Motivation

- Let \((X_1, X_2)\) be a random vector. Suppose we know the joint distribution of \((X_1, X_2)\) and we seek the distribution of a transformation of \((X_1, X_2)\), say, \(Y_1 = u_1(X_1, X_2)\) and \(Y_2 = u_2(X_1, X_2)\).

- We know how to do it for single random variable.
Discrete

• Discrete (Easy): Let $p_{X_1, X_2}(x_1, x_2)$ be the joint pmf of $X_1$ and $X_2$ with support $S$. Let $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one-to-one transformation that maps $S$ to $T$. The joint pmf of $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ is given by

$$p_{Y_1, Y_2}(y_1, y_2) = \begin{cases} p_{X_1, X_2}[w_1(y_1, y_2), w_2(y_1, y_2)] & (y_1, y_2) \in T \\ 0 & \text{elsewhere,} \end{cases}$$

where $x_1 = w_1(y_1, y_2), x_2 = w_2(y_1, y_2)$ is the single-valued inverse of $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2).$
Example: Let $X_1$ and $X_2$ have the joint pmf

$$p_{X_1,X_2}(x_1, x_2) = \frac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1! x_2!}, \ x_1 = 0, 1, 2, \ldots, x_2 = 0, 1, 2, \ldots,$$

Find the joint pmf of $Y_1 = X_1 + X_2$ and $Y_2 = X_2$. 

\text{MULTIVARIATE DISTRIBUTIONS}
Continuous

- cdf technique
- Simple example: Let pdf of $X$ and $Y$ is

$$f_{X,Y}(x, y) = \begin{cases} 
1 & 0 < x < 1, 0 < y < 1 \\
0 & \text{elsewhere}
\end{cases}$$

Find the pdf of $Z = X + Y$. $F_Z(z) = P(X + Y \leq z)$. Then

$$F_Z(z) = \begin{cases} 
0 & z < 0 \\
\frac{z^2}{2} & 0 \leq z < 1 \\
1 - \frac{(2-z)^2}{2} & 1 \leq z < 2 \\
1 & 2 \leq z.
\end{cases}$$
Since \( f_Z(z) = F'_Z(z) \), the pdf of \( Z \) is

\[
f_Z(z) = \begin{cases} 
  z & 0 < z < 1 \\
  2 - z & 1 \leq z < 2 \\
  0 & \text{elsewhere.}
\end{cases}
\]
• Consider more general case: Let \((X_1, X_2)\) have pdf \(f_{X_1,X_2}(x_1, x_2)\). Suppose \(Y_1 = u_1(X_1, X_2), Y_2 = u_2(X_1, X_2)\), where \(y_1 = u_1(x_1, x_2)\) and \(y_2 = u_2(x_1, x_2)\) define a one-to-one transformation from \(S \in R^2\) onto \(T \in R^2\). Write the inverse of transformation as \(x_1 = w_1(y_1, y_2)\) and \(x_2 = w_2(y_1, y_2)\). Define the **Jacobian** as

\[
J = \begin{vmatrix}
\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\
\frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}
\end{vmatrix}
\]
• Let \( A \) be a subset of \( S \) and let \( B \) denote the mapping of \( A \) under the one-to-one transformation. Because the transformation is one-to-one, the event \( \{(X_1, X_2) \in A\} \) and \( \{(Y_1, Y_2) \in B\} \) are equivalent. Hence,

\[
P[(Y_1, Y_2) \in B] = P[(X_1, X_2) \in A] = \int \int_A f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2
\]

\[
= \int \int_B f_{X_1, X_2}(w_1(y_1, y_2), w_2(y_1, y_2)) \, |J| \, dy_1 \, dy_2
\]

[check your multivariate analysis book] So,

\[
f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 
  f_{X_1, X_2}(w_1(y_1, y_2), w_2(y_1, y_2)) \, |J| & (y_1, y_2) \in T \\
  0 & \text{elsewhere}
\end{cases}
\]
Examples

• Let pdf of \((X_1, X_2)\) is

\[
f_{X_1,X_2}(x_1, x_2) = \begin{cases} 
1 & 0 < x_1 < 1, 0 < x_2 < 1 \\
0 & \text{elsewhere}
\end{cases}
\]

Find the joint pdf of \(Y_1 = X_1 + X_2\) and \(Y_2 = X_1 - X_2\). Find the marginal pdf of \(Y_1\) and \(Y_2\).
• Let $Y_1 = \frac{1}{2}(X_1 - X_2)$, where $X_1$ and $X_2$ have the joint pdf

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \frac{1}{4} \exp\left(-\frac{x_1 + x_2}{2}\right) & 0 < x_1 < \infty, 0 < x_2 < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Find the pdf of $Y_1$ (Double exponential or Laplace pdf).
Let $X_1$ and $X_2$ have the joint pdf

$$f_{X_1, X_2} = \begin{cases} 
10x_1x_2^2 & 0 < x_1 < x_2 < 1 \\
0 & \text{elsewhere.}
\end{cases}$$

Suppose $Y_1 = X_1/X_2$ and $Y_2 = X_2$. Find the marginal pdf of $Y_1$ and $Y_2$. 

Moment Generating Technique

- Let $Y = u(X_1, X_2)$. First find the mgf of $Y$, which is $E e^{tu(X_1, X_2)}$, then recognize this mgf as belonging to a certain distribution. Then, $Y$ would have that distribution since mgf uniquely determines the distribution.

- Here $X_1$ and $X_2$ have the joint pdf

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} \frac{1}{4} \exp\left(-\frac{x_1 + x_2}{2}\right) & 0 < x_1 < \infty, 0 < x_2 < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

So the mgf of $Y = \frac{1}{2}(X_1 - X_2)$ is given by

$$E e^{tY} = \int_0^\infty \int_0^\infty e^{t(x_1-x_2)/2} \frac{1}{4} \exp\left(-\frac{x_1 + x_2}{2}\right) dx_1 dx_2$$
\[
= \int_0^\infty \frac{1}{2} e^{-x_1(1-t)/2} \, dx_1 \int_0^\infty \frac{1}{2} e^{-x_2(1-t)/2} \, dx_2
\]
\[
= \left[ \frac{1}{1-t} \right] \left[ \frac{1}{1+t} \right] = \frac{1}{1-t^2},
\]
for \(-1 < t < 1\). However, the mgf of a double exponential distribution is
\[
\int_{-\infty}^{\infty} e^{tx} e^{-|x|/2} = \frac{1}{1-t^2},
\]
so \(Y\) has the double exponential distribution.