A localized version of the SK model with external field

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Abstract: In this note, we consider a Sherrington-Kirkpatrick- (SK)-type model on $\mathbb{Z}^d$ for $d \geq 1$, weighted by a function allowing for any single spin to interact with a small proportion of the other ones. Both mean field and diluted cases are considered, and we compute the replica symmetric solution at high temperature in those two cases. We are also able to study the influence of some boundary conditions on the replica symmetric solution under mild assumptions.

1 Introduction

Conceived originally as a simplified model that could describe the main behavior of some special alloys (see [8]), the Sherrington-Kirkpatrick (SK) model has become in fact a canonical example of disordered system, and the techniques introduced to handle it are also used in a wide number of applications, ranging from neural networks (cf. [10], [1]) to polymer measures [2] or biodiversity [4]. On the other hand, some substantial progress has been made in the understanding of this canonical model, through the introduction of Parisi's ansatz [6] and the smart path method (see [5], [10]).

In this paper, however, we would like to focus on one of the main simplifications that have been made in order to make the original SK model solvable, that is the mean field approximation. Indeed, letting all the spins interact with each other allows the limiting mixing behavior of the system to take place in a simple way. On the other hand, the results on short range models with random interactions are scarce (see however e.g. [7] for an account on the topic). Nevertheless, the mean field approximation is often seen as an oversimplification, physically unrealistic; in particular, it does not take into account the geometry of the system under consideration. Thus, our aim in this note is to introduce a kind of localized mean field model, that will partially respect the the geometric shape of our system, but also share some features of the original SK model with external field

For $N, d \geq 1$, our space of configurations will be $\Sigma = \Sigma_N = \{-1, 1\}^{C_N}$, where C_N is the finite lattice box $C_N = [-N; N]^d$ in $\mathbb{Z}^d$. For a given configuration $\sigma \in \Sigma_N$, we will consider the Hamiltonian

$$-H_N(\sigma) = \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j)\in C_N} q\left(\frac{i-j}{N}\right) g_{(i,j)}\sigma_i\sigma_j + h \sum_{i\in C_N} \sigma_i, \tag{1}$$

where β stands for the inverse of the temperature of the system, $\hat{N} = 2N+1$, (i,j) is our (abusive) notation for a pair of sites $i,j \in C_N$ (taken only once), $\left\{g_{(i,j)} : (i,j) \in C_N\right\}$ is a family of IID standard centered Gaussian random variables, and h represents a constant positive external field, under which the spins tend to take the value $+1$. Our localization is represented by the function q, which can be thought of as a smooth frame, and which is only assumed to be defined on $[-1,1]^d$ such that q^2 is of positive type, so that q^2 is non-negative and invariant by symmetry about the origin. We also assume q^2 is continuously differentiable, including at its periodic boundary. We refer to Section 2 for some more detailed information about our model. Notice that

1. When $h = 0$, this kind of localized model have been considered in [11], using the martingale method introduced by Comets and Neveu [3]. In our case, however, the results will have to be obtained thanks to the smart path and cavity methods.
2. Having a localizing weight of the form $q(\frac{\cdot}{N})$ implies that each spin will interact with at least a proportion of the $\hat{N}^d$ others, which means that we will stay in the mean field range.

This paper's aim is to get the replica solution for the model defined by the Hamiltonian (1) and some related ones, at high temperature: let $Z_N = \sum_{\sigma\in\Sigma_N} e^{-H_N(\sigma)}$ and $p_N(\beta) = \frac{1}{N}\mathbf{E}[\log(Z_N)]$. First, if γ_0 stands for the L^2 norm of q on $[-1,1]^d$, and if $SK(\hat{\beta},h)$ is the usual replica solution for the SK model at temperature $\hat{\beta}$, we will get that, for β small enough,

$$\left| p_N(\beta) - \frac{1}{\gamma_0} SK\left(\sqrt{\gamma_0}\beta, h\right) \right| \le \frac{C}{N},$$

for a constant C depending on the parameters of the model. Observe here that the speed of convergence obtained is of order $\hat{N}$ instead of $\hat{N}^d$ in case of the SK model. This is due to the slow rate of convergence of some Riemann sums, and we will also show how to obtain some better bounds by changing the assumptions made on q (namely by letting q having smoother oscillations). It is also worth noticing that the final result does not depend on the actual shape of q, but only on its L^2 norm γ_0. The proof of this result will heavily rely on some Fourier analysis performed on q, combined with the cavity and smart path methods. Note however that the usual cavity-method ingredient of "proof by induction on N" is not used; rather, some explicit calculations are performed for fixed N.

Once the geometry of the system is reintroduced by our weight q, it will also be natural to study the influence of the boundary conditions on the limiting behavior of the system: we will show that those boundary condition will not affect the replica symmetric solution at high temperature under very mild assumptions, a common fact shared with the Ising model.

We will then make another step towards a nearest neighbor type model, and consider on Σ_N the Hamiltonian

$$-H_N(\sigma) = \beta \sum_{(i,j)\in C_N} q_N(i-j)\gamma_{(i,j)} g_{(i,j)}\sigma_i\sigma_j + \beta h \sum_{i\in C_N} \sigma_i,$$

where $\gamma_{(i,j)}$ is an i.i.d. sequence of Bernoulli random variables with parameter $\frac{\gamma}{\hat{N}^d}$, i.e.

$$P\left(\gamma_{(i,j)} = 1\right) = \frac{\gamma}{\hat{N}^d}, \qquad P\left(\gamma_{(i,j)} = 0\right) = 1 - \frac{\gamma}{\hat{N}^d}.$$

This defines a diluted localized model, for which a given spin interacts (in mean) with a finite number of spins in its neighborhood defined by q. This is a natural extension of the previous model, also considered for the non-localized case in [9], [10], and has, from our point of view, one additional advantage: the limiting behavior of this system will involve the whole structure of q, through the introduction of a Poisson point process with intensity q, and not only its L^2 norm as in the mean field case. On the other hand, the formula obtained for the replica-symmetric solution becomes quite involved, and we refer to Theorem 3.16 for a precise description of the result we obtained.

Our paper is organized as follows: in Section 2, we deal with the mean field localized model , starting with some preliminary results on the cavity method and a definition of the overlap suited to our situation in Section 2.1, getting then the replica solution in Section 2.2, and considering the influence of the boundary conditions in Section 2.3. Then, we treat the diluted case in Section 3.

2 The localized mean field case

We consider a random Gibbs measure on the configurations of spins in the finite lattice box $C_N = [-N;N]^d$ in $\mathbb{Z}^d$ defined by its Hamiltonian $H = H_N$ on $\Sigma = \Sigma_N = \{-1,1\}^{C_N}$ as follows. For each $\sigma \in \Sigma_N$ let

$$-H_N(\sigma) = \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j)\in C_N} q\left(\frac{i-j}{N}\right) g_{(i,j)}\sigma_i\sigma_j + h\sum_{i\in C_N}\sigma_i$$

where β,h are fixed positive numbers, $\hat{N} = 2N+1$, (i,j) stands for a pair in C_N counted only once, the set $\left\{g_{(i,j)} : (i,j)\in C_N\right\}$ is a family of IID standard centered Gaussian random variables, and q is a function defined on $[-1,1]^d$ such that q^2 is of positive type, so that q^2 is non-negative and invariant by symmetry about the origin. We also assume q^2 is continuously differentiable, including at its periodic boundary. The Gibbs probability measure is defined for each $\sigma \in \Sigma_N$ by

$$G_N(\sigma) = B(\sigma)/Z$$

where

$$B(\sigma) = B_N(\sigma) = \exp(-H_N(\sigma)),$$
$$Z = Z_N = \sum_{\sigma\in\Sigma_N} B(\sigma).$$

Note that whether the sum over $(i,j)\in C_N$ includes the diagonal $i=j$ or not does not effect the Gibbs measure. The average with respect to the Gibbs measure is denoted by $\langle\cdot\rangle$, that is $\langle F\rangle = Z^{-1}\sum_{\sigma\in\Sigma_N} B(\sigma)F(\sigma)$, while the expectation with respect to the randomness of the $g_{(i,j)}$'s is denoted by $\mathbf{E}$. When a function F is defined on $(\Sigma_N)^n$, then $F\left(\sigma^1,\sigma^2,\cdots,\sigma^n\right)$ is a random variable on the product space of several independent replicas of the Gibbs measure, and we still use the notation $\langle F\rangle$ which now denotes $Z_N^{-n}\sum_{\sigma=(\sigma^1,\sigma^2,\cdots,\sigma^n)\in(\Sigma_N)^n}\prod_{k=1}^n B(\sigma_j)F(\sigma)$. Note however, that the same random variables $g_{(i,j)}$ are common to all the replicas of the Gibbs measures. We will commonly use the abusive notation $\langle F(\sigma)\rangle$ instead of $\langle F\rangle$.

The interaction function q can be taken to signify, for example, that interactions decay with the distance between sites of C_N. For example, $q^2(x) = 1/2 + 1/2\cos(\pi|x|)$. A function q that decays very fast near a peak at the origin signifies a very weak interaction except at neighboring sites. In all cases, denoting the Euclidean inner product in $\mathbf{R}^d$ by a dot $\cdot$, we have the following Fourier representation for q^2:

$$q^2(x) = \gamma_0 + \sum_{k\in\mathbb{Z}^d-\{0\}} \gamma_k\cos(\pi k\cdot x) \qquad (2)$$
$$= \sum_{k\in\mathbb{Z}^d}\gamma_k e^{\iota\pi k\cdot x}$$

where the greek letter ι denotes the imaginary unit, where $\{\gamma_k\}_{k\in\mathbb{Z}^d}$ is a family of nonnegative reals, where $\gamma_{-k} = \gamma_k$ for all $k \neq 0$, and we assume

$$\sum_k |k|\,\gamma_k < \infty$$

which ensures differentiability.

A quantity of interest is the expectation of the logarithm of the so-called *partition* function Z:

$$p_N = p_N(\beta,h) = \frac{1}{\hat{N}^d}\mathbf{E}[\log Z_N].$$

Lemma 2.1

$$\frac{\partial p_N}{\partial\beta} = \frac{\beta}{\hat{N}^{2d}}\sum_{(i,j)\in C_N} q^2\left(\frac{i-j}{N}\right)\mathbf{E}\left[1-\langle\sigma_i\sigma_j\rangle^2\right]$$
$$= \frac{\beta}{\hat{N}^{2d}}\sum_{(i,j)\in C_N} q^2\left(\frac{i-j}{N}\right) - \beta\sum_{k\in\mathbb{Z}^d}\gamma_k\mathbf{E}\left\langle\left|\frac{1}{\hat{N}^d}\sum_{i\in C_N} e^{\iota\pi i\cdot k/N}\sigma_i^1\sigma_i^2\right|^2\right\rangle.$$

Proof. Follows from the definitions with some simple algebra.

2.1 The Cavity method

The cavity method which we propose is influenced by the presentation of Talagrand in [10]. Some of the passages below are reminiscent of this work. In particular, the calculations in paragraph 2.1.2 seem to be fairly standard; a reader familiar with the subject can skip these. We have included these and other details in order to increase readability and self-containedness.

2.1.1 Notation

We introduce notation that will be useful in implementing the so-called *cavity* method for this problem. The idea is to single out a site m in C_N and create a cavity there, by progressively decorrelating the spin at m from the other spins. Any calculations to be performed on the Gibbs measure with decorrelated site m will be easier than with the full Gibbs measure.

Set $\hat{C}_N^m = \{i \in C_N; i \neq m\}$. We decompose, for all $\sigma \in \Sigma_N$,

$$-H_N(\sigma) = -H_N^m(\rho_m) + \sigma_m(h + g_m(\rho_m))$$

where ρ_m denotes the ordered spin values except for the m-th spin, that is, the $\left(\hat{N}^d - 1\right)$-tuple $\left(\sigma_i; i \in \hat{C}_N^m\right)$ and we denote

$$g_m(\rho_m) = \frac{\beta}{\hat{N}^{d/2}} \sum_{i \in \hat{C}_N^m} q\left(\frac{m-i}{N}\right) g_{(m,i)}\sigma_i,$$

and we also use the notation

$$-H_N^m(\rho_m) = \sum_{i,j \in \hat{C}_N^m} q\left(\frac{i-j}{N}\right) g_{(i,j)}\sigma_i\sigma_j + h \sum_{i \in \hat{C}_N^m} \sigma_i.$$

This new Hamiltonian is similar but not identical to the Hamiltonian $-H_{N-1}$ on Σ_{N-1}. Note in particular that H_N^m has $\hat{N}^d - 1$ variables, while H_{N-1} has $\widehat{N-1}^d$ variables. For a function $f = f(\sigma_m; \cdots)$ depending on σ_m and other variables, let $\mathbf{Av}\, f(\cdots)$ denote the average of f with respect to the two values $\sigma_m = \pm 1$. Then we can write, for any function f on Σ_N,

$$\begin{aligned}\langle f \rangle &= \frac{1}{Z_N} \sum_{\left\{\rho_m = \left(\sigma_i : i \in \hat{C}_N^m\right)\right\}} \left\{ \left[\sum_{\sigma_m = \pm 1} f(\sigma_m, \rho_m) e^{\sigma_m(h + g_m(\rho_m))} \right] \exp\left(-H_N^m(\rho_m)\right) \right\} \\ &= \frac{1}{Z} \left\langle \mathbf{Av}\, f(\cdot, \rho_m) e^{\cdot(h + g_m(\rho_m))} \right\rangle_{-,m}\end{aligned}$$

where $\langle \cdot \rangle_{-,m}$ denotes the average of a function of ρ_m only with respect to the Gibbs measure with Hamiltonian H_N^m, so that necessarily the normalizing function Z above must be equal to

$$Z = \left\langle \mathbf{Av}\, e^{\cdot(h + g_m(\rho_m))} \right\rangle_{-,m} = \langle \cosh(h + g_m(\rho_m)) \rangle_{-,m}.$$

We now introduce the decorrelation scale for the spin at site m, by progressively killing the only term that is responsible for correlation, namely $g_m(\rho_m)$, and replacing it with an independent Gaussian r.v. of appropriate magnitude, as follows. For any $t \in [0,1]$ let

$$g_{m,t}(\rho_m) = \sqrt{t} g_m(\rho_m) + \beta\sqrt{\hat{r}}\sqrt{1-t}G$$

where G is Gaussian $\mathcal{N}(0,1)$ independent of everything and $\hat{r}$ is a number that will be chosen later. For $t = 1$, this changes nothing. For $t = 0$, there is indeed no correlation left between σ_m and the other spins. More precisely we can now introduce the averaging with respect to a measure with this new partial correlation for fixed t. We do this in a multivariable setting, where a function f depends on an n-tuple of configurations of spins $\left(\sigma^1, \cdots, \sigma^l, \cdots, \sigma^n\right)$, and possibly some other variables. The averaging operator $\mathbf{Av}$

operates on $f = f(\sigma_m^1, \cdots, \sigma_m^n; \cdots)$ by calculating the average of f with respect to all 2^n possibilities for $(\sigma^1, \cdots, \sigma^l, \cdots, \sigma^n)$. Let

$$\begin{aligned}
\mathcal{E}_{m,n,t} &= \exp\left(\sum_{l=1}^{n} \sigma_m^l \left(h + g_{m,t}\left(\rho_m^l\right)\right)\right) \\
\mathcal{E}_{m,t} &= \mathcal{E}_{m,1,t} \\
Z_{m,t} &= \langle \mathbf{Av}\, \mathcal{E}_{m,t} \rangle_{-,m} \\
\langle f \rangle_{m,t} &= \frac{1}{\left(Z_{m,t}\right)^n} \langle \mathbf{Av}\, f \mathcal{E}_{m,n,t} \rangle_{-,m} \\
\nu_{m,t}(f) &= \mathbf{E}\left[\langle f \rangle_{m,t}\right] = \mathbf{E}\left[\frac{1}{\left(Z_{m,t}\right)^n} \langle \mathbf{Av}\, f \mathcal{E}_{m,n,t} \rangle_{-,m}\right].
\end{aligned}$$

2.1.2 Calculation of the variations of $\nu_{m,t}$

We have

$$\frac{\partial \mathcal{E}_{m,n,t}}{\partial t} = \mathcal{E}_{m,n,t}\left(\frac{1}{2\sqrt{t}} \sum_{l=1}^{n} \sigma_m^l g_m\left(\rho_m^l\right) - \frac{\beta\sqrt{\hat{r}}G}{2\sqrt{1-t}} \sum_{l=1}^{n} \sigma_m^l\right)$$

and

$$\frac{\partial Z_{m,t}}{\partial t} = \left\langle \mathbf{Av}\, \mathcal{E}_{m,t}\left(\frac{1}{2\sqrt{t}} \sigma_m g_m\left(\rho_m\right) - \frac{\beta\sqrt{\hat{r}}G}{2\sqrt{1-t}} \sigma_m\right)\right\rangle_{-,m}$$

and thus

$$\begin{aligned}
&\frac{\partial \nu_{m,t}(f)}{\partial t} \\
&= \mathbf{E}\left[\frac{1}{\left(Z_{m,t}\right)^n}\left\langle \mathbf{Av}\, f\left(\frac{1}{2\sqrt{t}} \sum_{l=1}^{n} \sigma_m^l g_m\left(\rho_m^l\right)\right) \mathcal{E}_{m,n,t}\right\rangle_{-,m}\right] \\
&- \mathbf{E}\left[\frac{1}{\left(Z_{m,t}\right)^n}\left\langle \mathbf{Av}\, f \frac{\beta\sqrt{\hat{r}}G}{2\sqrt{1-t}} \mathcal{E}_{m,n,t} \sum_{l=1}^{n} \sigma_m^l\right\rangle_{-,m}\right] \\
&- n\mathbf{E}\left[\frac{1}{\left(Z_{m,t}\right)^{n+1}} \langle \mathbf{Av}\, f \mathcal{E}_{m,n,t} \rangle_{-,m}\left\langle \mathrm{Av} \mathcal{E}_{m,t}\left(\frac{1}{2\sqrt{t}} \sigma_m g_m\left(\rho_m\right) - \frac{\beta\sqrt{\hat{r}}G}{2\sqrt{1-t}} \sigma_m\right)\right\rangle_{-,m}\right] \\
&= A - B - C.
\end{aligned}$$

To calculate A we use the following identity, valid if G is a Gaussian random variable: $\mathbf{E}\left[GF(G)\right] = \mathbf{E}\left[\partial F / \partial g\,(G)\right]$. Since we have for each $i \neq m$,

$$\begin{aligned}
\frac{\partial \mathcal{E}_{m,n,t}}{\partial g_{(m,i)}} &= \mathcal{E}_{m,n,t} \sum_{l=1}^{n} \sigma_m^l \frac{\partial g_m\left(\rho_m^l\right)}{\partial g_{(m,i)}} \\
&= \frac{\beta\sqrt{t}}{\hat{N}^{d/2}} \mathcal{E}_{m,n,t} \sum_{l=1}^{n} \sigma_m^l q\left(\frac{m-i}{N}\right) \sigma_i^l; \\
\frac{\partial Z_{m,t}}{\partial g_{(m,i)}} &= \left\langle \mathbf{Av}\, \frac{\partial \mathcal{E}_{m,t}}{\partial g_{(m,i)}}\right\rangle_{-,m} \\
&= \left\langle \mathbf{Av}\, \frac{\beta\sqrt{t}}{\hat{N}^{d/2}} \mathcal{E}_{m,t} \sigma_m q\left(\frac{m-i}{N}\right) \sigma_i\right\rangle_{-,m}
\end{aligned}$$

we get

$$A = A_1 - nA_2,$$

with

$$A_1 = \mathbf{E}\left[\frac{1}{(Z_{m,t})^n}\left\langle \mathbf{Av}\, f \frac{1}{2\sqrt{t}} \sum_{i\in\hat{C}_N^m} \frac{\partial \mathcal{E}_{m,n,t}}{\partial g_{(m,i)}} \sum_{l=1}^n \sigma_m^l \frac{\beta}{\hat{N}^{d/2}} q\left(\frac{m-i}{N}\right)\sigma_i^l \right\rangle_{-,m}\right],$$

and

$$A_2 = \sum_{i\in\hat{C}_N^m} \mathbf{E}\left[\frac{1}{(Z_{m,t})^{n+1}} \frac{\partial Z_{m,t}}{\partial g_{(m,i)}} \left\langle \mathbf{Av}\, f \frac{1}{2\sqrt{t}} \mathcal{E}_{m,n,t} \sum_{l=1}^n \sigma_m^l \frac{\beta}{\hat{N}^{d/2}} q\left(\frac{m-i}{N}\right)\sigma_i^l \right\rangle_{-,m}\right],$$

so

$$A_1 = \frac{\beta^2}{2\hat{N}^d} \sum_{i\in\hat{C}_N^m} q^2\left(\frac{|m-i|}{N}\right) \sum_{l,l'=1}^n \nu_{m,t}\left(f\sigma_i^{l'}\sigma_i^l\sigma_m^{l'}\sigma_m^l\right),$$

and

$$A_2 = \frac{\beta^2}{2\hat{N}^d} \sum_{i\in\hat{C}_N^m} q^2\left(\frac{m-i}{N}\right) \sum_{l=1}^n \nu_{m,t}\left(f\sigma_i^l\sigma_i^{n+1}\sigma_m^l\sigma_m^{n+1}\right),$$

where in the second line we combined the product of one average over n independent replicas of the Gibbs space with one average over an additional replica of the Gibbs space into just one average over $n+1$ independent replicas. This yields

$$A = \frac{\beta^2}{2\hat{N}^d} \sum_{i\in\hat{C}_N^m} q^2\left(\frac{m-i}{N}\right)\left[\sum_{l,l'=1}^n \nu_{m,t}\left(f\sigma_i^{l'}\sigma_i^l\sigma_m^{l'}\sigma_m^l\right) - n\sum_{l=1}^n \nu_{m,t}\left(f\sigma_i^l\sigma_i^{n+1}\sigma_m^l\sigma_m^{n+1}\right)\right].$$

For B, using again integration-by-parts, now for the Gaussian r.v. G itself, since

$$\begin{aligned}
\frac{\partial \mathcal{E}_{m,n,t}}{\partial G} &= \mathcal{E}_{m,n,t} \sum_{l=1}^n \sigma_m^l \beta\sqrt{\hat{r}}\sqrt{1-t},\\
\frac{\partial Z_{m,t}}{\partial G} &= \left\langle \mathbf{Av}\, \frac{\partial \mathcal{E}_{m,t}}{\partial G}\right\rangle_{-,m}\\
&= \left\langle \mathbf{Av}\, \mathcal{E}_{m,t}\sigma_m \beta\sqrt{\hat{r}}\sqrt{1-t}\right\rangle_{-,m}
\end{aligned}$$

we obtain

$$B = \frac{\beta^2\hat{r}}{2}\sum_{l=1}^n\sum_{l'=1}^n \nu_{m,t}\left(f\sigma_m^l\sigma_m^{l'}\right) - n\frac{\beta^2\hat{r}}{2}\sum_{l=1}^n \nu_{m,t}\left(f\sigma_m^l\sigma_m^{n+1}\right).$$

To calculate C we first write $C = C_1 - C_2$ with

$$\begin{aligned}
C_1 &= n\mathbf{E}\left[\frac{1}{(Z_{m,t})^{n+1}}\left\langle \mathbf{Av}\, f\mathcal{E}_{m,n,t}\right\rangle_{-,m}\left\langle \mathbf{Av}\, \mathcal{E}_{m,t}\frac{1}{2\sqrt{t}}\sigma_m g_m(\rho_m)\right\rangle_{-,m}\right],\\
C_2 &= n\mathbf{E}\left[\frac{1}{(Z_{m,t})^{n+1}}\left\langle \mathbf{Av}\, f\mathcal{E}_{m,n,t}\right\rangle_{-,m}\left\langle \mathbf{Av}\, \mathcal{E}_{m,t}\frac{\beta\sqrt{\hat{r}}G}{2\sqrt{1-t}}\sigma_m\right\rangle_{-,m}\right].
\end{aligned}$$

We first rewrite the products of Gibbs averages as single Gibbs average, and then use the same calculations that lead to the expression for A to get:

$$C_1 = \frac{n\beta^2}{2\hat{N}^d}\sum_{i\in\hat{C}_N^m} q^2\left(\frac{m-i}{N}\right)\left[\sum_{l=1}^{n+1}\nu_{m,t}\left(f\sigma_i^l\sigma_i^{n+1}\sigma_m^l\sigma_m^{n+1}\right) - (n+1)\,\nu_{m,t}\left(f\sigma_i^{n+1}\sigma_i^{n+2}\sigma_m^{n+1}\sigma_m^{n+2}\right)\right].$$

The same calculations as for B yields similarly that

$$C_2 = n\frac{\beta^2\hat{r}}{2}\sum_{l=1}^{n+1}\nu_{m,t}\left(f\sigma_m^l\sigma_m^{n+1}\right) - n\,(n+1)\,\frac{\beta^2\hat{r}}{2}\nu_{m,t}\left(f\sigma_m^{n+1}\sigma_m^{n+2}\right).$$

Grouping all our calculations, and some algebra, yield that the quantity $\partial_t \nu_{m,t}(f)$ is equal to

$$\frac{\beta^2}{2\hat{N}^d} \sum_{i \in C_N} q^2 \left(\frac{m-i}{N}\right) \left(2 \sum_{1 \le l < l' \le n} \nu_{m,t}\left(f \sigma_i^{l'} \sigma_i^l \sigma_m^{l'} \sigma_m^l\right)\right.$$
$$- 2n \sum_{l=1}^{n} \nu_{m,t}\left(f \sigma_i^l \sigma_i^{n+1} \sigma_m^l \sigma_m^{n+1}\right)$$
$$\left. + n(n+1)\nu_{m,t}\left(f \sigma_i^{n+1} \sigma_i^{n+2} \sigma_m^{n+1} \sigma_m^{n+2}\right)\right)$$
$$- \frac{\beta^2 \hat{r}}{2}\left(2 \sum_{1 \le l < l' \le n} \nu_{m,t}\left(f \sigma_m^{l'} \sigma_m^l\right) - 2n \sum_{l=1}^{n} \nu_{m,t}\left(f \sigma_m^l \sigma_m^{n+1}\right) + n(n+1)\nu_{m,t}\left(f \sigma_m^{n+1} \sigma_m^{n+2}\right)\right), \quad (3)$$

where we used the fact that for $i = m$, the corresponding term is zero.

We now identify *overlap*-type quantities. Let

$$R^{l,l'} = \frac{1}{\hat{N}^d} \sum_{i \in C_N} \sigma_i^{l'} \sigma_i^l$$

For $k \in \mathbb{Z}^d - \{0\}$, set now

$$R_k^{l,l'} = \frac{1}{\hat{N}^d} \sum_{i \in C_N} e^{\iota \pi i \cdot k/N} \sigma_i^{l'} \sigma_i^l \quad (4)$$

while for $k = 0$ we use instead

$$R_0^{l,l'} = R^{l,l'} - r, \qquad r := \frac{\hat{r}}{\gamma_0}. \quad (5)$$

We call $R^{l,l'}$ the basic *overlap* quantity for our problem as it measure the average overlap of spin values for a fixed configuration. The $R_k^{l,l'}$'s are also overlaps, albeit relative to a certain Fourier mode k. That all these overlaps are significant can already be seen immediately since, according to Lemma 2.1, a quantity of importance for the behavior of the partition function is precisely $\sum_{k \in \mathbb{Z}^d - \{0\}} \gamma_k \left|R_k^{1,2}\right|^2 + \gamma_0 \left|R^{1,2}\right|^2$. The sum of this last series can thus be considered as the *total overlap* relevant to the problem of studying the partition function. With these overlap notations in mind, the expression for $\partial \nu_{m,t}(f) / \partial t$ in (3) easily yields the following, which the reveals relevance of $\nu_{m,t}$ to the calculation of expected overlaps.

Proposition 2.2 *For any $f : \Sigma_N^n \to \mathbf{R}$ and $t \in [0,1]$, the derivative of $\nu_{m,t}(f)$ is given by*

$$\frac{\partial \nu_{m,t}(f)}{\partial t} = \beta^2 \sum_{k \in \mathbb{Z}^d} \gamma_k e^{\iota \pi m \cdot k/N} \left(\sum_{1 \le l < l' \le n} \nu_{m,t}\left(f \overline{R_k^{l,l'}} \sigma_m^{l'} \sigma_m^l\right)\right.$$
$$\left. - n \sum_{l=1}^{n} \nu_{m,t}\left(f \overline{R_k^{l,n+1}} \sigma_m^l \sigma_m^{n+1}\right) + \frac{n(n+1)}{2} \nu_{m,t}\left(f \overline{R_k^{n+1,n+2}} \sigma_m^{n+1} \sigma_m^{n+2}\right)\right) \quad (6)$$

2.1.3 Bounds on $\nu_{m,t}$

Lemma 2.3 *There exist positive constants $c_{n,q,\beta}$ and $c'_{n,q,\beta}$ that depend only on n, q and β, and are uniformly bounded in β for $\beta \in [0,1]$, such that if f is a positive function on $(\Sigma_N)^n$ then for all $m \in C_N$ and $t \in [0,1]$,*

$$\nu_{m,t}(f) \le c_{n,q,\beta} \nu(f) \quad (7)$$

and

$$\left|\nu_{m,t}(f) - \nu_{m,0}(f)\right| \le c'_{n,q,\beta} \beta^2 \nu^{1/2}\left(|f|^2\right) \left[\sum_{k \in \mathbb{Z}^d} \gamma_k \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right]. \quad (8)$$

Proof. See [10, Proposition 2.4.3]. □

This lemma, whose short proof follows essentially that of the corresponding result in the cavity method for the standard Sherrington-Kirkpatrick model, will be combined with the explicit expression for $\nu_{m,t}$ in Proposition 2.2 and a separate calculation of $\nu_{m,t}$ for $t=0$ to obtain information on the actual expected overlaps, under $\nu = \nu_{m,1}$, i.e. for $t = 1$.

2.1.4 Overlap calculation at $t = 0$

Lemma 2.4 *For fixed $m \in \Sigma_N$, let f be a function on $(\Sigma_N)^n$ that does not depend on the values $\sigma_m^1, \sigma_m^2, \cdots, \sigma_m^n$. Then for any subset I of $\{1, \cdots, n\}$ we have*

$$\nu_{m,0}\left(f\prod_{l\in I}\sigma_m^l\right) = \mathbf{E}\left[\tanh(Y)^{|I|}\right]\nu_{m,0}(f)$$

where Y is the Gaussian random variable defined as follows, with a standard normal variable z:

$$Y = \beta z\sqrt{\gamma_0 r} + h.$$

Proof. See [10, Lemma 2.4.1].

2.1.5 Overlap expressions at $t = 1$

The previous result, which states explicitly how $\nu_{m,0}$ separates the site m from the other ones, will be exploited in order to determine the most convenient value of r when estimating the expected-overlap-type quantity

$$\mathcal{O} := \nu\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\left|R_k^{1,2}\right|^2\right).$$

The first step is to notice the following consequence of a symmetry property among sites.

Lemma 2.5 *We have, for any fixed $m \in C_N$,*

$$\nu\left(\left|R^{1,2}-r\right|^2\right) = \nu\left(\left(R^{1,2}-r\right)\left(\sigma_m^1\sigma_m^2 - r\right)\right).$$

Proof. We can write by definition

$$\nu\left(\left|R^{1,2}-r\right|^2\right) = \nu\left(\left(R^{1,2}-r\right)\left(\frac{1}{\hat{N}^d}\sum_{i\in C_N}\sigma_i^1\sigma_i^2 - r\right)\right).$$

However, under ν, for fixed i, the joint law of $\sigma_i^1\sigma_i^2$ and $R^{1,2} = \frac{1}{\hat{N}^d}\sum_{j\in C_N}\sigma_j^1\sigma_j^2$ does not depend on i; indeed the variables $g_{(i,j)}$ are IID and the interactions under the Gibbs measure, modulo the values of the fixed $g_{(i,j)}$'s, are invariant under translation by definition. Therefore we have

$$\nu\left(\left|R^{1,2}-r\right|^2\right) = \nu\left(\left(R^{1,2}-r\right)\left(\sigma_m^1\sigma_m^2 - r\right)\right)$$

which is the lemma's assertion. □

Using Lemma 2.5 and expanding one of the two factors in $\left|R_k^{1,2}\right|^2$ in $\mathcal{O}$ we can now calculate $\mathcal{O}$ using a fixed arbitrary value $m \in C_N$, yielding the following.

Corollary 2.6

$$\begin{aligned}\mathcal{O} &= \gamma_0\nu\left(\left(R^{1,2}-r\right)\left(\sigma_m^1\sigma_m^2-r\right)\right)\\ &\quad+\sum_{k\in\mathbb{Z}^d-\{0\}}\frac{1}{\hat{N}^d}\sum_{i\in C_N}\gamma_k\nu\left(\overline{R_k^{1,2}}\sigma_i^1\sigma_i^2\right)e^{\iota\pi i\cdot k/N}.\end{aligned}$$

In order to exploit Lemma 2.4, we must modify the above expression for $\mathcal{O}$ by completing the following two tasks:

(i) estimate the error made by replacing the arguments of $\nu_{i,0}$ by functions that are of the same form as those in Lemma 2.4;

(ii) estimate the error made by replacing ν by $\nu_{m,0}$ (or $\nu_{i,0}$ as appropriate).

2.1.6 Task (i). Separation of cavity variable from others

We deal with task (i) first. It is sufficient to replace $R^{1,2}$ by the same quantity with the m-th term omitted: define

$$\left(R^{1,2}\right)^{-,m} := \frac{1}{\hat{N}^d} \sum_{i\in \hat{C}_N^m} \sigma_i^1\sigma_i^2 = R^{1,2} - \frac{1}{\hat{N}^d}\sigma_m^1\sigma_m^2 = R^{1,2} + 0\left(\frac{1}{\hat{N}^d}\right)$$

and similarly let

$$\begin{aligned}\left(R_k^{1,2}\right)^{-,m} &:= \frac{1}{\hat{N}^d} \sum_{i\in \hat{C}_N^m} \sigma_i^1\sigma_i^2 e^{\iota\pi i\cdot k/N} \\ &= R_k^{1,2} - \frac{1}{\hat{N}^d}\sigma_m^1\sigma_m^2 e^{\iota\pi m\cdot k/N} = R_k^{1,2} + 0\left(\frac{1}{\hat{N}^d}\right).\end{aligned}$$

Note that the "big oh" terms that are the difference of R and $R^{-,m}$, or of R_k and $(R_k)^{-,m}$, are uniform in all parameters, random or otherwise. Specifically let $\mathcal{O}_0$ denote the expected overlap quantity $\mathcal{O}$ with ν replaced by $\nu_{m,0}$ or $\nu_{i,0}$ as appropriate Thus, according to Corollary 2.6, and using the quantities $\left(R_k^{1,2}\right)^{-,m}$ with no m-spin dependence, we must have (or define):

$$\begin{aligned}\mathcal{O}_0 &= \gamma_0\nu_{m,0}\left(\left(\left(R^{1,2}\right)^{-,m} - r\right)\left(\sigma_m^1\sigma_m^2 - r\right)\right) \\ &+ \sum_{k\in\mathbb{Z}^d-\{0\}} \frac{1}{\hat{N}^d}\sum_{i\in C_N}\gamma_k\nu_{i,0}\left(\overline{\left(R_k^{1,2}\right)^{-,i}}\sigma_i^1\sigma_i^2\right)e^{\iota\pi i\cdot k/N} \\ &+ \gamma_0\nu_{m,0}\left(\frac{1}{\hat{N}^d}\sigma_m^1\sigma_m^2\left(\sigma_m^1\sigma_m^2 - r\right)\right) + \sum_{k\in\mathbb{Z}^d-\{0\}}\frac{1}{\hat{N}^d}\sum_{i\in C_N}\gamma_k\frac{1}{\hat{N}^d}\end{aligned}$$

The completion of task (i) can then immediately be summarized as the following.

Lemma 2.7

$$\begin{aligned}\mathcal{O}_0 &= \gamma_0\nu_{m,0}\left(\left(\left(R^{1,2}\right)^{-,m} - r\right)\left(\sigma_m^1\sigma_m^2 - r\right)\right) \\ &+ \sum_{k\in\mathbb{Z}^d-\{0\}} \frac{\gamma_k}{\hat{N}^d}\sum_{i\in C_N}\nu_{i,0}\left(\overline{\left(R_k^{1,2}\right)^{-,i}}\sigma_i^1\sigma_i^2\right)e^{\iota\pi i\cdot k/N} + 0\left(\hat{N}^{-d}\right).\end{aligned}$$

2.1.7 Task (ii). Estimating the difference between the overlaps at $t = 0$ and $t = 1$

We now prepare the tools for proving and exploiting task(ii), beginning with an elementary lemma, due to Lattala and Guerra, and whose proof can be found i.e. in [10, Proposition 2.4.5]:

Lemma 2.8 *For any choice of the parameters $\beta, \gamma_0, h > 0$, there is a unique solution r in $[0,1]$ to the equation*

$$r = \mathbf{E}\left[\tanh^2\left(\beta z\sqrt{\gamma_0 r} + h\right)\right]. \tag{9}$$

This lemma allows us to eliminate one of the terms in our overlap calculation $\mathcal{O}_0$ (at $t = 0$ with separated spins) by choosing r properly.

Lemma 2.9 *Let r be the solution of equation (9). Then, for any $m \in C_N$,*

$$\nu_{m,0}\left(\left(\left(R^{1,2}\right)^{-,m} - r\right)\left(\sigma_m^1\sigma_m^2 - r\right)\right) = 0.$$

Proof. By Lemma 2.4 we have

$$\begin{aligned}
&\nu_{m,0}\left(\left(\left(R^{1,2}\right)^{-,m} - r\right)\left(\sigma_m^1\sigma_m^2 - r\right)\right)\\
&= \nu_{m,0}\left(\left(R^{1,2}\right)^{-,m} - r\right)\left[\mathbf{E}\left[\tanh\left(\beta z\sqrt{\gamma_0 r} + h\right)^2\right] - r\right]\\
&= 0.
\end{aligned}$$

which finishes the proof. □

Similarly, with this choice of r, regarding the other term in $\mathcal{O}_0$ we can write

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d-\{0\}} \frac{\gamma_k}{\hat{N}^d}\sum_{i\in C_N}\nu_{i,0}\left(\overline{\left(R_k^{1,2}\right)^{-,i}}\sigma_i^1\sigma_i^2\right)e^{\iota\pi i\cdot k/N}\\
&= r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\frac{1}{\hat{N}^d}\sum_{i\in C_N}\nu_{i,0}\left(\overline{\left(R_k^{1,2}\right)^{-,i}}\right)e^{\iota\pi i\cdot k/N} \qquad (10)\\
&:= r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\left(A_k^1 + A_k^2 + A_k^3\right)
\end{aligned}$$

where

$$\begin{aligned}
A_k^1 &:= \nu\left(\overline{R_k^{1,2}}\right)\frac{1}{\hat{N}^d}\sum_{i\in C_N}e^{\iota\pi i\cdot k/N}\\
A_k^2 &:= \frac{1}{\hat{N}^d}\sum_{i\in C_N}\nu_{i,0}\left(\overline{\left(R_k^{1,2}\right)^{-,i}} - \overline{R_k^{1,2}}\right)e^{\iota\pi i\cdot k/N}\\
A_k^3 &:= \frac{1}{\hat{N}^d}\sum_{i\in C_N}\left[\nu_{i,0}\left(\overline{R_k^{1,2}}\right) - \nu\left(\overline{R_k^{1,2}}\right)\right]e^{\iota\pi i\cdot k/N}.
\end{aligned}$$

Let

$$S_k = \frac{1}{\hat{N}^d}\sum_{i\in C_N}e^{\iota\pi i\cdot k/N};$$

this is the Riemann sum approximation of

$$\int_{[-1,1]^d}dx e^{\iota\pi x\cdot k} = \prod_{l=1}^d\left(\int_{-1}^1 dx e^{\iota\pi x k_l}\right)$$

and since this product is zero because at least one of the components k_l of k is non-zero, we can bound $|S_k|$ by $|k|/N$, since $|k|$ is a bound on the integrand's gradient. However one may prefer to rewrite the Riemann sum as

$$S_k = \prod_{l=1}^d\left(\frac{1}{N}\sum_{i_l=-N}^N e^{\iota\pi i_l k_l/N}\right)$$

which shows that it is the product of the d Riemann sum approximations for the d integrals $\int_{-1}^1 dx e^{\iota\pi x k_l}$. Therefore, $|S_k|$ can also be bounded above by $\left(|k|/N\right)^{Z(k)}$ where $Z(k)$ is the number of components of k that are non-zero. We state this second result formally for future use.

Lemma 2.10 *Consider the following two conditions:*

(H) *There exists an integer $d' \in \{2, \cdots, d\}$ such that for every $k \in \mathbb{Z}^d - \{0\}$ such that $\gamma_k \neq 0$, the number $Z(k)$ of components of k that are non-zero satisfies $Z(k) \geq d'$.*

(H') *Condition (H) holds and we have*

$$\sum_{k\in\mathbb{Z}^d-\{0\}} \gamma_k \left|k\right|^{d'} < \infty. \tag{11}$$

Under Condition (H) we have for all $k \in \mathbb{Z}^d - \{0\}$

$$\left|A_k^1\right| \leq \frac{\left|k\right|^{d'}}{N^{d'}}.$$

For now, let us only use the first estimation on A_k^1, which does not require any additional assumptions on the structure of q, namely:

$$\left|A_k^1\right| \leq \frac{|k|}{N} \tag{12}$$

We easily obtain the following.

Lemma 2.11 *There exists a constant κ that depends on β and q but is bounded for β bounded such that*

$$\left|A_k^1 + A_k^2\right| \leq \frac{|k|}{N} + \frac{1}{\hat{N}^d} \leq 2\frac{|k|}{N}, \tag{13}$$

and

$$A_k^3 \leq \kappa\beta^2\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right]. \tag{14}$$

Proof. As noted before, $\left|\left(R_k^{1,2}\right)^{-,i} - R_k^{1,2}\right| \leq \hat{N}^{-d}$, and $\left|R_k^{1,2}\right| \leq 1$. Thus, ignoring Lemma 2.10, these facts yield (13). For the estimation (14) on A_k^3, we only need inequality (8) in Lemma 2.3. □

Remark 2.12 *In the remainder of the article, κ will stand for a positive constant depending on β and q and that is uniformly bounded in the range of our parameter β; we will allow κ to change from line to line.*

We are ready to state and prove the result which completes task (ii).

Lemma 2.13 *There exists a constant κ that depends on β and q but is bounded for β bounded, such that*

$$\left|\mathcal{O} - \mathcal{O}_0\right| \leq O\left(\hat{N}^{-d}\right) + \kappa\beta^2\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right)\right] \tag{15}$$

Proof. We can first write, using Lemma 2.3

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\left|\frac{1}{\hat{N}^d}\sum_{i\in C_N}\left[\nu\left(\overline{R_k^{1,2}}\sigma_i^1\sigma_i^2\right) - \nu_{i,0}\left(\overline{R_k^{1,2}}\sigma_i^1\sigma_i^2\right)\right]e^{\iota\pi i\cdot k/N}\right| \\
&\leq \kappa\beta^2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right].
\end{aligned}$$

Similarly

$$\begin{aligned}
&\gamma_0\left|\nu\left(\left(R^{1,2}-r\right)\left(\sigma_m^1\sigma_m^2 - r\right)\right) - \nu_{i,0}\left(\left(R^{1,2}-r\right)\left(\sigma_m^1\sigma_m^2-r\right)\right)\right| \\
&\leq \gamma_0\kappa\beta^2\nu^{1/2}\left(\left|\left(R^{1,2}-r\right)\left(\sigma_m^1\sigma_m^2-r\right)\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right] \\
&\leq \gamma_0\kappa\beta^2\nu^{1/2}\left(\left|R^{1,2}-r\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right] \\
&= \gamma_0\kappa\beta^2\nu^{1/2}\left(\left|R_0^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right],
\end{aligned}$$

where we used the fact, from Lemma 2.8, that $r \in [0,1]$.

We thus have

$$\begin{aligned}
&|\mathcal{O}-\mathcal{O}_0| \\
&\le O\left(\hat{N}^{-d}\right)+\kappa\beta^2 \sum_{k\in\mathbb{Z}^d-\{0\}} \gamma_k \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right] \\
&+\gamma_0\kappa\beta^2\nu^{1/2}\left(\left|R_0^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right] \\
&\le O\left(\hat{N}^{-d}\right)+\kappa\beta^2 \sum_{k\in\mathbb{Z}^d} \gamma_k \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right] \\
&= O\left(\hat{N}^{-d}\right)+\kappa\beta^2 \left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right]^2 \\
&\le O\left(\hat{N}^{-d}\right)+\kappa\beta^2 \left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right)\right],
\end{aligned}$$

which proves the lemma. □

2.1.8 Self-averaging overlap limit

We are now in a position to estimate $\mathcal{O}$. We show that for small β, this expected total overlap, which is recentered using the value r, converges to 0 at the speed $1/N$ as long as q has a continuous gradient. Since r does not depend on the random media g, the appellation "self-averaging" is used.

Proposition 2.14 *Let $\beta > 0$ and let $r = r(\beta)$ be the solution of*

$$r = \mathbf{E}\left[\tanh\left(\beta z\sqrt{\gamma_0 r}+h\right)^2\right]$$

where z is a standard normal random variable. Let κ be the constant defined in Lemma 2.13, i.e. κ is a constant that depends on β and q but is bounded for β bounded. Assume that

$$\sum_{k\in\mathbb{Z}^d}\gamma_k|k| < \infty, \tag{16}$$

and that β is so small that

$$\kappa\beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k|k| < 1.$$

In that case, we have, for N large enough, with $R_k^{1,2}$ defined by (4) and (5),

$$0 \le \nu\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\left|R_k^{1,2}\right|^2\right) \le \frac{r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k|k|}{\left(1-\kappa\beta^2\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\right)N}$$

Proof. We have, using (13), (14), and (15)

$$\begin{aligned}
0 \le \mathcal{O} &= \nu\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\left|R_k^{1,2}\right|^2\right) \\
&\le |\mathcal{O}-\mathcal{O}_0| + r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\left(A_k^1+A_k^2+A_k^3\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \kappa\beta^2\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right)\right]+0\left(\hat{N}^{-d}\right)\\
&+r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\frac{2\,|k|}{N}\\
&+r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\kappa\beta^2\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right]\\
&=0\left(\hat{N}^{-d}\right)+\kappa\beta^2\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\mathcal{O}+r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\frac{2\,|k|}{N}\\
&+\kappa\beta^2 r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\left[\sum_{k\in\mathbb{Z}^d}\gamma_k\nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right] \qquad (17)\\
&\leq 0\left(\hat{N}^{-d}\right)+\kappa\beta^2\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\mathcal{O}+\frac{r}{N}\sum_{k\in\mathbb{Z}^d}\gamma_k 2\,|k|\,. \qquad (18)
\end{aligned}
$$

where we used Jensen's inequality to identify the presence of $\mathcal{O}$ in line (17), and we recall that $O\left(\hat{N}^{-d}\right)$ is a function that tends to zero as fast as $\hat{N}^{-d}$ and that this convergence holds uniformly in all parameters. We now assume the C^1 condition (16). Thus since κ is bounded for β bounded, for β sufficiently small we can make $\kappa\beta^2\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)$ smaller than 1. The result of the proposition follows. □

Corollary 2.15 *Under the same assumptions as in Proposition 2.14, assuming additionally that Condition (H') holds, we have*

$$0\leq\nu\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\left|R_k^{1,2}\right|^2\right)\leq\frac{r\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\,|k|^{d'}}{\left(1-\kappa\beta^2\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\right)N^{d'}}.$$

Proof. This follows trivially from the proof of Proposition 2.14 if we use Lemma 2.10 instead of the bound (12). □

2.2 Consequence for the partition function

Let s be the solution of the equation

$$s=\mathbf{E}\left[\tanh^2\left(\beta z\sqrt{s}+h\right)\right] \qquad (19)$$

where z is a standard normal r.v. This $s=s(\beta)$ is uniquely defined by Lemma 9. Let

$$F(\beta,h,s)=\beta^2(1-s)^2/4+\log 2+\mathbf{E}\left[\log\left[\cosh\left(\beta z\sqrt{s}+h\right)\right]\right] \qquad (20)$$

and

$$SK(\beta,h)=F(\beta,h,s(\beta))\,. \qquad (21)$$

The function SK is known to be the limit of p_N in the standard Sherrington-Kirkpatrick model. The calculations of the previous sections imply the following theorem on the behavior of the expected logarithm of the partition function p_N in our situation.

Theorem 2.16 *Under the hypotheses of Proposition 2.14, we have*

$$\left|p_N(\beta)-\frac{1}{\gamma_0}SK\left(\sqrt{\gamma_0}\beta,h\right)\right|\leq\frac{\beta C(\beta)}{N} \qquad (22)$$

where the constant C depends on h, q, and β, and is bounded for $\beta\in[0,\beta_0]$.

Proof. Recall from Lemma 2.1 that

$$\begin{aligned}\frac{\partial p_N}{\partial\beta} &= \frac{\beta}{\hat{N}^{2d}}\sum_{(i,j)\in C_N} q^2\left(\frac{i-j}{N}\right) + \beta\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right) + \beta\gamma_0\nu\left(\left|R_0^{1,2}+r\right|^2\right)\\ &= \beta\left(B_1+B_2+B_3\right).\end{aligned}$$

We have that B_1 is a Riemann sum approximation of

$$\begin{aligned}&4^{-d}\int_{[-1,1]^d\times[-1,1]^d} dxdyq^2\left(x-y\right)\\ &= 4^{-d}\int_{[-1,1]^d} dy\int_{[-1,1]^d-y} dx' q^2\left(x'\right)\\ &= 4^{-d}\int_{[-1,1]^d} dy\int_{[-1,1]^d} dx' q^2\left(x'\right)\\ &= 2^{-d}\int_{[-1,1]^d} dx' q^2\left(x'\right)\\ &= \gamma_0.\end{aligned}$$

where we used the fact that q^2 is periodic in all variables, with period 2. Therefore, since by the C^1 condition (2) on q, the gradient in $[-1,1]^d\times[-1,1]^d$ of $q^2\left(x-y\right)$ is bounded by

$$c''_q := 2\sum_{k\in\mathbb{Z}^d}\gamma_k\left|k\right| < \infty$$

we have

$$\left|B_1-\gamma_0\right| \le \frac{c''_q}{N}. \tag{23}$$

Using this and Proposition 2.14, we can write for $\beta<\beta_0$,

$$\begin{aligned}&\left|\frac{\partial p_N}{\partial\beta}-\gamma_0\beta-r^2\gamma_0\beta\right|\\ &\le \frac{c''_q}{N}+\beta\left|\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right)+\gamma_0\nu\left(\left|R_0^{1,2}+r\right|^2\right)-r^2\gamma_0\right|\\ &= \frac{c''_q}{N}+\beta\left|\sum_{k\in\mathbb{Z}^d-\{0\}}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right)+\gamma_0\nu\left(\left|R_0^{1,2}\right|^2+r^2+2rR_0^{1,2}\right)-r^2\gamma_0\right|\\ &= \frac{c''_q}{N}+\beta\left|\sum_{k\in\mathbb{Z}^d}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right)+2\gamma_0 r\nu\left(R_0^{1,2}\right)\right|\\ &\le \frac{c''_q}{N}+\frac{2\beta rc''_q}{1-\kappa\beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k}\frac{1}{N}+2\gamma_0 r\beta\left|\nu\left(R_0^{1,2}\right)\right|.\end{aligned} \tag{24}$$

To estimate the last term, if we simply say that

$$\left|\nu\left(R_0^{1,2}\right)\right| \le \nu\left(\left|R_0^{1,2}\right|^2\right)^{1/2} \le \left(\sum_{k\in\mathbb{Z}^d}\gamma_k\nu\left(\left|R_k^{1,2}\right|^2\right)\right)^{1/2} \le K_{\beta,h,q}\frac{1}{\sqrt{N}}, \tag{25}$$

this does not allow us to get the announced convergence of order $1/N$. The following improvement is necessary. It is proved by showing how our choice for r allows the terms of order $1/\sqrt{N}$ to get killed off.

Lemma 2.17 *For $\beta<\beta_0$, $\left|\nu\left(R_0^{1,2}\right)\right| \le K\left(\beta,q\right)/N$ where $K\left(\beta,q\right)$ is bounded for β bounded.*

Proof. By symmetry among sites, we have

$$\begin{aligned}\nu\left(R_0^{1,2}\right) &= \nu\left(R^{1,2}\right) - r \\ &= \nu\left(\frac{1}{\hat{N}^d}\sum_{i\in C_N}\sigma_i^1\sigma_i^2\right) \\ &= \nu\left(\sigma_m^1\sigma_m^2 - r\right) = \nu\left(f\right)\end{aligned}$$

for any fixed value of m, where we define the function $f = \sigma_m^1\sigma_m^2 - r$. Recall also in general that from Lemma 2.4 we have for any function g defined on Σ_N (that depends on spin values from n independent replicas) but that does not depend on any replicas of σ_m,

$$\nu_{m,0}\left(g\sigma_m^1\sigma_m^2\right) = \nu_{m,0}\left(g\right)\mathbf{E}\left[\tanh^n\left(Y\right)\right] = \nu_{m,0}\left(g\right) r \tag{26}$$

where $Y = \beta\sqrt{\gamma_0 r}z + h$ with z a standard normal r.v. In particular we obtain

$$\nu_{m,0}\left(f\right) = 0. \tag{27}$$

This is important because of the following observation: we have

$$\begin{aligned}\nu\left(f\right) &= \nu_{m,0}\left(f\right) + \nu'_{m,0}\left(f\right) + \int_0^1\left(\int_0^t \nu''_{m,s}\left(f\right)ds\right)dt \\ &= \nu'_{m,0}\left(f\right) + \int_0^1\left(\int_0^t \nu''_{m,s}\left(f\right)ds\right)dt.\end{aligned} \tag{28}$$

We see that our task is to control $\nu'_{m,0}\left(f\right)$ and $\nu''_{m,s}\left(f\right)$.

From Proposition 2.2 we have

$$\begin{aligned}\nu'_{m,s}\left(f\right) &= \beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k e^{\iota\pi m\cdot k/N}\nu_{m,s}\left(f\overline{R_k^{1,2}}\sigma_m^1\sigma_m^2\right) \\ &\quad - 4\beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k e^{\iota\pi m\cdot k/N}\nu_{m,s}\left(f\overline{R_k^{1,3}}\sigma_m^1\sigma_m^3\right) \\ &\quad + 3\beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k e^{\iota\pi m\cdot k/N}\nu_{m,s}\left(f\overline{R_k^{3,4}}\sigma_m^4\sigma_m^4\right).\end{aligned} \tag{29}$$

Next, in the above formula, replace the terms $R_k^{l,l'}$ by their non-m dependent terms $\left(R_k^{l,l'}\right)^{-,m}$

$$\left(R_k^{l,l'}\right)^{-,m} = R_k^{l,l'} - \frac{1}{\hat{N}^d}\sigma_m^l\sigma_m^{l'}e^{\iota\pi m\cdot k/N}.$$

It is easy to check that, because of fact (27), this will not change the values of any of the terms. I.e.

$$\begin{aligned}\nu'_{m,s}\left(f\right) &= \beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k e^{\iota\pi m\cdot k/N}\nu_{m,s}\left(\left(\overline{R_k^{1,2}}\right)^{-,m} f\sigma_m^1\sigma_m^2\right) \\ &\quad - 4\beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k\nu_{m,s}\left(\left(\overline{R_k^{1,3}}\right)^{-,m} f\sigma_m^1\sigma_m^3\right) \\ &\quad + 3\beta^2\sum_{k\in\mathbb{Z}^d}\gamma_k\nu_{m,s}\left(\left(\overline{R_k^{3,4}}\right)^{-,m} f\sigma_m^3\sigma_m^4\right)\end{aligned}$$

which, after expanding f with the m-dependent factors above, and using again the result (26) from Lemma 2.4, yields

$$\nu'_{m,0}\left(f\right) = \beta^2\check{c}(\beta)\sum_{k\in\mathbb{Z}^d}\gamma_k e^{\iota\pi m\cdot k/N}\nu_{m,0}\left(\left(\overline{R_k^{1,2}}\right)^{-,m}\right), \tag{30}$$

where

$$\check{r} = \mathbf{E}\left[\tanh^4 Y\right], \qquad \check{c}(\beta) = 1 - 4r + 3\check{r}.$$

By iterating the formula (29) for $\nu'_{m,s}(f)$ we can calculate $\nu''_{m,s}(f)$. Rather than writing down all the terms in $\nu''_{m,s}(f)$ we will simply note that we will obtain a sum of terms of the form

$$c(\beta)\,\nu_{m,s}\left(f\sigma_m^l\sigma_m^{l'}\overline{R_k^{l,l'}}\sigma_m^{l''}\sigma_m^{l'''}\overline{R_{k'}^{l'',l'''}}\right)$$

where l, l', l'' and l''' are integer numbers between 1 and 6 and the constants $c(\beta)$, which may be complex numbers, are bounded when β is bounded. We then estimate each of these terms as follows:

$$\begin{aligned}
&c(\beta)\,\nu_{m,s}\left(f\sigma_m^l\sigma_m^{l'}\overline{R_k^{l,l'}}\sigma_m^{l''}\sigma_m^{l'''}\overline{R_{k'}^{l'',l'''}}\right)\\
&\leq c(\beta)\,|f|_\infty\,\nu_{m,s}\left(\left|R_k^{l,l'}\right|^2\right)^{1/2}\nu_{m,s}\left(\left|R_k^{l'',l'''}\right|^2\right)^{1/2}\\
&\leq c(b)\,(1+r)\left[\nu_{m,s}\left(\left|R_k^{1,2}\right|^2\right)+\nu_{m,s}\left(\left|R_{k'}^{1,2}\right|^2\right)\right]/2.
\end{aligned}$$

Putting all these estimates together yields:

$$\begin{aligned}
\left|\nu''_{m,s}(f)\right| &\leq K(\beta)\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\nu_{m,s}\left(\left|R_k^{1,2}\right|^2\right)\right)\\
&\leq K(\beta)\left(\sum_{k\in\mathbb{Z}^d}\gamma_k\right)2r\left(\sum_{k\in\mathbb{Z}^d}|k|\,\gamma_k\right)\frac{1}{N}\\
&= \kappa\frac{1}{N}
\end{aligned} \tag{31}$$

where we used Proposition 2.14 with β small enough and $\kappa$is a constant that depends only on β and q and is bounded when β is small enough. Again, we will allow κ below to change from line to line.

The estimate (31) on the second derivative of $\nu_{m,t}(f)$, together with (28), imply immediately

$$\left|\nu(f) - \nu'_{m,0}(f)\right| \leq \frac{\kappa}{N}. \tag{32}$$

We now rewrite $\nu'_{m,0}(f)$ with the intention of controlling the errors made by replacing $t=0$ with $t=1$, and replacing $\left(\overline{R_k^{1,2}}\right)^{-,m}$ with $\overline{R_k^{1,2}}$ for every k. The first replacement will be handled thanks to Lemma 2.3 and Proposition 2.14, while the second will only require noticing that we trivially have

$$\left|\nu_{m,0}\left(\left(\overline{R_k^{1,2}}\right)^{-,m}\right) - \nu_{m,0}\left(\overline{R_k^{1,2}}\right)\right| \leq \frac{1}{N^d}. \tag{33}$$

Accordingly we write

$$\nu'_{m,0}(f) = \beta^2\check{c}(\beta)\sum_{k\in\mathbb{Z}^d}\gamma_k\nu\left(\overline{R_k^{1,2}}\right) + D_1 + D_2 \tag{34}$$

where

$$D_1 = \beta^2\check{c}(\beta)\sum_{k\in\mathbb{Z}^d}\gamma_k\left[\nu_{m,0}\left(\overline{R_k^{1,2}}\right) - \nu\left(\overline{R_k^{1,2}}\right)\right]$$

and

$$D_2 = \beta^2\check{c}(\beta)\sum_{k\in\mathbb{Z}^d}\gamma_k\left[\nu_{m,0}\left(\left(\overline{R_k^{1,2}}\right)^{-,m}\right) - \nu_{m,0}\left(\overline{R_k^{1,2}}\right)\right].$$

We have by Lemma 2.3 and Jensen's inequality

$$\begin{aligned}|D_1| &\leq \beta^2 \check{c}(\beta) \sum_{k\in\mathbb{Z}^d} \gamma_k \beta^2 \kappa \left(\sum_{k'\in\mathbb{Z}^d} \gamma_{k'} \sqrt{\nu\left(\left|R_{k'}^{1,2}\right|^2\right)} \right) \sqrt{\nu\left(\left|R_k^{1,2}\right|^2\right)} \\ &= \beta^4 \kappa \left(\sum_{k\in\mathbb{Z}^d} \gamma_k \sqrt{\nu\left(\left|R_k^{1,2}\right|^2\right)} \right)^2 \\ &\leq \beta^4 \kappa \left(\sum_{k\in\mathbb{Z}^d} \gamma_k \right) \sum_{k\in\mathbb{Z}^d} \gamma_k \nu\left(\left|R_k^{1,2}\right|^2\right)\end{aligned}$$

so that by Proposition 2.14,

$$|D_1| \leq \frac{\kappa}{N}. \tag{35}$$

Now by (33) we immediately get

$$\begin{aligned}|D_2| &\leq \beta^2 \sum_{k\in\mathbb{Z}^d} \gamma_k \frac{1}{N^d} \check{c}(\beta) &(36)\\ &\leq \frac{\kappa}{N}. &(37)\end{aligned}$$

Now combining (32), (34), (35) and (37) we obtain

$$|\nu(f)| = \left|\nu(f) - \nu'_{m,0}(f) + \nu'_{m,0}(f)\right| \leq \frac{\kappa}{N} + \beta^2\gamma_0 \check{c}(\beta) |\nu(f)| + \beta^2 \check{c}(\beta) \sum_{k\in\mathbb{Z}^d-\{0\}} \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right) \leq \frac{\kappa}{N} + \beta^2\gamma_0 \check{c}(\beta) |\nu(f)|,$$

where we used again Jensen's inequality to apply Proposition 2.14. Therefore

$$|\nu(f)| \left(1 - \beta^2\gamma_0 |1 - 4r + 3\check{r}|\right) \leq \frac{\kappa}{N}$$

which proves the lemma for β small enough. □

Thanks to this lemma we now obtain from (24):

$$\left| \frac{\partial p_N}{\partial \beta} - \gamma_0 \beta \left(1 - r^2\right) \right| \leq \frac{\kappa}{N}$$

where κ depends only on β, h, q and is bounded for $\beta \in [0, \beta_0]$.

The first step to conclude the proof of the theorem it to perform the following integration:

$$\begin{aligned}&\left| p_N(\beta) - p_N(0) - \gamma_0 \int_0^\beta \left(1 - r(b)^2\right) b\, db \right| \\ &= \left| \int_0^\beta \left[\frac{\partial p_N}{\partial \beta}(b) - \gamma_0 \left(1 - r(b)^2\right) b \right] db \right| \\ &\leq \beta \frac{\kappa}{N}. &(38)\end{aligned}$$

Therefore we need to calculate two terms. Calculating $p_N(0)$ is trivial because we have by definition

$$\begin{aligned}p_N(0) &= \frac{1}{\hat{N}^d} \log\left[\sum_{\sigma\in\Sigma_N} \prod_{i\in C_N} e^{h\sigma_i} \right] \\ &= \frac{1}{\hat{N}^d} \log\left[\prod_{i\in C_N} \left(\sum_{\sigma_i=\pm1} e^{h\sigma_i} \right) \right] \\ &= \frac{1}{\hat{N}^d} \sum_{i\in C_N} \log(2\cosh(h)) \\ &= \log 2 + \log(\cosh(h)). &(39)\end{aligned}$$

For the final calculation we can use known facts about the standard Sherrington-Kirkpatrick model. Let s be the unique solution of the equation (19). We see that

$$s(\beta) = q(\sqrt{\gamma_0}\beta). \tag{40}$$

It is known (see [10, Lemma 2.4.5]) that with F and SK defined by (20) and (21), we have

$$\int_0^\beta \left(1 - s(x)^2\right) x\, dx = SK(\beta, h) - \log(2\cosh(h)).$$

By performing the trivial change of variable $b = \sqrt{\gamma_0}x$ demanded by relation (40) we obtain

$$\int_0^\beta \left(1 - r(b)^2\right) b\, db = \frac{1}{\gamma_0} SK(\sqrt{\gamma_0}\beta, h) - \log(2\cosh(h)).$$

Now combining this (38) and (39) we obtain the theorem. □

The final result we present in this section shows that while the complete structure of q does not seem to effect the limiting behavior of the partition function beyond the average value γ_0 of q^2, the speed of convergence towards this value may depend heavily on the behavior of q. We show that the speed can be increased to the order $N^{-d'}$ as long as Condition (H') from Lemma 2.10 holds. This condition restricts the Fourier decomposition of q^2 to containing only terms of the form $\cos(k \cdot x)$ where $k \cdot x$ is the sum of at least d' terms $k_l x_l$. Moreover the summability part (11) of Condition (H') means that q^2 is d' times differentiable.

Corollary 2.18 *Under the hypotheses of Corollary 2.15, we have*

$$\left| p_N(\beta) - \frac{1}{\gamma_0} SK(\sqrt{\gamma_0}\beta, h) \right| \leq \frac{\beta C(\beta)}{N^{d'}}$$

where the constant C depends on h, q, and β, and is bounded for $\beta \in [0, \beta_0]$.

Proof. In the proof of Theorem 2.16, including Lemma 2.17, some estimates are already of order $N^{-d} = 0\left(N^{d'}\right)$. For the others, we may use Corollary 2.15 instead of Proposition (2.14) in all its occurrences. This improves all estimates that were originally of order N^{-1} to the order $N^{-d'}$, with the exception of the estimation (23) for the term B_1. To deal with this last term, we only need to use Condition (H'). Indeed we write

$$\begin{aligned} |B_1 - \gamma_0| &= \left| \frac{1}{\hat{N}^{2d}} \sum_{(i,j) \in C_N} q^2\left(\frac{i-j}{N}\right) - \gamma_0 \right| \\ &= \frac{1}{\hat{N}^{2d}} \sum_{k \in \mathbb{Z}^d - \{0\}} \gamma_k \sum_{(i,j) \in C_N} e^{\iota \pi k \cdot (i-j)/N} \\ &= \frac{1}{\hat{N}^{2d}} \sum_{k \in \mathbb{Z}^d - \{0\}} \gamma_k \left(\sum_{i \in C_N} e^{\iota \pi k \cdot i/N} \right) \left(\sum_{j \in C_N} e^{-\iota \pi k \cdot j/N} \right) \end{aligned}$$

The conclusion of Lemma 2.10 also means that

$$\left| \frac{1}{\hat{N}^d} \sum_{i \in C_N} e^{\iota \pi i \cdot k/N} \right| \leq 2 \frac{|k|^{d'}}{N^{d'}}.$$

which implies immediately that

$$|B_1 - \gamma_0| \leq \frac{2}{N^{d'}} \sum_{k \in \mathbb{Z}^d - \{0\}} \gamma_k |k|^{d'}.$$

Thanks to Condition (11), the corollary is proved. □

2.3 Boundary conditions

In this section, we will show that the previous results are not affected if we take into account some boundary conditions in the Hamiltonian H_N: let $\{\omega_i : i \in \mathbb{Z}^d\}$ be an arbitrary deterministic family of real numbers, such that $|\omega_i| \le M$ for a given constant $M > 0$ (in fact, ω_i could also be allowed to depend on N, but we will not try to go deeper into that possibility). For $N \ge 1$, consider the Hamiltonian:

$$-H_N^*(\sigma) = -H_N(\sigma) + \sum_{i\in\partial C_N} \omega_i \sigma_i,$$

where

$$\partial C_N = \{i = (i_1, \ldots, i_d); \exists k \le d, |i_k| = N\}.$$

Notice that $\{\omega_i : i \in \partial C_N\}$ stands for the boundary condition, acting on each σ_i on ∂C_N. It can also be observed that H_N^* can be written as

$$-H_N^*(\sigma) = \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j)\in C_N} q\left(\frac{i-j}{N}\right) g_{(i,j)} \sigma_i \sigma_j + h \sum_{i\in C_N^\circ} \sigma_i + \sum_{i\in\partial C_N} h_i \sigma_i,$$

where $C_N^\circ = C_N \backslash \partial C_N$, and $h_i = h + \omega_i$.

Denote by $p_N^*(\beta)$, $\langle f\rangle_*$, ν_t^* the equivalent to $p_N(\beta)$, $\langle f\rangle$, ν_t related to the Hamiltonian H_N^*. Then we have the

Theorem 2.19 *Under the hypotheses of Proposition 2.14, we have*

$$\left| p_N^*(\beta) - \frac{1}{\gamma_0} SK\left(\sqrt{\gamma_0}\beta, h\right)\right| \le \frac{\beta C(\beta)}{N}$$

where the constant C depends on h, q, and β, and is bounded for $\beta \in [0, \beta_0]$.

Proof: The result is of course easily obtained once the self-averaging property for $R_k^{1,2}$ is established. Thus, we will just try to stress the main differences between the current case and the one dealt with at Proposition 2.14: first, note that the symmetry property (among sites) is no longer true, and hence the equivalent to Corollary 2.6 is

$$\mathcal{O} = \frac{1}{\hat{N}^d} \sum_{m\in C_N} \Bigg[\gamma_0 \nu\left(\left(R^{1,2} - r\right)\left(\sigma_m^1 \sigma_m^2 - r\right)\right) + \sum_{k\in\mathbb{Z}^d - \{0\}} \sum_{i\in C_N} \gamma_k \nu\left(\overline{R_k^{1,2}} \sigma_m^1 \sigma_m^2\right) e^{\iota\pi i\cdot k/N} \Bigg].$$

Thus Lemma 2.7 becomes

$$\mathcal{O}_0 = \frac{1}{\hat{N}^d} \sum_{m\in C_N} \Bigg[\gamma_0 \nu_{m,0}\left(\left(\left(R^{1,2}\right)^{-,m} - r\right)\left(\sigma_m^1 \sigma_m^2 - r\right)\right) + \sum_{k\in\mathbb{Z}^d - \{0\}} \gamma_k \sum_{m\in C_N} \nu_{m,0}\left(\overline{\left(R_k^{1,2}\right)^{-,m}} \sigma_m^1 \sigma_m^2\right) e^{\iota\pi m\cdot k/N} \Bigg] + 0\left(\hat{N}^{-d}\right).$$

As far as Lemma 2.9 is concerned, it is only valid for $m \in C_N^\circ$. However, invoking the fact that $\frac{|\partial C_N|}{|C_N|} = O(N^{-1})$, relation 10 becomes

$$\sum_{k\in\mathbb{Z}^d - \{0\}} \frac{\gamma_k}{\hat{N}^d} \sum_{i\in C_N} \nu_{i,0}\left(\overline{\left(R_k^{1,2}\right)^{-,i}} \sigma_i^1 \sigma_i^2\right) e^{\iota\pi i\cdot k/N} = r \sum_{k\in\mathbb{Z}^d - \{0\}} \gamma_k \left(A_k^1 + A_k^2 + A_k^3\right) + 0\left(\frac{1}{N}\right).$$

Up to those terms of order N^{-1}, the proof goes now along the same lines as for Proposition 2.14. □

3 The diluted case

The model under consideration here will be of the form

$$-H_N(\sigma) = \beta \sum_{(i,j)\in C_N} q_N(i-j)\gamma_{(i,j)} g_{(i,j)}\sigma_i\sigma_j + \beta h \sum_{i\in C_N} \sigma_i, \tag{41}$$

where β, h and q have been defined in the previous section, $q_N(x) = q(x/N)$ and $\gamma_{(i,j)}$ is an i.i.d. sequence of Bernoulli random variables with parameter $\gamma/\hat{N}^d$, i.e.

$$P\left(\gamma_{(i,j)} = 1\right) = \frac{\gamma}{\hat{N}^d}, \qquad P\left(\gamma_{(i,j)} = 0\right) = 1 - \frac{\gamma}{\hat{N}^d},$$

with $\hat{N}^d$ standing for $(2N+1)^d$, which is the size of C_N. We will also set $M_q = \|q\|_\infty$.

Contrary to the method we used for the mean-field case, in which calculations were performed for each N independently, the study of our diluted model will be based on an induction procedure on N, and this will force us to consider the Hamiltonian H on a general subset $F \subset C_N$, of size K, that is the quantity

$$-H_F(\sigma) = -H_{N,F}(\sigma) = \beta \sum_{(i,j)\in F} q_N(i-j)\gamma_{(i,j)} g_{(i,j)}\sigma_i\sigma_j + \beta h \sum_{i\in F} \sigma_i.$$

Having this fact in mind, our basic method will continue to be the cavity method, inspired again by the presentation in [10]; it will take the following form in our current situation: for any $m \in F$, notice that

$$-H_F(\sigma) = -H_{\hat{F}^m}(\sigma) + \beta\sigma_m \left(\sum_{i\in\hat{F}^m} \gamma_i^m W_i^m(\sigma_i) + h \right),$$

where $\gamma_i^m = \gamma_{(i,m)}$, $g_i^m = g_{(i,m)}$ and

$$\hat{F}^m = \{i \in F;\, i \neq m\}, \qquad W_i^m(\sigma_i) = q_N(i-m)\, g_{(i,m)}\sigma_i.$$

For any $F \subset C_N$, denote by $\langle . \rangle_F$ the Gibbs average on F with respect to the Hamiltonian H_F. Then, for $f : F^n \to \mathbf{R}$, we have

$$\langle f \rangle_F = \frac{\left\langle \mathbf{Av}\, f \hat{\mathcal{E}}_{m,n} \right\rangle_{\hat{F}^m}}{\left\langle \mathbf{Av}\, \hat{\mathcal{E}}_m \right\rangle_{\hat{F}^m}^n},$$

where

$$\hat{\mathcal{E}}_{m,n} = \exp\left(\beta \sum_{l=1}^n \sigma_m^l \left(\sum_{i\in\hat{F}_m} \gamma_i^m W_i^m(\sigma_i^l) + h \right) \right),$$

and $\hat{\mathcal{E}}_m = \hat{\mathcal{E}}_{m,1}$. We will denote by $\mathbf{E}_m^{\gamma,g}$ the expectation given the randomness contained in $\{\gamma_i^m, g_i^m; i \in \hat{F}^m\}$ and by $\mathbf{E}_m^\gamma$ (resp. $\mathbf{E}_m^g$) the expectation when the given the randomness contained in $\{\gamma_i^m; i \in \hat{F}^m\}$ (resp. in $\{g_i^m; i \in \hat{F}^m\}$).

3.1 Preliminary results

Let $n, k \geq 1$ and $f : F^n \to \mathbf{R}$, depending only on a given set of coordinates $(\sigma_{p_1}, \ldots, \sigma_{p_k})$, where $\pi \equiv (p_1, \ldots, p_k) \in F^k$ (here we suppose $|F| \geq k$). We will first show that if f is antisymmetric with respect to one of its coordinates, then $\mathbf{E}[\langle f \rangle_F]$ is asymptotically small. Let us begin with an easy lemma quantifying the probability of an interaction within π, whose proof is left to the reader.

Lemma 3.1 *Define $A_{N,\pi} \subset \Omega$ by*

$$A_{N,\pi} = \left\{\omega;\ \text{there exists } (i,j) \in \pi^2 : i \neq j, \gamma_{(i,j)} = 1\right\}.$$

Then

$$P\left(A_{N,\pi}\right) \leq \frac{k^2\gamma}{\hat{N}^d}.$$

We also label the following Poisson law bound on the number of interactions with a fixed site (whose proof is postponed until the proof of Lemma 3.10 Step 1) for further use.

Lemma 3.2 *Let $p \in C_N$, J_p defined by*

$$J_p = \cup\{i \in C_N;\, \gamma_i^p = 1\}, \tag{42}$$

and set $r = r(p) = |J_p|$. Then, for N large enough, with $\kappa_\gamma = \exp((5\gamma)/2)$ and $\hat\gamma = 2\gamma$, we have

$$P(r = u) \le \kappa_\gamma e^{-\hat\gamma} \frac{\hat\gamma^u}{u!}$$

for any $u \ge 0$.

Let $f : F^n \to \mathbf{R}$, with $n \ge 2$, and let us write, for $j \le k$,

$$f = f\left(\sigma^1_{\hat F^{p_j}}, \sigma^1_{p_j}; \sigma^2_{\hat F^{p_j}}, \sigma^2_{p_j}; \ldots; \sigma^n_{\hat F^{p_j}}, \sigma^n_{p_j}\right),$$

with obvious notations. Let us call T_j the that transformation that permutes the first two occurrences of the jth spin in π:

$$f \circ T_j \left(\sigma^1_{\hat F^{p_j}}, \sigma^1_{p_j}; \sigma^2_{\hat F^{p_j}}, \sigma^2_{p_j}; \ldots; \sigma^n_{\hat F^{p_j}}, \sigma^n_{p_j}\right)$$
$$= f\left(\sigma^1_{\hat F^{p_j}}, \sigma^2_{p_j}; \sigma^2_{\hat F^{p_j}}, \sigma^1_{p_j}; \ldots; \sigma^n_{\hat F^{p_j}}, \sigma^n_{p_j}\right)$$

The next proposition on the behavior of antisymmetric functions, adapted from [10, Theorem 4.2.1], will be useful in order to get some *pure state* results.

Proposition 3.3 *Let $F \subset C_N$ such that $|F| = K$, and $f, f' : F^n \to \mathbf{R}$ be two functions depending on coordinates in $\pi = (p_1, \ldots, p_k)$ such that $f' \circ T_k = -f'$ and $|f'| \le af$ (in particular, we assume f to be a positive function). Then, given a constant $\gamma > 0$, there exists strictly positive constants $\beta_0 = \beta_0(\gamma, M_q)$ and $b = 2\gamma$ such that if $\beta \le \beta_0$,*

$$\mathbf{E}\left[\frac{|\langle f'\rangle_F|}{\langle f\rangle_F}\right] \le \frac{k^2 ab}{\hat N^d}.$$

Proof: First, observe that, using Lemma 3.1, we have

$$\mathbf{E}\left[\frac{|\langle f'\rangle_F|}{\langle f\rangle_F}\right] \le \frac{k^2\gamma}{\hat N^d} a + \mathbf{E}\left[\mathbf{1}_{A^c_{N,\pi}} \frac{|\langle f'\rangle_F|}{\langle f\rangle_F}\right]. \tag{43}$$

We will now work on the last term of the right hand side, and always assume that $A^c_{N,\pi}$ is satisfied.

Step 1: Let J_{p_k} be the set defined by (42), and note that $J_{p_k} \cap \pi = \emptyset$ on $A^c_{N,\pi}$. Since we will use the cavity method with respect to σ_{p_k}, we will call U_j the equivalent of transformation T_j, performed on a function f defined on $\hat F^{p_k}$. Then, it is easily seen that, on $A^c_{N,\pi}$,

$$\left[\mathbf{Av}\, f' \hat{\mathcal{E}}_{p_k,n}\right] \circ \prod_{i \in J_{p_k}} U_i = -\mathbf{Av}\, f' \hat{\mathcal{E}}_{p_k,n}.$$

Consequently, writing $J_{p_k} = \{i_1, \ldots, i_r\}$ (here notice that r is a random integer depending on γ), we also have

$$\mathbf{Av}\, f' \hat{\mathcal{E}}_{p_k,n} = \frac{1}{2}\left[\left(\mathbf{Av}\, f' \hat{\mathcal{E}}_{p_k,n}\right) - \left(\mathbf{Av}\, f' \hat{\mathcal{E}}_{p_k,n}\right) \circ \prod_{s=1}^r U_{i_s}\right] = \frac{1}{2}\sum_{s=1}^r f_s,$$

where

$$f_s = \left(\mathbf{Av}\, f' \hat{\mathcal{E}}_{p_k,n}\right) \circ \prod_{v=1}^{s-1} U_{i_v} - \left(\mathbf{Av}\, f' \hat{\mathcal{E}}_{p_k,n}\right) \circ \prod_{v=1}^{s} U_{i_v} \equiv f_s^{(1)} - f_s^{(2)}.$$

Note that f_s satisfies $f_s \circ U_{i_s} = -f_s$, which means that f_s is still an antisymmetric function.

Step 2: We will now get an estimate of f_s in terms of $\mathbf{Av}\, f\hat{\mathcal{E}}_{p_k,n}$, similar to the relation between f' and f. Indeed, we have

$$
\begin{aligned}
f_s^{(1)} &= \mathbf{Av}_{\sigma^1_{p_k},\ldots,\sigma^n_{p_k}} \left[f'(\sigma^1,\ldots,\sigma^n) \exp\left(\beta \sum_{l=1}^n \sigma^l_{p_k} \left(q_N(p_k - i_v)\, g^{p_k}_{i_v} \sigma^l_{i_v} + h \right) \right) \circ \prod_{v=1}^{q-1} T_{i_v} \right] \\
&= \mathbf{Av}_{\sigma^1_{p_k},\ldots,\sigma^n_{p_k}} f'(\sigma^1,\ldots,\sigma^n) \hat{\mathcal{E}}_{p_k,n} \exp\left(L^{(1)} \right),
\end{aligned}
$$

with

$$
L^{(1)} = \beta \sum_{l=1}^2 \sigma^l_{p_k} \sum_{v=1}^{s-1} q_N(m - i_v) g^{p_k}_{i_v} \left(\sigma^l_{i_v} - \sigma^{3-l}_{i_v} \right).
$$

Similarly,

$$
f_s^{(2)} = \mathbf{Av}_{\sigma^1_{p_k},\ldots,\sigma^n_{p_k}} f'(\sigma^1,\ldots,\sigma^n) \hat{\mathcal{E}}_{p_k,n} \exp\left(L^{(2)} \right),
$$

with

$$
L^{(2)} = \beta \sum_{l=1}^2 \sigma^l_{p_k} \sum_{v=1}^{s} q_N(m - i_v) g^{p_k}_{i_v} \left(\sigma^l_{i_v} - \sigma^{3-l}_{i_v} \right).
$$

Then, it is easily checked that

$$
|L^{(1)}| \le 2\beta \sum_{l=1}^2 \sum_{v=1}^{s-1} M_q |g^{p_k}_{i_v}| \le 4\beta M_q \sum_{v=1}^{r} |g^{p_k}_{i_v}|.
$$

The same estimate holds for $L^{(2)}$, and invoking the fact that

$$
|e^x - e^y| \le e^a |x - y|
$$

if $x, y \in [-a, a]$, we get

$$
|f_s| \le 4\beta M_q |g^{p_k}_{i_s}| \exp\left(4\beta M_q \sum_{v=1}^{r} |g^{p_k}_{i_v}| \right) \left[\mathbf{Av}\, f\hat{\mathcal{E}}_{p_k,n} \right].
$$

Step 3: We are now ready to state our induction hypothesis. We will assume that for $K \ge 1$, β small enough, any subset F of size K and any functions f, f' fulfilling the hypothesis of our proposition, then

$$
\mathbf{E}\left[\frac{|\langle f' \rangle_F|}{\langle f \rangle_F} \right] \le \frac{kab}{\hat{N}^d}. \tag{44}
$$

This is true if $K = 1$. Indeed, then, F is a singleton $\{p\}$ and $\pi = \{p\}$ so that the event $A^c_{N,\pi}$ is empty, and the second term on the right-hand side of (43) is null, so (44) is true with $b = \gamma$. Consider then a subset F of C_N of size $K + 1$, and f, f' satisfying the assumptions of our proposition. Then

$$
\frac{\langle f' \rangle_F}{\langle f \rangle_F} = \sum_{s=1}^{r} \frac{\langle f_s \rangle_{\hat{F}^{p_k}}}{\left\langle \mathbf{Av}\, f\hat{\mathcal{E}}_{p_k,n} \right\rangle_{\hat{F}^{p_k}}},
$$

and applying the induction hypothesis, we have, on $A^c_{N,\pi}$,

$$
\mathbf{E}^{\gamma,g}_{p_k}\left[\frac{\langle f' \rangle_F}{\langle f \rangle_F} \right] \le \frac{4k^2 \beta M_q ab}{\hat{N}^d} \left(\sum_{s=1}^{r} \left| g^{p_k}_{i_s} \right| \right) \exp\left(4\beta M_q \sum_{s=1}^{v} \left| g^{p_k}_{i_v} \right| \right).
$$

Thus, integrating with respect to the randomness in g yields

$$
\mathbf{E}^{\gamma}_{p_k}\left[\frac{\langle f' \rangle_F}{\langle f \rangle_F} \right] \le \frac{\kappa k^2 \beta M_q abr(1 + \beta M_q)}{\hat{N}^d} \exp\left(8r\beta^2 M_q^2 \right).
$$

Now, using Lemma 3.2, some elementary computations of exponential moments for a Poisson measure, and combining with (43), we get

$$\mathbf{E}\left[\frac{\langle f'\rangle_F}{\langle f\rangle_F}\right] \leq a\frac{\gamma k^2}{\hat{N}^d}\left[1+\kappa\beta\gamma b(1+\beta M_q)\right.$$
$$\left.\exp\left(\frac{5\gamma}{2}+8(\beta M_q)^2+2\gamma\left(e^{8(\beta M_q)^2}-1\right)\right)\right].$$

Hence, if a, b, β, γ are such that

$$\gamma\left[1+\kappa\beta\gamma b(1+\beta M_q)\exp\left(\frac{5\gamma}{2}+8(\beta M_q)^2+2\gamma\left(e^{8(\beta M_q)^2}-1\right)\right)\right]$$

does not exceed b, our proposition will be shown. This occurs for small β and $b = b(\gamma)$ as announced. Indeed, take $b = 2\gamma$. Then it is sufficient to take

$$2\kappa\beta\gamma^2(1+\beta M_q)\exp\left(\frac{5\gamma}{2}+8(\beta M_q)^2+2\gamma\left(e^{8(\beta M_q)^2}-1\right)\right) \leq 1,$$

which is obviously true for small β. □

Remark 3.4 *Notice that, applying this first result to $f' = \sigma_i^1(\sigma_j^1-\sigma_j^2)$ and $f = \mathbf{1}$ for arbitrary $i, j \in C_N$, we get, for a constant $\kappa > 0$, a mean measure of how fast spins decorrelate as the system size increases, which is intuitively consistent with the fact that we started off by assuming that each correlation probability is $\gamma/\hat{N}^d$:*

$$\mathbf{E}\left[\left|\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle\right|\right] \leq \frac{\kappa}{\hat{N}^d}.$$

Remark 3.5 *In many of our future computations, a rate of $1/N^d$ could be attained. However, some Riemann sums convergence will push this rate down to $1/N$. This is why, from now, we will often use the trivial bound $1/N^d \leq 1/N$ for our estimates.*

We will turn now to some preliminary results in order to get the limiting behavior of the magnetization, and let us introduce first some notation: for two $[-1,1]^k$-valued random variables X and Y, define the Monge-Kantorovich distance between the law of X and the law of Y by

$$\rho(\mathcal{L}(X),\mathcal{L}(Y)) = \sup\left\{\mathbf{E}[f(X)]-\mathbf{E}[f(Y)];\ f\in L_{(1)}\right\} \tag{45}$$
$$= \inf\left\{\mathbf{E}[|X_1-X_2|];\ (X_1,X_2)\in\mathcal{M}_{X,Y}\right\}, \tag{46}$$

where $L_{(1)}$ denotes the set of Lipschitz functions bounded by 1, with Lipschitz constant lesser than 1, and $\mathcal{M}_{X,Y}$ is the set of random vectors in ^{2k} with marginal laws $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively. For a vector $\xi = (\xi_1,\ldots,\xi_v)\in[-1,1]^v$, a function ϕ on $\{-1,1\}^v$, define $\langle\phi\rangle_\xi$ by

$$\langle\phi\rangle_\xi = \int_{\{-1,1\}^v}\phi(y_1,\ldots,y_v)d\nu_{\xi_1}(y_1)\ldots d\nu_{\xi_v}(y_v), \tag{47}$$

where ν_{ξ_j} is a Bernoulli measure on $\{-1,1\}$ with mean ξ_j.

Let $\pi = (p_1,\ldots,p_k)$ be a sequence in F^k, where $F\subset C_N$. We will also need to introduce a generalized cavity method, singling out all the elements of π at once, and changing the integration over C_N for an integration over C_{N-1}: set

$$\hat{F}^\pi = \{i\in F;\ i\notin\pi\},$$

and for $j\in\{1,\ldots,k\}$ define J_{p_j} as in (42). Define also $\hat{\mathcal{E}}_{\pi,n}$ by

$$\hat{\mathcal{E}}_{\pi,n} = \exp\left(\beta\sum_{l=1}^n\sum_{j=1}^k\sigma_{p_j}^l\left(\sum_{i\in J_{p_j}}W_i^{p_j}(\sigma_i^l)+h\right)\right) \tag{48}$$

As usual, we will designate $\hat{\mathcal{E}}_{\pi,1}$ by $\hat{\mathcal{E}}_{\pi}$. Then, for any $f : \pi^n \to \mathbf{R}$,

$$\langle f \rangle_F = \frac{\left\langle \mathbf{Av}\, f \hat{\mathcal{E}}_{\pi,n} \right\rangle_{\hat{F}^\pi}}{\left\langle \mathbf{Av}\, \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^\pi}}.$$

The expectation given the randomness in π will be written as $\mathbf{E}_{\pi}^{\gamma,g}$ and, when conditioning only on the values taken by γ, by $\mathbf{E}_{\pi}^{\gamma}$. We will consider the set $B_{N,\pi} = A_{N,\pi} \cup A_{N,\pi}^{(1)} \cup A_{N,\pi}^{(2)}$, where

$$A_{N,\pi}^{(1)} = \left\{\omega;\ \text{there exist } i \in \partial C_N, l \in \cup_{j=1}^k J_{p_j} \cup \pi,\ \gamma_{(i,l)} = 1\right\}$$
$$A_{N,\pi}^{(2)} = \left\{\omega;\ \text{there exist } i \in F, j \neq j', \gamma_{i,p_j} = \gamma_{i,p_{j'}} = 1\right\}.$$

Then $B_{N,\pi}$ satisfies the following property, whose proof relies on results in the proof of Lemma 3.10 Step 1.a.

Lemma 3.6 *The probability of the set $B_{N,\pi}$ is bounded as follows:*

$$P(B_{N,\pi}) \leq \kappa\gamma k \frac{\log(N)}{N}. \tag{49}$$

Proof: We use the results of Step 1.a in the proof of Lemma 3.10 with $\delta = 0$. Therein, with p a fixed site, the quantity A_k denotes the probability $\mathbf{P}\left[r(p) = k\right]$, and is estimated as $A_k \leq \kappa_\gamma e^{-2\gamma}(2\gamma)^k/k!$, so that $\mathbf{P}\left[r(p) > 2\log \hat{N}^d\right] \leq \kappa_\gamma/\hat{N}^d$. Then using the trivial fact that the cardinality of ∂C_N is of order $\hat{N}^{d-1}$ up to a constant κ_d, we write

$$\begin{aligned}\mathbf{P}\left[A_{N,\pi}^{(1)}\right] &\leq \sum_{i\in\partial C_N}\left(\sum_{j=1}^k \mathbf{P}\left[\exists l \in J_{p_j} : \gamma_{(i,l)} = 1\right] + \sum_{l\in\pi}\mathbf{P}\left[\gamma_{(i,l)} = 1\right]\right)\\ &\leq \frac{\kappa_d k\gamma}{\hat{N}} + \sum_{i\in\partial C_N}\sum_{j=1}^k \mathbf{P}\left[\exists l \in J_{p_j} : \gamma_{(i,l)} = 1\right].\end{aligned}$$

We estimate the last probability above as follows:

$$\begin{aligned}&\mathbf{P}\left[\exists l \in J_{p_j} : \gamma_{(i,l)} = 1\right]\\ &\leq \mathbf{P}\left[r(p_j) \leq 2\log \hat{N}^d; \exists l \in J_{p_j} : \gamma_{(i,l)} = 1\right] + \mathbf{P}\left[r(p_j) > 2\log\hat{N}^d\right]\\ &\leq 2\log\hat{N}^d \frac{\gamma}{\hat{N}^d} + \frac{\kappa_\gamma}{\hat{N}^d}.\end{aligned}$$

Therefore

$$\mathbf{P}\left[A_{N,\pi}^{(1)}\right] \leq \frac{\kappa_d k\gamma}{\hat{N}} + \kappa_d \hat{N}^{d-1} k \left(2\log\hat{N}^d\frac{\gamma}{\hat{N}^d} + \frac{\kappa_\gamma}{\hat{N}^d}\right),$$

which proves the lemma, given that the calculation for $A_{N,\pi}^{(2)}$ is easier, and that it is trivial for $A_{N,\pi}$. $\square$

Remark 3.7 *The lack of interaction between ∂C_N and $\cup_{j=1}^k J_{p_j} \cup \pi$, responsible for the additional term* $\log(N)$ *in (49), is only needed in order to get Lemma 3.9.*

Since $B_{N,\pi}$ is an exceptional set, we will always work (unless specified) on $B_{N,\pi}^c$. In that case, set

$$r = \sum_{i\in F}\sum_{j=1}^k \gamma_{i,p_j}, \tag{50}$$

and

$$\hat{F}^{\pi,-} = \left\{ i \in \hat{F}^{\pi};\ i \notin \partial C_N \right\}.$$

Then, on $B^c_{N,\pi}$,

$$\langle f \rangle_F = \frac{\left\langle \mathbf{Av}\, f \hat{\mathcal{E}}_{\pi,n} \right\rangle_{\hat{F}^{\pi,-}}}{\left\langle \mathbf{Av}\, \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^{\pi,-}}},$$

and notice that now $\hat{F}^{\pi,-} \subset C_{N-1}$. In particular, on $B^c_{N,\pi}$,

$$\begin{aligned}
&\mathcal{L}^{\gamma}_{\pi}\left(\langle \sigma_{p_1} \rangle_F, \dots, \langle \sigma_{p_k} \rangle_F \right) \\
&= \mathcal{L}^{\gamma}_{\pi}\left(\frac{\left\langle \mathbf{Av}\, \sigma_{p_1} \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^{\pi,-}}}{\left\langle \mathbf{Av}\, \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^{\pi,-}}}, \dots, \frac{\left\langle \mathbf{Av}\, \sigma_{p_k} \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^{\pi,-}}}{\left\langle \mathbf{Av}\, \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^{\pi,-}}} \right).
\end{aligned}$$

The next lemmas are then a first step towards the limiting behavior of $\langle \sigma_i \rangle$.

Lemma 3.8 *Let Y be the random vector defined by*

$$Y = \left(\langle \sigma_j \rangle_{\hat{F}^{\pi,-}}; j \in \hat{F}^{\pi,-} \right),$$

and set, for $i \in \{1, \dots, k\}$,

$$u_i = \frac{\left\langle \mathbf{Av}\, \sigma_{p_i} \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^{\pi,-}}}{\left\langle \mathbf{Av}\, \hat{\mathcal{E}}_{\pi} \right\rangle_{\hat{F}^{\pi,-}}}, \quad v_i = \frac{\left\langle \mathbf{Av}\, \sigma_{p_i} \hat{\mathcal{E}}_{\pi} \right\rangle_{Y}}{\left\langle \mathbf{Av}\, \hat{\mathcal{E}}_{\pi} \right\rangle_{Y}},$$

where the numerator and denominator of v_i are defined by (47). Then, on $B^c_{N,\pi}$, we have, for a constant $\kappa = \kappa_{\beta_0} > 0$,

$$\rho\left(\mathcal{L}^{\gamma}_{\pi}(u_1, \dots, u_k); \mathcal{L}^{\gamma}_{\pi}(v_1, \dots, v_k) \right) \le \frac{\kappa k^2 r^2}{\hat{N}^d} \exp\left(2\beta(h+\beta) \right).$$

Proof: The definition of the distance ρ easily implies that

$$\rho\left(\mathcal{L}^{\gamma}_{\pi}(u_1, \dots, u_k); \mathcal{L}^{\gamma}_{\pi}(v_1, \dots, v_k) \right) \le \sum_{i=1}^{k} \mathbf{E}^{g,\gamma}_{\pi}\left[|u_i - v_i| \right].$$

Let $\{s_j; j \le K\}$ be an enumeration of $\hat{F}^{\pi,-}$, set $f = \mathbf{Av}\, \hat{\mathcal{E}}_{\pi}$, $f' = \mathbf{Av}\, \sigma_{p_i} \hat{\mathcal{E}}_{\pi}$, and for $j \le K$,

$$f_j = f\left(\sigma^1_{s_1}, \dots, \sigma^j_{s_j}, \sigma^1_{s_{j+1}}, \dots, \sigma^1_{s_K} \right),$$

and f'_j similarly. Then, notice that

$$u_i = \frac{\langle f'_1 \rangle_{\hat{F}^{\pi,-}}}{\langle f_1 \rangle_{\hat{F}^{\pi,-}}}, \qquad v_i = \frac{\langle f'_K \rangle_{\hat{F}^{\pi,-}}}{\langle f_K \rangle_{\hat{F}^{\pi,-}}}.$$

Thus,

$$\mathbf{E}^{\gamma,g}_{\pi}\left[|u_i - v_i| \right] \le \sum_{j=1}^{K} \mathbf{E}^{\gamma,g}_{\pi}\left[\frac{\langle f'_{j-1} \rangle_{\hat{F}^{\pi,-}}}{\langle f_{j-1} \rangle_{\hat{F}^{\pi,-}}} - \frac{\langle f'_j \rangle_{\hat{F}^{\pi,-}}}{\langle f_j \rangle_{\hat{F}^{\pi,-}}} \right].$$

The lemma is now obtained using the antisymmetry of $f'_{j-1} - f'_j$, the fact that all the J_{p_j} are disjoint on $B^c_{N,\pi}$, and that $|\sum_j J_{p_j}| = r$ where r is defined by (50). □

We will show in the next section that $\langle\sigma_j\rangle_F$ converges in law to a certain random variable with law η_γ by a fixed point argument. Let then $X=\{X_j; j\in\hat{F}^\pi\}$ be an i.i.d. sequence of law η on $[-1,1]$, and set

$$w_i=\frac{\left\langle \mathbf{Av}\,\sigma_{p_i}\hat{\mathcal{E}}_\pi\right\rangle_X}{\left\langle \mathbf{Av}\,\hat{\mathcal{E}}_\pi\right\rangle_X}.$$

Define now, for $\delta\in\mathbf{N}$, $K=|F|$,

$$I_N(\delta)=\left\{F\subset C_N;\ K\geq \hat{N}^d-\delta\right\},$$

and for $N\geq 1$, $\delta\in\mathbf{N}$, $k\leq\hat{N}^d$

$$D(N,\delta,k)$$
$$=\sup\left\{\rho\left(\mathcal{L}\left(\langle\sigma_{t_1}\rangle_F,\ldots,\langle\sigma_{t_k}\rangle_F\right);\eta^{\otimes k}\right);\ |F|\in I_N(\delta),t_1,\ldots,t_k\in F\right\}.$$

We also set $D(N,\delta,k)=0$ if $k>\hat{N}^d$. Then we have the following result, for which we refer to [10, Lemma 4.4.2], and from which there is little doubt that our strategy should include an induction on the value of $D\left(N,\delta,k\right)$ with respect to N.

Lemma 3.9 *Suppose that $F\in I_N(\delta)$. Then outside $B_{N,\pi}$, we have*

$$\rho\left(\mathcal{L}(v_1,\ldots,v_k);\mathcal{L}(w_1,\ldots,w_k)\right)$$
$$\leq\kappa(1+\beta)\exp\left(2\beta(h+\beta)-2\gamma\right)\beta\cdot\sum_{r=0}^{\infty}\frac{(2\gamma)^r}{r!}D(N-1,\delta+k,r)+2P(B_{N,\pi}).$$

3.2 Limit law for the magnetization

Our first task here will be to give an asymptotic expression for the law of $(w_1,\ldots,w_k)$. Set then, for $r\geq 1$, $N\geq 1$ and $\tau=(i_1,\ldots,i_s)\in F^s$,

$$\mathcal{G}_{K,r,\tau}=\mathcal{G}_{K,r,\tau}\left(\sigma_1,\ldots,\sigma_r,\varepsilon\right)=\exp\left(\beta\sum_{s=1}^{r}q_N(i_s)g_s\sigma_s\varepsilon+\beta h\varepsilon\right),$$

where $(g_1,\ldots,g_s)$ are i.i.d. standard Gaussian random variables and ε represents as usual an independent spin.

On the other hand, if $x=(x_1,\ldots,x_r)$ is a sequence in $[-1,1]^d$, set

$$\mathcal{G}_{r,x}=\exp\left(\beta\sum_{s=1}^{r}q(x_s)g_s\sigma_s\varepsilon+\beta h\varepsilon\right),\tag{51}$$

and define, for a given $p\in[-1,1]^r$, the quantity

$$U_{r,x}(p)=\frac{\langle\mathbf{Av}\,\varepsilon\mathcal{G}_{r,x}\rangle_p}{\langle\mathbf{Av}\,\mathcal{G}_{r,x}\rangle_p}=\frac{\langle\sinh\left(\beta\sum_{s=1}^{r}q(x_s)g_s\sigma_s+\beta h\right)\rangle_p}{\langle\cosh\left(\beta\sum_{s=1}^{r}q(x_s)g_s\sigma_s+\beta h\right)\rangle_p}.\tag{52}$$

In the sequel, we will consider the random variable $U_{r,x}(X^{(r)})$, where $X^{(r)}=(X_1,\ldots,X_r)$ is an i.i.d. sequence of law η on $[-1,1]$. Let now Z be a Poisson point process on $[-1,1]^d$ with intensity $\gamma\,dx$. A given realization of Z is of the form (r,ξ), where $r\in$ and $\xi=(\xi_1,\ldots,\xi_r)\in[0,1]^{d\times r}$. Define then $\hat{\mathcal{G}}_Z=\mathcal{G}_{r,\xi}$ and $\hat{U}_Z=U_{r,\xi}(X^{(r)})$. If η is a probability law on $[-1,1]^d$, one can construct the transform $T\eta$ as follows: consider $\{X_n;n\geq 1\}$ an i.i.d. sequence of random variables of law η. Then, for a realization (r,ξ) of Z, set $V_Z\eta$ as the law of the random variable $\hat{U}_Z$, and if $\hat{\mathbf{E}}_Z$ designates the expectation with respect to the Poisson point process Z, put

$$T\eta=\hat{\mathbf{E}}_Z\left[V_Z\eta\right],$$

which means that if S is a random variable with law $T\eta$, and if ϕ is a test function on $[-1,1]$, then

$$\begin{aligned}\mathbf{E}\left[\phi(S)\right] &= \mathbf{E}\left[\hat{\mathbf{E}}_Z\left[\phi\left(\hat{U}_Z\right)\right]\right] \\ &= \sum_{r=0}^{\infty} e^{-\gamma}\frac{\gamma^r}{r!}\int_{[-1,1]^{d\times r}} \mathbf{E}\left[\phi\left(U_{r,\xi}(X^{(r)})\right)\right] d\xi.\end{aligned} \tag{53}$$

We then have the following.

Lemma 3.10 *There exists a constant $\kappa > 0$ such that for any $F \in I_N(\delta)$,*

$$\rho\left(\mathcal{L}(w_1,\ldots,w_k);(T\eta)^{\otimes k}\right) \le \frac{\kappa k}{N}\left[k+1+\gamma\left(1+\delta+\beta\|\nabla q\|_\infty\right)+\gamma^2\right].$$

Proof: Notice first that, outside $B_{N,\pi}$, the random variables $w_1, \ldots, w_k$ are independent. Hence, using the fact that $|w_j| \le 1$ and the expression (46) defining ρ, we get

$$\rho\left(\mathcal{L}(w_1,\ldots,w_k);(\mathcal{L}(w_1))^{\otimes k}\right) \le \frac{\kappa(k^2+1)\gamma}{\hat{N}^d}. \tag{54}$$

We will now turn to some estimates on the law of $w = w_1$. For notational sake, we will set $p_1 = p$, $w_1 = w$, $g_i^p = g_i$. Then, if ϕ is a function in $L_{(1)}$, recalling definition (48), we get

$$\begin{aligned}\mathbf{E}\left[\phi(w)\right] &= \mathbf{E}\left[\phi\left(\frac{\left\langle \mathbf{Av}\,\varepsilon\exp\left(\beta\varepsilon\sum_{i\in J_p} W_i^p(\sigma_i)+\beta h\varepsilon\right)\right\rangle_X}{\left\langle \mathbf{Av}\,\exp\left(\sum_{i\in J_p}\beta W_i^p(\sigma_i)\varepsilon+\beta h\varepsilon\right)\right\rangle_X}\right)\right] \\ &= \mathbf{E}\left[\psi_{X,g,r}\left(\frac{i_1-p}{N},\ldots,\frac{i_r-p}{N}\right)\right],\end{aligned}$$

where

$$\begin{aligned}\psi_{X,g,r}(x_1,\ldots,x_r) &= \phi\left(U_{r,x}(X^{(r)})\right) \\ &= \phi\left(\frac{\left\langle \sinh\left(\beta\left(\sum_{i=1}^r q(x_i)g_i+h\right)\right)\right\rangle_X}{\left\langle \cosh\left(\beta\left(\sum_{i=1}^r q(x_i)g_i+h\right)\right)\right\rangle_X}\right),\end{aligned}$$

and where $r = |J_p|$ and $i_1, \cdots, i_r$ are an enumeration of the r sites in F that interact with p.

Then, using some standard calculations for the binomial distribution,

$$\mathbf{E}^{g,X}\left[\phi(w)\right] = \sum_{r=0}^{\infty} A_r B_r,$$

with $A_r = 0$ for $r \ge K$, and for $r \le K-1$,

$$\begin{aligned}A_r &= \binom{K-1}{r}\left(\frac{\gamma}{\hat{N}^d}\right)^r\left(1-\frac{\gamma}{\hat{N}^d}\right)^{K-r-1} \\ B_r &= \sum_{(i_1,\ldots,i_r)\in A(\hat{F}^p,r)} \frac{\psi_{X,g,r}\left(\frac{i_1-p}{N},\ldots,\frac{i_r-p}{N}\right)}{A_{K-1}^r},\end{aligned} \tag{55}$$

where $A(\hat{F}^p, r)$ stands for the set of ordered combinations of r elements among those of $\hat{F}^p$, and

$$A_{K-1}^r = \frac{(K-1)!}{(K-1-r)!} = \left|A(\hat{F}^p,r)\right|.$$

Note in particular that $A_k = \mathbf{P}\left[|J_p| = k\right]$, and is indeed the binomial probability of k successes (interactions from within F to p) among $K-1$ trials with success probability $\gamma/\hat{N}^d$.

We will now look for the limit of A_r and B_r separately in Steps 1 and 2 respectively.

Step 1.a: From the Poisson approximation of the binomial distribution, it is clear that

$$\lim_{N\to\infty} A_r = e^{-\gamma}\frac{\gamma^r}{r!},$$

but we will look for some sharper estimates for that convergence. First, remark that

$$\binom{K-1}{r} \geq \frac{\hat{N}^{dr}}{r!}\prod_{j=1}^{r}\left(1-\frac{\delta+j}{\hat{N}^d}\right).$$

Hence, using the fact that $(1-x)(1-y) \geq 1-(x+y)$ for $x, y \geq 0$, we get

$$\frac{\hat{N}^{dr}}{r!}\left(1-\frac{r(2\delta+r+1)}{2\hat{N}^d}\right) \leq C_{K-1}^r \leq \frac{\hat{N}^{dr}}{r!}. \tag{56}$$

Invoking now the fact that $\log(1-y) \leq -y$ for $y \in [0,1)$, if N is large enough, we have

$$A_r \leq \frac{\gamma^r}{r!}\exp\left(-\frac{(K-1)\gamma}{\hat{N}^d}\right)\left(1-\frac{\gamma}{\hat{N}^d}\right)^{-r}.$$

Recall that $\frac{K-1}{\hat{N}^d} \geq 1 - \frac{\delta+1}{\hat{N}^d}$. Hence

$$A_r \leq e^{-\gamma}\frac{\gamma^r}{r!}\exp\left(\frac{(\delta+1)\gamma}{\hat{N}^d}\right)\left(1-\frac{\gamma}{\hat{N}^d}\right)^{-r}. \tag{57}$$

We will first use a crude bound on A_r: if N is large enough, then

$$\frac{\gamma}{1-\frac{\gamma}{\hat{N}^d}} \leq 2\gamma,$$

and if $\frac{(\delta+1)\gamma}{\hat{N}^d} < \frac{\gamma}{2}$, which of course occurs for N large enough, we get

$$A_r \leq \kappa_\gamma e^{-\hat{\gamma}}\frac{\hat{\gamma}^r}{r!},$$

with $\kappa_\gamma = \exp(5\gamma/2)$ and $\hat{\gamma} = 2\gamma$. Now, if V is a Poisson random variable with parameter $\hat{\gamma}$, the following exponential bound holds true, for $v > 0$:

$$P(V > v) \leq \frac{e^{(e-1)\hat{\gamma}}}{e^v}.$$

In particular, if $v \geq \log(\hat{N}^d) + (e-1)\hat{\gamma}$, which is true for $v \geq 2\log(\hat{N}^d)$ when N is large enough, we have

$$P(V > v) \leq \frac{1}{\hat{N}^d}.$$

Thus, up to an exceptional set of probability less than $\kappa_\gamma/\hat{N}^d$, we can assume that $r \leq 2\log(\hat{N}^d)$. For notational convenience, we will set $G_N = \{\omega; r \geq 2\log(\hat{N}^d)\}$.

Step 1.b: Recall that (57) holds true for any r. Hence, if $r \leq 2\log(\hat{N}^d)$,

$$A_r - e^{-\gamma}\frac{\gamma^r}{r!} \leq e^{-\gamma}\frac{\gamma^r}{r!}\left(\frac{\exp\left(\frac{(\delta+1)\gamma}{\hat{N}^d}\right)-1}{\left(1-\frac{\gamma}{\hat{N}^d}\right)^r} + \frac{1}{\left(1-\frac{\gamma}{\hat{N}^d}\right)^r} - 1\right).$$

Furthermore, it is easily checked that, for N large enough, and $r \le 2\log(\hat{N}^d)$,

$$\left(1-\frac{\gamma}{\hat{N}^d}\right)^r \ge \frac{1}{2}, \qquad \frac{1}{\left(1-\frac{\gamma}{\hat{N}^d}\right)^r} - 1 \le \frac{2r}{\hat{N}^d}\gamma$$

and

$$\exp\left(\frac{(\delta+1)\gamma}{\hat{N}^d}\right) - 1 \le \frac{2(\delta+1)\gamma}{\hat{N}^d},$$

since $e^u - 1 \le 2u$ for $u \le 1$. Hence, for N large enough, on G_N^c, we have

$$A_r - e^{-\gamma}\frac{\gamma^r}{r!} \le 2e^{-\gamma}\frac{\gamma^r}{r!}\left[\frac{2(\delta+1)+r}{\hat{N}^d}\right]\gamma. \tag{58}$$

On the other hand, it is clear from (56) that

$$\begin{aligned} A_r &\ge \frac{\gamma^r}{r!}\left(1-\frac{\gamma}{\hat{N}^d}\right)^K \left(1-\frac{\gamma}{\hat{N}^d}\right)^{-(r+1)}\left[1-\frac{r(2\delta+r+1)}{2\hat{N}^d}\right] \\ &\ge \frac{\gamma^r}{r!}\left(1-\frac{\gamma}{\hat{N}^d}\right)^K \left[1-\frac{r(2\delta+r+1)}{2\hat{N}^d}\right]. \end{aligned}$$

Moreover, if $x > 0$, it holds that $1 - x \ge e^{-x}(1-x^2)$, and hence

$$\begin{aligned} A_r &\ge \frac{\gamma^r}{r!}\exp\left(-\frac{\gamma K}{\hat{N}^d}\right)\left(1-\frac{\gamma^2}{\hat{N}^{2d}}\right)\left[1-\frac{r(2\delta+r+1)}{2\hat{N}^d}\right] \\ &\ge \frac{\gamma^r}{r!}e^{-\gamma}\left[1-\frac{2\gamma^2+r(2\delta+r+1)}{2\hat{N}^d}\right], \end{aligned}$$

which means that

$$A_r - e^{-\gamma}\frac{\gamma^r}{r!} \ge -e^{-\gamma}\frac{\gamma^r}{r!}\left[\frac{2\gamma^2+r(2\delta+r+1)}{2\hat{N}^d}\right]. \tag{59}$$

Now, putting together (58) and (59), we get, on G_N^c,

$$\left|A_r - e^{-\gamma}\frac{\gamma^r}{r!}\right| \le e^{-\gamma}\frac{\gamma^r}{r!}\left[\frac{4\gamma(\delta+1+\gamma)+r(1/2+\delta+2\gamma)+r^2/2}{\hat{N}^d}\right].$$

Plugging the values of the first two moments of a Poisson law into this last equation, and using the fact that G_N is an exceptional set, we get

$$\sum_{r=0}^{\infty}\left|A_r - e^{-\gamma}\frac{\gamma^r}{r!}\right| \le \frac{4+\gamma(10\delta+12)+5\gamma^2}{2\hat{N}^d}.$$

Step 2.a: Our next task will be to obtain some sharp estimates on B_r, defined by (55). Set first

$$B_r^{(1)} = \frac{1}{\hat{N}^{dr}}\sum_{(i_1,\ldots,i_r)\in A(\hat{F}^p,r)}\psi_{X,g,r}\left(\frac{i_1-p}{N},\ldots,\frac{i_r-p}{N}\right).$$

Then

$$\left|B_r - B_r^{(1)}\right| \le \|\psi_{X,g,r}\|_\infty \sum_{(i_1,\ldots,i_r)\in A(\hat{F}^p,r)}\frac{\hat{N}^{dr}-A_{K-1}^r}{\hat{N}^{dr}A_{K-1}^r}.$$

However, by inequality (56), we have

$$0 \le \frac{\hat{N}^{dr}-A_{K-1}^r}{\hat{N}^{dr}} \le \frac{r(2\delta+r+1)}{2\hat{N}^d}. \tag{60}$$

Hence

$$\left|B_r - B_r^{(1)}\right| \le \|\psi_{X,g,r}\|_\infty \frac{r(2\delta + r + 1)}{2\hat{N}^d}. \tag{61}$$

Furthermore, inequality (60) also yields

$$\left|\hat{B}_r - B_r^{(1)}\right| \le \|\psi_{X,g,r}\|_\infty \frac{r(2\delta + r + 1)}{2\hat{N}^d}, \tag{62}$$

where

$$\hat{B}_r = \frac{1}{\hat{N}^{dr}} \sum_{(i_1,\dots,i_r)\in(C_N)^r} \psi_{X,g,r}\left(\frac{i_1 - p}{N}, \dots, \frac{i_r - p}{N}\right).$$

Because of the periodicity of q, we can identifying B_r as the Riemann sum corresponding to the integral of $\psi_{X,g,r}$ over $[-1,1]^{dr}$. Thus, setting

$$I_{X,g,r} = \int_{[-1,1]^{dr}} \psi_{X,g,r}(x_1,\dots,x_r)\, dx_1 \cdots dx_r,$$

we get that

$$|\hat{B}_r - I_{X,g,r}| \le \frac{\|\psi_{X,g,r}\|_{L^{(1)}}}{2N}. \tag{63}$$

Putting together (61), (62) and (63), we obtain

$$|B_r - I_{X,g,r}| \le \frac{1}{N}\left[\|\psi_{X,g,r}\|_{L^{(1)}} + \|\psi_{X,g,r}\|_\infty r(2\delta + r + 1)\right]. \tag{64}$$

Step 2.b: Let us compute now $\|\psi_{X,g,r}\|_\infty$ and $\|\psi_{X,g,r}\|_{L^{(1)}}$. The first one is easily obtained, since

$$\|\psi_{X,g,r}\|_\infty \le \|\phi\|_\infty \le 1.$$

To estimate the Lipschitz constant of $\psi_{X,g,r}$, for $r \ge 1$ and $x = (x_1,\dots,x_r) \in [-1,1]^{dr}$, set

$$\ell_r(x) = \frac{\langle \sinh(v_r(x))\rangle_X}{\langle \cosh(v_r(x))\rangle_X},$$

where

$$v_r(x) = \beta\left(\sum_{i=1}^r \sigma_i q(x_i) g_i + h\right).$$

For each $i \in \{1,\dots,r\}$, notice that x_i is of the form $\{x_i(j); j \le d\}$, with $x_i(j) \in [-1,1]$. Thus, with obvious notations,

$$\partial_{x_i(j)}\ell_r(x) = \beta g_i \partial_{x_i(j)} q(x_i) \left[\frac{\langle \sigma_i \cosh(v_r(x))\rangle_X}{\langle \cosh(v_r(x))\rangle_X} - \frac{\langle \sinh(v_r(x))\rangle_X \langle \sigma_i \sinh(v_r(x))\rangle_X}{\langle \cosh(v_r(x))\rangle_X^2}\right],$$

and thus

$$\left|\partial_{x_i(j)}\ell_r(x)\right| \le 2\beta\|\nabla q\|_\infty |g_i|,$$

which yields

$$\|\psi_{X,g,r}\|_{L^{(1)}} \le 2\|\phi\|_{L^{(1)}}\beta\|\nabla q\|_\infty \sum_{i=1}^r |g_i| \le 2\beta\|\nabla q\|_\infty \sum_{i=1}^r |g_i|,$$

and hence, taking the expectation with respect to the g's only,

$$\mathbf{E}^r\left[\|\psi_{X,g,r}\|_{L^{(1)}}\right] \le 2(2\pi)^{1/2}\beta\|\nabla q\|_\infty r. \tag{65}$$

Plugging this estimate into (64), we get, for a constant $\kappa > 0$,

$$\mathbf{E}^r \left[|B_r - I_{X,g,r}| \right] \leq \frac{\kappa}{N} \left[\beta \|\nabla q\|_\infty + 2\delta + r + 1 \right] r.$$

On the other hand, we will also need the following trivial bound on B_r:

$$|B_r| \leq \|\psi_{X,g,r}\|_\infty \leq \|\phi\|_\infty \leq 1. \tag{66}$$

Step 3 (conclusion): We are now ready to estimate the distance $\rho(\mathcal{L}(w_1, \ldots, w_k); (T\eta)^{\otimes k})$. Note first that

$$\rho \left((\mathcal{L}(w))^{\otimes k}; (T\eta)^{\otimes k} \right) \leq k \rho \left(\mathcal{L}(w); T\eta \right).$$

Let now ϕ be a function in $L^{(1)}$ and S be a random variable of law $T\eta$. Invoking the definition (52), relations (53), (65), and (66), we have proved that

$$\begin{aligned}
&|\mathbf{E}[\phi(w)] - \mathbf{E}[\phi(S)]| \\
&= \left| \sum_{r=0}^{K-1} \mathbf{E}[A_r B_r] - \sum_{r=0}^{\infty} e^{-\gamma} \frac{\gamma^r}{r!} \mathbf{E}\left[I_{X,g,r} \right] \right| \\
&\leq \sum_{r=0}^{\infty} \left| A_r - e^{-\gamma} \frac{\gamma^r}{r!} \right| + \sum_{r=0}^{\infty} e^{-\gamma} \frac{\gamma^r}{r!} \mathbf{E}\mathbf{E}^r \left[|B_r - I_{X,g,r}| \right] \\
&\leq \frac{\kappa}{\hat{N}^d} \left[1 + \gamma(\delta + 1) + \gamma^2 \right] + \frac{\kappa\gamma}{N} \left[\beta \|\nabla q\|_\infty + \gamma + 1 + 2\delta \right] \\
&\leq \frac{\kappa}{N} \left[1 + \gamma(1 + \delta + \beta \|\nabla q\|_\infty) + \gamma^2 \right].
\end{aligned}$$

This, together with (54), ends the proof. □

Recalling the definition of $D(N, \delta, k)$, Lemma 3.9 and 3.10, it is obvious that the natural candidate for the limit law of $\langle \sigma_i \rangle$ is the solution to the equation $\eta = T\eta$. We will now show that this equation has a unique solution.

Proposition 3.11 *Suppose that β, γ and M_q satisfy*

$$\kappa \beta M_q (1 + \beta M_q) \exp\left(2(\beta M_q)^2 \right) (\gamma + \gamma^2) \leq \frac{1}{2}, \tag{67}$$

for a constant κ large enough. Then there exists a unique solution η_γ to the equation $\eta = T\eta$.

Proof: We will first state some elementary results, and then apply them to prove that T is a contracting map for the distance ρ.

Step 1: It can be shown, by a slight variation of [10, Lemma 4.3.1], that, $\{a_r; r \geq 0\}$ being a sequence of positive numbers such that $\sum_{r \geq 0} a_r = 1$, we have

$$\rho \left(\sum_{r=0}^{\infty} a_r \int_{[-1,1]^{dr}} \mu_{r,\xi} d\xi; \sum_{r=0}^{\infty} a_r \int_{[-1,1]^{dr}} \nu_{r,\xi} d\xi \right) \leq \sum_{r=0}^{\infty} a_r \int_{[-1,1]^{dr}} \rho \left(\mu_{r,\xi}; \nu_{r,\xi} \right) d\xi. \tag{68}$$

Furthermore, if $f : \{-1, 1\}^r \mapsto \mathbf{R}$, and $p \in [-1, 1]^r$, $\langle f \rangle_p$ being defined by (47), we have (cf. [10, Lemma 4.3.3]), for $i \in \{1, \ldots, r\}$,

$$\partial_{p_i} \langle f \rangle_p = \langle \Delta_i f \rangle_p,$$

where, for $\eta \in \{-1, 1\}^r$,

$$\Delta_i f(\eta) = \frac{1}{2} \left(f(\eta_i^+) - f(\eta_i^-) \right),$$

with

$$\eta_i^{\pm} = (\eta_1, \dots, \eta_{i-1}, \pm 1, \eta_{i+1}, \dots, \eta_r).$$

Step 2: From the previous step, for any $p = (p_1, \dots, p_r) \in [-1,1]^r$, and $U_{r,x}(p)$ defined by (52), we have

$$\partial_{p_i}\left[U_{r,x}(p)\right] = \frac{\langle \Delta_i \left(\mathbf{Av}\, \varepsilon \mathcal{G}_{r,x}\right)\rangle_p}{\langle \mathbf{Av}\, \mathcal{G}_{r,x}\rangle_p} - \frac{\langle \mathbf{Av}\, \varepsilon \mathcal{G}_{r,x}\rangle_p \langle \Delta_i \left(\mathbf{Av}\, \mathcal{G}_{r,x}\right)\rangle_p}{\langle \mathbf{Av}\, \mathcal{G}_{r,x}\rangle_p^2}.$$

Moreover

$$\left|\Delta_i \left(\mathbf{Av}\, \varepsilon \mathcal{G}_{r,x}\right)\right| = \left|\mathbf{Av}\, \left(\varepsilon \Delta_i \mathcal{G}_{r,x}\right)\right| \le \mathbf{Av}\, \left|\Delta_i \mathcal{G}_{r,x}\right|,$$

and having in mind equation (51) defining $\mathcal{G}_{r,x}$, it is easily seen that

$$|\Delta_i \mathcal{G}_{r,x}| \le 2\beta M_q |g_i| \exp\left(2\beta M_q |g_i|\right),$$

from which the upper bound

$$|\partial_i U_{r,x}(p)| \le 2\beta M_q |g_i| \exp\left(2\beta M_q |g_i|\right) \tag{69}$$

follows.

Step 3: Let p^1, p^2 be two elements of $[-1,1]^r$. Then, from (69), we have

$$\left|U_{r,x}(p^2) - U_{r,x}(p^1)\right| \le 2\beta M_q |g_i| \exp\left(2\beta M_q |g_i|\right)\left|p^2 - p^1\right|.$$

Consider now μ_1, μ_2 two probability measures on $[-1,1]$, and $\{X_i^{(1)}, X_i^{(2)}; i \le r\}$ some independent copies of random variables of law μ_1 and μ_2 respectively. Set, for $j \in \{1,2\}$,

$$T_{r,x}\left(\mu_j\right) = \mathcal{L}\left(U_{r,x}\left(X^{(j)}\right)\right).$$

Then we have

$$\begin{aligned}
&\rho\left(T_{r,x}\left(\mu_1\right); T_{r,x}\left(\mu_2\right)\right) \\
&\le 2\beta M_q r \mathbf{E}\left[|g| \exp\left(2\beta M_q |g|\right)\right] \mathbf{E}\left[|X^{(1)} - X^{(2)}|_1\right] \\
&\le \kappa \beta M_q (1 + M_q) e^{2(\beta M_q)^2} r \mathbf{E}\left[|X^{(1)} - X^{(2)}|_1\right],
\end{aligned}$$

where $|z|_1$ is defined on ^{r} by $|z|_1 = \sum_{i\le r} |z_i|$. Hence, taking the infimum of this last quantity over $\mathcal{M}_{X^{(1)},X^{(2)}}$, we get

$$\rho\left(T_{r,x}\left(\mu_1\right); T_{r,x}\left(\mu_2\right)\right) \le \kappa \beta M_q (1 + M_q) e^{2(\beta M_q)^2} r^2 \rho\left(\mu_1; \mu_2\right).$$

Now, using (68), we have

$$\begin{aligned}
&\rho\left(T\left(\mu_1\right); T\left(\mu_2\right)\right) \\
\le&\kappa \beta M_q (1 + M_q) e^{2(\beta M_q)^2} \sum_{r=1}^{\infty} r^2 e^{-\gamma} \frac{\gamma^r}{r!} \int_{[-1,1]^{rd}} \rho(\mu_1; \mu_2) d\xi \\
=&\kappa \beta M_q (1 + M_q) e^{2(\beta M_q)^2} (\gamma + \gamma^2) \rho(\mu_1; \mu_2).
\end{aligned}$$

Hence, under the hypothesis (67) of our proposition, T is a contracting map defined on the probability measures on $[-1,1]$, which shows our claim. □

We can now turn to the main result of this section, that is the limit law for the magnetization.

Theorem 3.12 *Let $k \ge 1$. Then, for a strictly positive constant $\kappa = \kappa_{\beta,\gamma,q}$, we have*

$$\rho\left(\mathcal{L}\left(\langle \sigma_1 \rangle, \dots, \langle \sigma_k \rangle\right); \eta_\gamma^{\otimes k}\right) \le \kappa k \frac{k + \log(N)}{N},$$

provided

$$\gamma \exp\left(2\beta(h+\beta)\right)\left(1+\gamma+4(1+\beta)\beta\right) < 1. \tag{70}$$

Proof: We assume the following induction hypothesis: for some constant $\kappa = \kappa_{\beta,\gamma,q}$ and for all $\delta \in \mathbf{N}$, $k \in \mathbf{N} - \{0\}$, and $F \in I_N(\delta)$,

$$D(N-1,\delta,k) \leq \kappa k (1+\delta) \frac{k+\log N - 1}{N-1}. \tag{71}$$

We can begin the induction at $N = 3$ by choosing the constant κ large enough, since the above inequality needs only to be checked for $k, \delta \leq \hat{2}^d$, and its left-hand side is some fixed finite value that can be absorbed into κ. Note first that using $\pi = \{1, \cdots, k\}$ and $F = C_N$ itself, then we have that the sequence $\langle\sigma_1\rangle, \cdots, \langle\sigma_k\rangle$ is identical to $\langle\sigma_{p_1}\rangle_F, \cdots, \langle\sigma_{p_k}\rangle_F$. From Lemma 3.8 and 3.9, we then have, for any $\delta \geq 0$ and $F \in I_N(\delta)$,

$$\begin{aligned}&\rho\left(\mathcal{L}\left(\langle\sigma_1\rangle, \ldots, \langle\sigma_k\rangle\right); \mathcal{L}\left(w_1, \ldots, w_k\right)\right)\\ &\quad\leq \kappa \exp\left(2\beta(h+\beta)\right) \times \left(\frac{k(\gamma+\gamma^2)(1+\delta)(k+\log(N))}{N} + (1+\beta)\beta \sum_{r=0}^{\infty} e^{-2\gamma}\frac{(2\gamma)^r}{r!} D(N-1,\delta+k,r)\right),\end{aligned}$$

and invoking now Lemma 3.10, we get, if η_γ is the solution to the equation $T\eta = \eta$,

$$\begin{aligned}&\rho\left(\mathcal{L}\left(\langle\sigma_1\rangle, \ldots, \langle\sigma_k\rangle\right); \eta_\gamma^{\otimes k}\right)\\ &\quad\leq \kappa_q \exp\left(2\beta(h+\beta)\right) \times \left(\frac{k(\gamma+\gamma^2)(1+\delta)\,(k+\log(N))}{N} + (1+\beta)\beta \sum_{r=0}^{\infty} e^{-2\gamma}\frac{(2\gamma)^r}{r!} D(N-1,\delta+k,r)\right)\\ &\quad\leq \kappa_q \exp\left(2\beta(h+\beta)\right) \times \frac{k(1+\delta)(k+\log(N))}{N}\gamma\left(1+\gamma+4(1+\beta)\beta + 0\left(\frac{1}{\log N}\right)\right).\end{aligned}$$

In the last step we used the induction hypothesis (71) and the fact that $k+1+\delta \leq 2k(1+\delta)$. If condition (70) is satisfied, this allows to propagate the induction hypothesis to the next step N, which ends the proof. □

3.3 The replica symmetric solution

As usual, set $Z_N = \sum_{\sigma\in\Sigma_N} \exp(-H_N(\sigma))$. In our context, it will be easier to consider this quantity as a function of the parameter γ, instead of β. Set then

$$p_N(\gamma) = \frac{1}{\hat{N}^d}\mathbf{E}\left[\log(Z_N)\right],$$

and call $p'_N(\gamma)$ the right derivative of p_N with respect to γ. It can be shown, as in [10, Lemma 4.4.5], that for a constant $\kappa > 0$,

$$|p'_N(\gamma) - A_N| \leq \frac{\kappa}{N},$$

where

$$A_N = \frac{1}{\hat{N}^{2d}} \sum_{(i,j)\in C_N} \mathbf{E}\left[\log\left(\left\langle e^{q_N(i-j)g_{(i,j)}\sigma_i\sigma_j}\right\rangle\right)\right].$$

Note that, setting $C_N^* = C_N \backslash \{0\}$, $g_j = g_{(0,j)}$, this last expression can be symmetrized to obtain

$$A_N = \frac{1}{\hat{N}^d} \sum_{j\in C_N^*} \mathbf{E}\left[\log\left(\left\langle e^{q_N(j)g_j\sigma_0\sigma_j}\right\rangle\right)\right].$$

Denote by $\mathbf{E}^{\gamma;g}_{C_N^*}$ the expectation on C_N^* conditioned on the values of $g_{(0,j)}$, $\gamma_{(0,j)}$. The following Lemma will be essential in the computation of $p'_N(\gamma)$.

Lemma 3.13 *Let $X^{(j)} = (\langle\sigma_0\rangle, \langle\sigma_j\rangle)$. Then, for any $j \in C_N^*$, if $\gamma \leq \gamma_0$ and β is small enough, we have*

$$\mathbf{E}_{C_N^*}^{\gamma,g}\left[\left|\log\left(\left\langle e^{q_N(j)g_j\sigma_0\sigma_j}\right\rangle\right) - \log\left(\left\langle e^{q_N(j)g_j\sigma_0\sigma_j}\right\rangle_{X^{(j)}}\right)\right|\right]$$
$$\leq \frac{\kappa_{\gamma_0}e^{\beta(M_q|g_j|+h)}\log(N)}{N},$$

for a strictly positive constant κ_{γ_0}

Proof: This result is obtained using the same techniques as in Lemma 3.8. □

One can also go one step further, and replace $X^{(j)}$ by $\eta^{\otimes 2}$:

Lemma 3.14 *Let $j \in C_N^*$ and $Y^{(j)} = (Y_0, Y_j)$ be two independent random variables of law η_γ, such that*

$$\mathbf{E}\left[|\langle\sigma_0 - Y_0\rangle|\right] + \mathbf{E}\left[|\langle\sigma_j - Y_j\rangle|\right] \leq \frac{\kappa\log(N)}{N}.$$

Then

$$E_{C_N^*}\left[\left|\log\left(\left\langle e^{q_N(j)g_j\sigma_0\sigma_j}\right\rangle\right) - \log\left(\left\langle e^{q_N(j)g_j\sigma_0\sigma_j}\right\rangle_{Y^{(j)}}\right)\right|\right]$$
$$\leq \frac{\kappa_{\gamma_0}e^{\frac{\beta}{2}(\beta M_q+2h)}\log(N)}{N},$$

Proof: By standard methods, using Theorem 3.12 and the bound

$$|\log(z_1) - \log(z_2)| \leq \frac{|z_1 - z_2|}{a},$$

that holds for $z_1, z_2 \in [a, \infty)$ with $a > 0$, we have

$$\mathbf{E}_{C_N^*}^{\gamma,g}\left[\left|\log\left(\left\langle e^{q_N(j)g_j\sigma_0\sigma_j}\right\rangle\right) - \log\left(\left\langle e^{q_N(j)g_j\sigma_0\sigma_j}\right\rangle_{Y^{(j)}}\right)\right|\right] \leq \frac{\kappa_{\gamma_0}e^{\beta(M_q|g_j|+h)}\log(N)}{N}.$$

Integrating this result with respect to the remaining disorder in g, γ, we get the desired result. □

We can now state our main result concerning the derivative of $p_N(\gamma)$.

Proposition 3.15 *For any $\gamma_0 > 0$, there exists a constant κ_{γ_0} such that for $\gamma \leq \gamma_0$ and beta small enough,*

$$\left|p_N'(\gamma) - \int_{[-1,1]^d} dx \int_{[-1,1]^2} \mathbf{E}\left[\log\left(\left\langle e^{q(x)g\varepsilon_1\varepsilon_2}\right\rangle_y\right)\right] d\eta_\gamma(y_1)d\eta_\gamma(y_2)\right| \leq \frac{\kappa_{\gamma_0}e^{2\beta M_q(\beta M_q+h)}\log(N)}{N}.$$

Proof: Lemma 3.14 implies directly that

$$\left|p_N'(\gamma) - \frac{1}{\hat{N}^d}\sum_{j\in C_N^*}\int_{[-1,1]^2} \mathbf{E}\left[\log\left(\left\langle e^{q_N(j)g\varepsilon_1\varepsilon_2}\right\rangle_y\right)\right] d\eta_\gamma(y_1)d\eta_\gamma(y_2)\right| \leq \frac{\kappa_{\gamma_0}e^{2\beta M_q(\beta M_q+h)}\log(N)}{N}.$$

The result is then obtained by Riemann sums approximation, once the gradient of

$$x \mapsto \mathbf{E}\left[\log\left(\left\langle e^{q(x)g\varepsilon_1\varepsilon_2}\right\rangle_y\right)\right]$$

is controlled, which can be done as in the proof of Lemma 3.10, Step 2.b. □

There is now little to change to the proof of [10, Theorem 4.4.2] to obtain the following

Theorem 3.16 *Under the conditions of Proposition 3.15, we have*

$$|p_N(\gamma) - F(\gamma)| \leq \frac{K \log N}{N},$$

with

$$F(\gamma) = \log(2) - \frac{\gamma}{2} \int_{[-1,1]^d} dx \int_{[-1,1]^2} L\,(x,y)\, d\eta_\gamma^{\otimes 2}(y_1,y_2) + \sum_{r=1}^{\infty} \frac{\gamma^r e^{-\gamma}}{r!} \int_{[-1,1]^d} dx \int_{[-1,1]^r} H\,(x,y)\, d\eta_\gamma^{\otimes r}(y_1,\ldots,y_r).$$

where

$$L\,(x,y) = \mathbf{E}\left[\log\left(\left\langle e^{q(x) g \varepsilon_1 \varepsilon_2}\right\rangle_y\right)\right],$$
$$H\,(x,y) = \mathbf{E}\left[\log\left(\left\langle e^{\sum_{s=1}^r q(x) g_s \sigma_s \varepsilon}\right\rangle_y\right)\right].$$

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