t_i$ by the formula $\lambda_i = t_i/\theta$ (so the mean of $X_i$ is the reciprocal $\theta/t_i$), where $\theta$ is an unknown parameter. So $X_i$ has density $(t_i/\theta)e^{-t_i x/\theta}$, $x > 0$.

(a) Find a complete sufficient statistic for the family of joint distributions.
(b) Determine the UMVU estimator of $\theta$.

(2) Suppose $X$ is absolutely continuous with density

$$p_\alpha(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, \quad x > 0.$$ 

Find expressions for the mean and variance of log $X$. These expressions will involve the gamma function and its derivatives.

(3) Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be independent normal variables, each with mean $\theta$, but with different variances: $\text{Var}(X_i) = 1$ and $\text{Var}(Y_i) = 2$, for $i = 1, \ldots, n$.

(a) Show that the joint distributions for all $2n$ variables form a one-parameter exponential family and identify the complete sufficient statistic.
(b) Find UMVU estimators for $\theta$ and $\theta^2$.

(4) Suppose our data is a single observation $X$, uniformly distributed on the interval $(0, \theta)$. Find the UMVU of $\sin(\theta)$.

(5) Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli variables with success probability $p$. Then $X = X_1 + \cdots + X_n$ is complete sufficient and has a Binomial distribution:

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, \ldots, n,$$

where $q = 1 - p$. Find the UMVU of $pq$.

(6) Let $X$ and $Y$ be independent with $X \sim N(\theta, 1)$ and $Y \sim N(\theta^2, 1)$.

(a) Find a minimal sufficient statistic $T$.
(b) Is the minimal sufficient statistic $T$ complete? If so, explain why. If not, find a function $g$ with $E_{\theta} g(T) = 0$ for all $\theta$, but $g(T) \neq 0$ with positive probability.
1. The joint distribution:

\[ \frac{n}{\theta} \sum_{i=1}^{n} \frac{t_i}{\theta} e^{-\frac{t_i}{\theta}} = \frac{n}{\theta} \sum_{i=1}^{n} \frac{t_i}{\theta} e^{-\frac{1}{\theta} \sum t_i x_i} \]

Suff. statistic: \( T = \sum_{i=1}^{n} t_i x_i \)

\[ \theta > 0 \quad \mathcal{N} = \{ \theta : \theta > 0 \} \quad \gamma(\varphi) - \text{interior not empty} \]

\( T \) does not satisfy linear constraints.

\( T \) — complete.

b) \( \mathbb{E} [T] = \frac{n}{\theta} \mathbb{E} [t_i x_i] = \frac{n}{\theta} \frac{\theta}{t_i} = n \theta \)

\[ \mathbb{E} \left( \frac{T}{n} \right) = \theta. \quad \frac{T}{n} \text{ is UMVUE of } \theta \]

2. \( p_x(x) = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} = \exp \left( (\alpha-1) \log x - x - 1 \gamma(\alpha) \right) \)

\[ = \exp \left( \alpha \log x - (\log x + x) - 1 \gamma(\alpha) \right) \]

\( T = \log x, \quad \eta = \alpha \quad A(\eta) = \log \Gamma(\eta) \)

Cumulant generating function: \( K_T(u) = A(u + \eta) - A(\eta) \)

\[ K_T(u) = (\log M_T(u)) = \mathbb{E} e^{uT} = (\log \Gamma(u + \eta)) - \log \Gamma(\eta) \]

\( K_1 = \mathbb{E} T \quad K_2 = \text{Var}(T) \)
3. \( X_1 \ldots X_n \overset{iid}{\sim} N(\theta, 1) \)
\( Y_1 \ldots Y_n \overset{iid}{\sim} N(\theta, 2) \)

Joint:
\[
\frac{n}{i^n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(X_1-\theta)^2}{2}} \frac{1}{i^n} \frac{1}{\sqrt{4\pi}} e^{-\frac{(Y_1-\theta)^2}{4}}
\]
\[
= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{n}{4} \left( \frac{(X_1-\theta)^2}{2} + \frac{(Y_1-\theta)^2}{4} \right)}
\]
\[
+ \theta \frac{n}{i^n} (X_1 + \frac{Y_1}{2}) \hat{=} \frac{3}{4} n \theta^2 - \frac{n}{i^n} \left( \frac{X_1^2 + Y_1^2}{2} \right)
\]

\( \overline{T} = \frac{n}{i^n} \left( X_1 + \frac{Y_1}{2} \right) \) — complete and suff.

\[
E \overline{T} = \frac{n}{i^n} \theta + \frac{\theta}{2} = \frac{3n}{2} \theta.
\]
\[
\frac{2 \overline{T}}{3n} \quad \text{umvue of } \theta
\]

\[
\text{Var}(\overline{T}) = \frac{n}{i^n} (1 + \frac{2}{4}) = \frac{3}{2} n
\]
\[
= E \overline{T}^2 - \frac{9n^2}{4} \theta^2
\]
\[
\left( \overline{T}^2 - \frac{3}{2} \theta \right) \cdot \frac{4}{9n^2} \quad \text{umvue of } \theta.
\]
4. \[ E g(x) = \sin \theta \]
\[ \int_{\theta}^{0} g(x) \frac{1}{\theta} \, dx = \sin \theta \]
\[ \int_{0}^{\pi} g(x) \, dx = \theta \sin \theta \quad g(x) = \sin B + B \cos x \]

5. \[ p_a = p (1-p) = p - p^2 \]
\[ E X = n p \]
\[ E X^2 = n p (1-p) + n^2 p^2 = np + (n^2-n) p^2 \]
\[ E \left( \frac{X^2 - X}{n^2-n} \right) = p^2 \]
\[ \frac{X}{n} - \frac{X^2 - X}{n^2-n} = \frac{(n-1)X - X^2 + X}{n(n-1)} = \frac{nX - X^2}{n(n-1)} \]

6. Joint: \[ \frac{1}{\sqrt{2\pi} e^{-\frac{(x-\theta)^2}{2}}} \cdot \frac{1}{\sqrt{2\pi} e^{-\frac{(y-\theta)^2}{2}}} = \frac{1}{2\pi} e^{-\frac{(X-Y)^2}{2}} \]
\[ T_1 = x, \quad \eta_1 = \theta \]
\[ T_2 = -Y, \quad \eta_2 = \theta^2 \]
\[ T = (-X, -Y) \quad \text{curved exponential family.} \]
\[ \text{minimal suff. not complete.} \]
\[ E T_1^2 = 1 + \theta^2 \quad E_0 T_1^2 + T_2 - 1 = 0 \]
7. \[ \text{Joint.} \]
\[ \frac{2X}{\theta^2} I(x), \frac{2Y}{\theta^2} I(y) \]
\[ = 4 \frac{XY}{\theta^4} I(\max(x,y)) I(\min(x,y)) \]
\[ \max(x, y) \quad \text{suff.} \quad \text{complete (by def.)} \]

\[ P\left( \frac{X}{\theta} \leq x \right) = P(X \leq \theta x) = \int_0^{\theta x} \frac{\alpha x}{\theta^2} dx = x^2, \quad 0 < x < 1 \]

\[ \frac{X}{Y} = \frac{X/\theta}{Y/\theta} \quad \text{ancillary} \]

By Basu, \( \max\{X, Y\} \perp X/Y \)