1. (10 Points) Let \( X_1, X_2, \ldots, X_n \) be a random sample from a \( \text{Bernoulli}(\theta) \) distribution with parameter \( 0 \leq \theta \leq 1 \). If \( X \sim \text{Bernoulli}(\theta) \), \( \mathbb{P}(X = 1) = \theta \) and \( \mathbb{P}(X = 0) = 1 - \theta \). Find the maximum likelihood estimator (MLE) \( \hat{\theta} \) of \( \theta \). What is the mean squared error (MSE) of \( \hat{\theta} \)?

\[
L(\theta) = \frac{n}{\prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}} = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}
\]

\[
\ell(\theta) = \sum_{i=1}^{n} x_i \ln \theta - \sum_{i=1}^{n} x_i \ln (1-\theta)
\]

\[
\ell'(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1-\theta} = 0
\]

\[
\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}
\]

\[
\text{MSE}(\hat{\theta}) = \text{bias}^2 + \text{var} = 0 + \frac{\theta(1-\theta)}{n} = \frac{\theta(1-\theta)}{n}
\]
2. (10 Points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with parameter $\lambda$. The Poisson probability mass function is given by
\[ f(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots, \text{ and } \lambda > 0. \]

Both the mean and the variance of a Poisson random variable are $\lambda$, i.e., $E(X_1) = \text{Var}(X_1) = \lambda$. If we want to estimate $\theta = \lambda^2$. What is the MLE of $\theta$ and check whether it is unbiased. Please justify your answer.

\[ L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{X_i} e^{-\lambda}}{X_i!} = \frac{\lambda^n e^{-\lambda}}{\prod_{i=1}^{n} X_i!} \]

\[ \log L(\lambda) = \sum_{i=1}^{n} X_i \lambda - n \lambda - \log \frac{n!}{X_i!} \]

MLE:
\[ \lambda^* = \frac{\sum_{i=1}^{n} X_i}{n} - n = 0 \]
\[ \hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{n} \]

2. Plug-in:
\[ \hat{\theta} = \frac{\sum_{i=1}^{n} X_i}{n}^2 \]

3. Biased:
\[ E(\hat{\theta}) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^2 = \left(\bar{X}\right)^2 + \text{Var}(\bar{X}) = \frac{\lambda^2 + \lambda}{n} + \lambda^2 \]
3. (10 Points) A coin was tossed $n = 1000$ times independently, and we observe 510 heads. Let $p$ be the success probability to obtain a head in one toss.

(a). Do you have evidence to conclude that the coin is fair? Perform a two-sided test to assess $H_0 : p = 0.5$. Report the P-value and your conclusion.

\[
\text{Obs. } \frac{X}{n} = \frac{510}{1000} = 0.51
\]

\[
3 \quad \begin{cases} 
\text{under } H_0 : \quad Z = \frac{X - 0.5}{\sqrt{0.5(0.5)}} \sim N(0,1) \\
\text{obs. } Z = \frac{0.51 - 0.5}{\sqrt{0.005}} = 0.6325
\end{cases}
\]

\[
2' \quad \text{P-value } = 2 \left( 1 - \Phi(1.21) \right) = 0.5286
\]

(b). Construct an approximate 0.95-confidence interval for $p$.

\[
\frac{X - p}{\sqrt{\frac{X(1-X)}{n}}} \sim N(0,1)
\]

\[
0.51 \pm \begin{pmatrix} Z_{0.025} \\ 0.5 \end{pmatrix} \sqrt{\frac{0.51 \times 0.49}{1000}} = (0.47902, 0.54098)
\]
4. (15 Points) A randomly selected sample of \( n = 30 \) students at a university is asked, “How much did you spend for textbooks this semester?” The responses, in dollars, are 200, 175, 450, 300, 350, 250, 150, 200, 320, 370, 404, \( \cdots \), 250.

The sample mean \( \bar{x} \) is 284.9 and the sample variance \( s^2 \) is 96.1.

(a) Construct an approximate 90% confidence interval for the population mean using the central limit theorem.

\[
\bar{x} \pm t_{0.05, 29} \frac{s}{\sqrt{n}} = 284.9 \pm 1.645 \sqrt{\frac{96.1}{30}}
\]

\[
= 284.9 \pm 1.645 \sqrt{3.20}
\]

\[
= 284.9 \pm 1.645 \cdot 5.61
\]

\[
= (281.65, 288.15)
\]

(b) If the data are assumed from \( N(\mu, \sigma^2) \), construct an exact 95% confidence interval for the population mean.

\[
\bar{x} \pm t_{0.025, 29} \frac{s}{\sqrt{n}}
\]

\[
= 284.9 \pm 2.042 \sqrt{\frac{96.1}{30}}
\]

\[
= 284.9 \pm 2.042 \cdot 3.66047
\]

\[
= (281.2355, 288.5605)
\]
5(c). For Part (b), it is required that the margin error of a 95% confidence interval for $\mu$ is no greater than 1 dollar under the same sample variance. What is the required sample size?

$$2.0452 \frac{961}{\sqrt{n}} \leq 1 \quad \Rightarrow \quad n \geq \frac{2^2}{2.0452 \times 961} = 401.971216.$$

5. (15 Points) Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. The normal pdf is given by

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty.$$

(a). If $\mu = 1$ is known, what is the MLE of $\sigma^2$?

\[\hat{\lambda}(\sigma^2) = \frac{n}{2\pi\sigma^2} e^{-\frac{(\bar{x} - 1)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(\bar{x} - 1)^2}{2\sigma^2}}\]

\[\lambda'(\sigma^2) = -\frac{n}{2\sigma^2} \bar{x} - \frac{n}{2} (\sigma^2 - \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - 1)^2)
\]

\[\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - 1)^2\]
(b). If \( \mu \) is unknown, what is the MLE of \( \sigma^2 \)?

\[
L(\mu, \sigma^2) = \frac{n}{\sigma^2} \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n e^{-\frac{(x_i - \mu)^2}{2\sigma^2}}
\]

\[
\ln(L(\mu, \sigma^2)) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2
\]

\[
\frac{\partial L}{\partial \mu} = 0 \quad \Rightarrow \quad \hat{\mu} = \bar{x}
\]

\[
\frac{\partial L}{\partial \sigma^2} = 0 \quad \Rightarrow \quad \hat{\sigma}^2 = \frac{1}{n} \sum \frac{n}{i=1} (x_i - \bar{x})^2
\]

(c). Are estimators in (a) and (b) unbiased estimators of \( \sigma^2 \)? Verify your answer.

2.5' a) \( E \hat{\sigma}^2 = \frac{1}{n} \sum \frac{1}{i=1} E(x_i - \bar{x})^2 = \sigma^2 \) unbiased!

2.5' b) \( E \hat{\sigma}^2 = E \frac{n-1}{n} S^2 = \frac{n-1}{n} \sigma^2 \) biased!
Bonus. (10 Points) Let \( X_1, \ldots, X_n \) be a random sample from the distribution with the pdf \( f(x) = \frac{3x^2}{\theta^3}, 0 \leq x \leq \theta \), zero elsewhere. What is the MLE of \( \theta \)? Is the MLE of \( \theta \) an unbiased estimator? If not, can you construct an unbiased estimator of \( \theta \) based on the MLE?

\[
L(\theta) = \frac{n}{\theta^n} \left( \frac{3}{\theta} \right)^{n X_i^2} I(\theta) I(\theta \leq \min X_i, \theta \geq 0)
\]

MLE: \( \hat{\theta} = \max (X_1, \ldots, X_n) \)

\[
P(\hat{\theta} \leq x) = \left( \frac{x}{\theta} \right)^n \left( \frac{3}{\theta^3} \right) = \frac{x^{3n}}{\theta^{3n}}
\]

\[
\hat{f}(x) = \frac{3n x^{3n-1}}{\theta^{3n}} \quad 0 \leq x \leq \theta
\]

\[
E \hat{\theta} = \int_0^\theta x \frac{3n x^{3n-1}}{\theta^{3n}} \, dx = \frac{3n}{\theta^{3n}} \frac{\theta^{3n+1}}{3n+1} = \frac{3n}{3n+1} \theta \neq \theta
\]

\[
\frac{3n+1}{3n} \hat{\theta} = \frac{3n+1}{3n} \max (X_1, \ldots, X_n) \text{ unbiased!}
\]
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