Some Special Distributions

- The Binomial and Related Distributions
- The Poisson Distribution
- The $\Gamma$, $\chi^2$, and $\beta$ Distributions
- The Normal Distributions
- The Multivariate Normal Distribution
- $t$ and $F$ — Distribution
- Mixture Distributions
Bernoulli distribution

- A **Bernoulli experiment** is a random experiment, the outcome is one of two mutually exclusive and exhaustive ways.

- For example: success or failure; female or male; life or death; nondefective or defective.

- A sequence of **Bernoulli trials** occurs when a Bernoulli experiment is performed several independent times.

- Let $X$ be a random variable associated with a Bernoulli trial as follows:

  $$X(success) = 1, \quad X(failure) = 0.$$ 

- Bernoulli distribution: The pmf of $X$ can be written as

  $$p(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$$
• The expected value of $X$ is

$$\mu = E(X) = \sum_{x=0}^{1} xp^x (1-p)^{1-x} = 0 \times (1-p) + 1 \times p = p.$$  

• The variance:

$$\sigma^2 = var(X) = \sum_{x=0}^{1} (x-p)^2 p^x (1-p)^{1-x}$$

$$= p^2(1-p) + (1-p)^2 p = p(1-p).$$

• It follows that the standard deviation of $X$ is $\sigma = \sqrt{p(1-p)}$. 
Binomial distribution

- A sequence of $n$ Bernoulli trials: An observed sequence of $n$ Bernoulli trials will be a $n$-tuple of zeros and ones. We are interested in the total number of successes and not in the order of their occurrence.

- Binomial pmf:

$$p(x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

- Recall, if $n$ is a positive integer, that

$$(a + b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}.$$
So, 
\[
\sum_x p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [1 - p + p]^n = 1.
\]

- Binomial Distribution: \( b(n, p) \). The constant \( n \) and \( p \) are called the parameters of the binomial distribution.

- For example, \( X \) is \( b(5, \frac{1}{3}) \), we mean \( X \) has pmf

\[
p(x) = \binom{5}{x} \left( \frac{1}{3} \right)^x \left( \frac{2}{3} \right)^{5-x}
\]
• mgf:

\[ M(t) = [(1 - p) + pe^t]^n \]

• Mean and variance:

\[ M'(t) = n[(1 - p) + pe^t]^{n-1}(pe^t) \]
\[ M''(t) = n[(1-p)+pe^t]^{n-1}(pe^t)+n(n-1)[(1-p)+pe^t]^{n-2}(pe^t)^2 \]
\[ \mu = M'(0) = np, \quad \sigma^2 = M''(0) - \mu^2 = np(1 - p). \]
Examples

• Let $X$ be the number of the heads (success) in $n = 7$ independent tosses of a coin. The pmf is

$$p(x) = \binom{7}{x} \left( \frac{1}{2} \right)^x \left( 1 - \frac{1}{2} \right)^{7-x}, \quad x = 0, 1, 2, \ldots, 7.$$ 

Then $X$ has the mgf $M(t) = \left( \frac{1}{2} + \frac{1}{2} e^t \right)^7$, has mean $\mu = np = \frac{7}{2}$ and has variance $\sigma^2 = np(1 - p) = \frac{7}{4}$. Further,

$$P(0 \leq X \leq 1) = \sum_{x=0}^{1} p(x) = \frac{1}{128} + \frac{7}{128} = \frac{8}{128}.$$
• If the mgf of a random variable $X$ is

$$M(t) = \left( \frac{2}{3} + \frac{1}{3} e^t \right)^5,$$

then $X$ has a binomial distribution with $n = 5$ and $p = \frac{1}{3}$; that is the pmf of $X$ is

$$p(x) = \binom{5}{x} \left( \frac{1}{3} \right)^x \left( 1 - \frac{1}{3} \right)^{5-x}, x = 0, 1, \ldots, 5.$$  

Here $\mu = np = \frac{5}{3}$ and $\sigma^2 = np(1 - p) = \frac{10}{9}$. 
Let the independent random variables $X_1, X_2, X_3$ have the same cdf $F(x)$. Let $Y$ be the middle value of $X_1, X_2, X_3$. Find the cdf of $Y$. $Y \leq y$ if and only if at least two of the random variables $X_1, X_2, X_3$ are less than or equal to $y$. Thus,

$$F_Y(y) = \binom{3}{2} [F(y)]^2[1 - F(y)] + [F(y)]^3.$$  

If $F$ is a continuous cdf so that the pdf of $X$ is $f$, then the pdf of $Y$ is

$$f_Y(y) = F_Y(y)' = 6F(y)[1 - F(y)]f(y).$$
Some Special Distributions

- **Negative Binomial Distribution**: Consider a sequence of independent repetitions of a random experiment with constant probability $p$ of success. Let the random variable $Y$ denote the total number of failures in this sequence before the $r$th success.

$$p_Y(y) = \binom{y + r - 1}{r - 1} p^r (1 - p)^y$$

In this special case $r = 1$, we say $Y$ has a **geometric distribution**,

$$P_Y(y) = p(1 - p)^y, \quad y = 0, 1, 2, ...$$
Theorem 3.1.1

Let $X_1, X_2, \ldots, X_m$ be independent random variables such that $X_i$ has binomial $b(n_i, p)$ distribution, for $i = 1, 2, \ldots, m$. Let $Y = \sum_{i=1}^{m} X_i$. Then $Y$ has a binomial $b(\sum_{i=1}^{m} n_i, p)$ distribution.
Some Special Distributions

Multinomial Distribution

• Let a random experiment be repeated \( n \) independent times. On each repetition, the experiment results in but one of \( k \) mutually exclusive and exhaustive says, say \( C_1, C_2, \ldots, C_k \). Let \( p_i \) be the probability that the outcome is an element \( C_i \). Define the random variable \( X_i \) to be equal to the number of outcomes that are elements of \( C_i \), \( i = 1, \ldots, k \). Let \( x_1, \ldots, x_k \) be nonnegative integers so that \( x_1 + x_2 + \cdots + x_k = n \).

\[
P(X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \cdots x_{k-1}! x_k!} p_1^{x_1} \cdots p_{k-1}^{x_{k-1}} p_k^{x_k}
\]

\[
\binom{n}{x_1} \binom{n-x_1}{x_2} \cdots \binom{n-x_1-\cdots-x_{k-2}}{x_{k-1}} = \frac{n!}{x_1! \cdots x_{k-1}! x_k!}
\]
• Trinomial distribution: When $k = 3$, we often let $X = X_1$ and $Y = X_2$ then $X_3 = n - X - Y$. We say $X$ and $Y$ have a trinomial distribution with pmf

$$p(x, y) = \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y p_3^{n-x-y},$$

where $x$ and $y$ are nonnegative integers with $x + y \leq n$ and $p_1, p_2$ and $p_3$ are positive proper fractions with $p_1 + p_2 + p_3 = 1$.

• mgf:

$$M(t_1, t_2) = (p_1 e^{t_1} + p_2 e^{t_2} + p_3)^n.$$  

$$M(t_1, 0) = ((1-p_1)+p_1 e^{t_1})^n; \quad M(0, t_2) = ((1-p_2)+p_2 e^{t_2})^n.$$
• $X$ and $Y$ are dependent binomial distributions. $X$ is $b(n, p_1)$ and $Y$ is $b(n, p_2)$, respectively. Consider the conditional pmf of $Y$ given $X = x$. We have

$$p_{2|1}(y|x) = \frac{(n - x)!}{y!(n - x - y)!} \left( \frac{p_2}{1 - p_1} \right)^y \left( \frac{p_3}{1 - p_1} \right)^{n-x-y},$$

where $y = 0, 1, \ldots, n - x$. $Y|X = x$ is $b[n - x, p_1/(1 - p_1)]$.

Hence, $E(Y|x) = (n - x)(\frac{p_2}{1 - p_1})$ and $E(X|y) = (n - y)(\frac{p_1}{1 - p_2})$,

$$\rho = -\sqrt{\frac{p_1 p_2}{(1 - p_1)(1 - p_2)}}$$