Poisson Distribution

• Recall

\[ e^m = 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots = \sum_{x=0}^{\infty} \frac{m^x}{x!}. \]

• Consider the function \( p(x) \) defined by

\[ p(x) = \begin{cases} \frac{m^x e^{-m}}{x!}, & x = 0, 1, 2, 3, \ldots \\ 0 & \text{elsewhere,} \end{cases} \]

where \( m > 0 \).

• Since \( m > 0 \) then \( p(x) \geq 0 \) and

\[
\sum_x p(x) = \sum_{x=0}^{\infty} \frac{m^x e^{-m}}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!} = e^{-m} e^m = 1
\]
• Poisson Distribution: a random variable that has a pmf $p(x)$ is said to have Poisson distribution with parameter $m$.

• Poisson distribution has many applications in a lot of areas.
  – The number of defects on refrigerator doors;
  – The number of accidents in a unit of time;
  – The number of insurance claims in some unit of time.
Some Special Distributions

- mgf:

\[ M(t) = \sum_x e^{tx} p(x) = e^{m(e^t - 1)} \]

for all real values of \( t \). Since

\[ M'(t) = e^{m(e^t - 1)} me^t \]
\[ M''(t) = e^{m(e^t - 1)} me^t + e^{m(e^t - 1)}(me^t)^2, \]

then

\[ \mu = M'(0) = m \]

and

\[ \sigma^2 = M''(0) - \mu^2 = m \]
Some Special Distributions

Examples

- Suppose that $X$ has a Poisson distribution with $\mu = 2$. Then the pmf of $X$ is

$$p(x) = \begin{cases} \frac{2^x e^{-2}}{x!}, & x = 0, 1, 2, 3, \ldots \\ 0 & \text{elsewhere,} \end{cases}$$

The variance of $X$ is $\sigma^2 = \mu = 2$.

$$P(X \geq 1) = 1 - P(X = 0) = 1 - p(0) = 1 - e^{-2} = 0.865$$
• If the mgf of a random variable $X$ is $M(t) = e^{4(e^t-1)}$, then $X$ has a Poisson distribution with $\mu = 4$.

$$P(X = 3) = \frac{4^3 e^{-4}}{3!} = \frac{32}{3} e^{-4},$$

or by Table I,

$$P(X = 3) = P(X \leq 3) - P(X \leq 2) = 0.433 - 0.238 = 0.195.$$
Some Special Distributions

Theorem 3.2.1

- Suppose $X_1, \ldots, X_n$ are independent random variables and suppose $X_i$ has a Poisson distribution with parameter $m_i$. Then $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with parameter $\sum_{i=1}^{n} m_i$. 

Exercise

- If the random variable $X$ has a Poisson distribution such that $P(X = 1) = P(X = 2)$, find $P(X = 4)$. 