MA/STAT 598 T. Lyapunov Exponents for Stochastic PDEs.

Purdue University, Fall 2002.

Professor Frederi Viens

Classroom and time: UNIV 319, Tu Th 10:15 - 11:45.

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- **Primary audience**. Ph.D. students and advanced M.S. students interested in stochastic processes.
- **General Course Description**. A basic course which introduces the topic of stochastic partial differential equations (SPDEs) via some simple examples that are amenable to detailed calculations, focusing mainly on parabolic equations. The prerequisite for the course being only a graduate course in probability theory, our "Chapter Zero" will be a tailored crash course on stochastics. We will then cover: existence and uniqueness questions, Feynman-Kac-type and particle representation, local (Holder continuity) and long-term (Lyapunov exponents) behavior via the theory of Gaussian regularity, and the brand new topic of SPDEs driven by fractional Brownian motion. We will look at specific applications in finance (theory of stochastic interest rates) and/or filtering (optimal nonlinear stochastic filtering theory). The first half of the semester will be spent covering the basics of the theory. In the second half, the instructor will continue to present lectures, now on more advanced material. Simultaneously each student will choose or will be assigned a short research article (or a portion of a longer article) to read and present in class. The list of articles to choose from are the most representative of the main recent contributions to the theory. If the class is large, groups will be allowed for the oral presentations. An important consequence of this course is that several students should be able to clearly identify PhD topics.
- **Prerequisites**: A non-measure-theoretic working knowledge of probability theory AND stochastic calculus, contained for example in a solid performance in 598F or in 532, or in the first few chapters of Oksendal's textbook on Stochastic Differential Equations. Because most of this course will be based heavily on explicit Gaussian tools, it will be accessible to a large audience of students with a working knowledge of stochastic calculus.
- **List of topics.** This list is non-exhaustive, and we reserve the right to skip some of the topics listed, depending on student interest. Although we will attempt to give detailed proofs for a number of topics, some proofs will be abridged and/or left as homework problems for the students.
  - Background material in stochastic analysis
    - basic stochastic calculus (along the lines of [6], Chapter 3),
    - stochastic differential equations and Feynman-Kac formulas for PDEs (e.g. [6], Ch 4)
    - extension to SPDEs, and evolution equations,

- Branching and interacting particle systems and representations of SPDEs
    * the Zakai equation for filtering and the particle system of Crisan-Gaines-Lyons [11]
    * extension to nonlinear SPDEs: a largely open problem, from [34]
- Lyapunov Lower bound in discrete space: from the material in Chapter 4 of [7]
- Lyapunov Upper bound in discrete space: from [8]
    * Basic Gaussian supremum estimates – Borell-type inequality (from [1])
    * application to the SPDE
- Exact Lyapunov exponent result of Cranston, Mountford and Shiga in discrete space: [10]
- Upper bound in continuous space
    * discretization and control of the error: from [33]
    * upper bound calculation: from [9]
- Fernique-Talagrand regularity theory
    * Fernique's upper bound on almost-sure moduli of continuity: from [1]
    * Talagrand's lower bound: from [28]
- Application of regularity theory to additive stochastic heat equation
    * Holder Regularity theory for SPDEs (simplified version of [30], [31])
    * SPDEs with fractional Brownian motion (from [29])
    * Beyond Holder regularity (from [29])
- Lower bound in continuous space
    * extension of regularity theory to linear multiplicative SPDE (conjecture)
    * Using the Gaussian supremum estimates to estimate the lower bound (from [33])

- **Open problems.** They will be presented at the most appropriate places in the course of the lectures; they will include:
    - examples in discrete space where percolation theory may be used for Lyapunov exponents, beyond the case of space-time white noise: appropriate for a Ph.D. dissertation;
    - using spin glasses for the upper bound in the Hölder case: a hard problem;
    - understanding the logarithmic corrections in the Hölder case, a good topic with ramifications that could result in a Ph.D. dissertation;
    - Lyapunov upper bound for less-than-Hölder cases: may be very difficult;
    - Lyapunov lower bounds for less-than-Hölder cases: perhaps an exercise, unless the case of discontinuous processes is considered;
    - Lyapunov exponents for the case of continuous space-time white noise: the issue of the Feynman-Kac formula itself seems to be at the core of the problem; may require a competely new approach;
    - a myriad of problems for linear systems of SPDEs, or for SPDEs with random differential operators: plenty of room for a variety of levels of difficulty...

The following bibliography contains the articles cited above, and some other nice references. Please see the instructor before jumping into any of these articles, for pointers on how to read them.

References

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