# Evolution equation of a stochastic semigroup with white-noise drift

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**Abstract**

We study the existence and uniqueness of the solution of a function-valued stochastic evolution equation based on a stochastic semigroup whose kernel $p(s,t,y,x)$ is Brownian in $s$ and $t$. The kernel $p$ is supposed to be measurable with respect to the increments of an underlying Wiener process in the interval $[s,t]$. The evolution equation is then anticipative and choosing the Skorohod formulation we establish existence and uniqueness of a continuous solution with values in $L^2(\mathbb{R}^d)$.

As an application we prove the existence of a mild solution of the stochastic parabolic equation

$$du_t = \Delta_x u dt + v\,(dt,x) \cdot \nabla u + F\,(t,x,u)\, W\,(dt,x)$$

where $v$ and $W$ are Brownian in time with respect to a common filtration. In this case, $p$ is the formal backward heat kernel of $\Delta_x + v\,(dt,x) \cdot \nabla_x$.

*Supported by the DGICYT grant no. PB96-0087

†Supported by NSF International Opportunities Postdoctoral Fellowship no. INT-9600278 and NSF-NATO Postdoctoral Fellowship no. DGE-9633937

[0]AMS Subject Classification: Primary 60H15, secondary 60H07

Key words and phrases: Stochastic parabolic equations. Anticipating stochastic calculus. Skorohod integral. Stochastic semigroups.

# 1 Introduction

The main goal of this paper is to establish the existence and uniqueness of solution for the following *anticipative stochastic evolution equation* :

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} p(0,t,y,x)\, u_0(y)\, dy \\ &\quad + \int_{\mathbb{R}^d}\int_0^t p(s,t,y,x)\, F(s,y,u(s,y))\, W(ds,y)\, dy. \end{aligned} \tag{1.1}$$

Here, the random field $W = \left\{W(t,x), t \geq 0, x \in \mathbb{R}^d\right\}$ is Gaussian and centered with covariance $\min(s,t)Q(x,y)$, where $Q$ is a bounded covariance function. For any $0 \leq s < t$ let $\mathcal{F}_{s,t}$ be the $\sigma$-field generated by the family of random variables $\left\{W(r,x) - W(s,x), s \leq r \leq t, x \in \mathbb{R}^d\right\}$. We require $p$ to be *a stochastic kernel* (see Definition 1 below). This means that $p(s,t,y,x)$ is measurable with respect to respect to the $\sigma$-field $\mathcal{F}_{s,t}$, the mapping $y \longrightarrow p(s,t,y,x)$ is a probability density on $\mathbb{R}^d$, and the following semigroup property is satisfied:

$$\int_{\mathbb{R}^d} p(s,r,y,z)p(r,t,z,x)\, dz = p(s,t,y,x),$$

for any $s \leq r \leq t$. On the other hand, we assume that $F(s,y,u)$ is $\mathcal{F}_{0,s}$ -measurable, and satisfies the usual Lipschtiz and linear growth conditions with respect to the variable $u$, uniformly in the other variables. This implies that, even if $u$ is adapted, the stochastic integral under the space integral of Equation (1.1) is anticipative. The integrand is the product of an adapted factor $A(s) := F(s,y,u(s,y))$ times a term $B(s) := p(s,t,y,x)$ which is adapted to the future increments of the random field $W$. For integrands of this type Pardoux and Protter in [24] introduced a stochastic integral called the two-sided stochastic integral, which is defined as the limit of Riemann sums of the following type

$$\int A(s)B(s)dW_s = \lim \sum_i A(s_i)B(s_{i+1})\left[W(s_{i+1}) - W(s_i)\right].$$

Later it was proved in [19] that this stochastic integral coincides with the Skorohod integral, which is an extension of the Itô integral that can be interpreted as the adjoint of the derivative operator on the Wiener space. Here we choose this type of stochastic integral in the formulation of the evolution equation (1.1). This choice is justified by the concrete example of application to an SPDE.

Equation (1.1) can be considered as an example of the following abstract stochastic evolution equation of a *random semigroup*

$$u(t) = T_{t,0}(u_0) + \int_0^t T_{t,s}\left[F(s, u(s)) W(ds)\right], \tag{1.2}$$

where $u(s)$ and $W_s$ are processes taking values in a Hilbert or Banach space $\mathbb{B}$. In this equation $\{T_{t,s}, t \geq s\}$ is a family of random linear operators on $\mathbb{B}$, satisfying the *backward flow* property $T_{t,s} = T_{t,u} \circ T_{u,s}$. In our case, the semigroup is defined on the space of bounded continuous functions on $\mathbb{R}^d$ by

$$(T_{t,s}f)(x) = \int_{\mathbb{R}^d} p(s, t, y, x) f(y) dy. \tag{1.3}$$

Stochastic evolution equations with nonrandom semigroups have been extensively studied (see [6] and references therein). Only recently has the question of using random semigroups been addressed. In [15], the authors study the existence and uniqueness of the solution of a stochastic evolution equation with a random semigroup $T_{t,s}$ that is $\mathcal{F}_{0,t}$-measurable. Its generator is the heat kernel of a second order elliptic differential operator whose coefficients are random and adapted. We consider here the case of a *stochastic semigroup*, that is, we assume that $T_{t,s}$ is $\mathcal{F}_{s,t}$-measurable. This implies that the semigroup has independent increments, and its infinitesimal generator may be, in general, a differential operator whose coefficients are white-noise in time. For this reason, the kernel $p(s, t, y, x)$ may be irregular (like Brownian motion) in the variables $s$ and $t$. A class of such semigroups has been constructed in [10] using stochastic flows. In comparison with the results proved in [15], the $\mathcal{F}_{s,t}$ -measurablility hypothesis on the semigroup allows us to get suitable estimates for the Skorohod integral in terms of the first derivative of $p(s, t, y, x)$, while in [15] two derivatives are required, and the differentiability of $p$ in the time variables is necessary.

Our motivation to study this kind of equation is the analysis of a stochastic parabolic equation of the form:

$$\begin{aligned} u(dt, x) &= \Delta_x u(t, x) dt + v(dt, x) \cdot \nabla_x u(t, x) \\ &\quad + F(t, x, u(t, x)) W(dt, x), \end{aligned} \tag{1.4}$$

where the processes $v(t, x)$ and $W(t, x)$ are Brownian in time with respect to a common filtration. If these random fields are not spatially smooth, the equation cannot have a meaning in the strong sense.

One must define a weaker sense of solution. This paper chooses to understand Equation (1.4) in the *evolution* sense: this is simply a stochastic evolution equation like Equation (1.1) above, the kernel $p$ being the backward heat kernel of the formal operator $\Delta + \dot{v}.\nabla$.

Several other weak senses for this Cauchy problem have been investigated in the literature. They include the so-called *Martingale* problem (of Stroock and Varadhan) of which a masterful treatment can be found in the recent monograph chapter [17], in which the regularity conditions on $W$ and $v$ are extremely weak, the trade-off being that one can only guarantee the existence of the law of a solution, rather than a solution itself as a function of the processes $W$ and $v$. The deterministic notion of *viscosity solution*, which made its appearance in the theory of stochastic processes for its relation to the non-linear Feynman-Kac formula (see [23]), has been adapted to the stochastic setting, of which the most recent treatment, in [5], appears to be quite general. The theory of *white-noise analysis* has been used to define distribution-valued solutions to special types of SPDEs, via the so-called *Wick* products (see the book [8]).

A solution of a stochastic PDE in the *weak* sense can be introduced using test functions and integrating by parts. Usually the evolution solution is also a solution in the weak sense (see, for instance, [26]). Weak solutions have seen a recent renewal of interest in the setting of measure-valued solutions ([14], [9]; also see the introductory remarks in [10]), as well as in the connection of SPDEs with superprocesses. These works reveal that the weak sense is not tailored to deciding when an SPDE has a function-valued solution.

Most recently, a very successful attempt to solve SPDEs by means of analytical methods was completed by Krylov in [11]. In this monograph, the authors give very weak assumptions on coefficients similar to $W$ and $v$ in Equation (1.4) guaranteeing that the solution is in some Sobolev space of distribution-valued processes. On the other hand, stochastic Sobolev embedding theorems are used in [11] to obtain continuity results.

Our approach to constructing an evolution solution treats the case of $L^2\left(\mathbb{R}^d\right)$ using the so-called *factorization method*, in the spirit of the work with non-random semigroups in [6]. In comparison with the analytical method used in [11] this approach provides an explicit integral expression for the solution, and it has some advantages like the possibility to handle without additional effort equations on bounded domains with Dirichlet or Neumann boundary conditions. Another interest of the evolution solution is found in the multiplicative linear case ($F(t, x, u) = u$) in seeking a Feynman-Kac formula. A forthcoming article ([25]) will show how this formula comes for free in the evolution setting, thanks to the existence of the kernel $p$ alluded to above, and its representation in terms of the "stochastic"

Markov process $\varphi$ with generator $\Delta + \dot{v} \cdot \nabla$. This study will use this formula to investigate the solution's Lyapunov exponent. Another approch to the Feynman-Kac formula can be found in [16], for a different form

Let us also remarks that the authors of the present paper, together with a collaborator, have shown in [1] that the approach used in this paper can be extended, in the one-dimensional case, to the case where $W$ is a space-time white noise.

We now explain the structure of this paper. Following the general scheme for evolution equations with non-random semigroups (see [6]), we establish in Section 2 the existence and uniqueness of a solution with values in $L^2(\mathbb{R}^d)$ for Equation (1.1) (Theorem 3 ) by a fixed point method. This theorem is a consequence of the estimate given in Proposition 4 which, under regularity and integrability conditions (v), (vi) and (vii) on the stochastic kernel $p(s,t,y,x)$, follows from the isometry property of the Skorohod integral and the semigroup property. Section 3 is devoted to establishing the continuity of the solution as an $L^2(\mathbb{R}^d)$-valued process. For this we need the maximal inequality for the Skorohod integral stated in Proposition 6, which requires slightly stronger ($L^p$) integrability conditions on $p(s,t,y,x)$ ($(\mathrm{v})_p$, $(\mathrm{vi})_p$ and $(\mathrm{vii})_p$ for some $p > 2$), and use of Itô's formula for the Skorohod integral following the approach introduced in [15].

In order to analyze Equation (1.4) we construct such as stochastic kernel in Section 4, i.e. a kernel $p$ which satisfies the "forward" Kolmogorov equation

$$p(ds, y) = \Delta_y p(s,y)ds + p(s,y)v(ds,y) \cdot \nabla_y,$$

where $v(t,x)$ is a $d$-dimensional centered Gaussian random field with covariance $\min(s,t)G(x,y)$, and we assume that the matrix $G$ satisfies the coercitivy assumption $I - 2^{-1}G(x,x) > 0$. The construction of this kernel follows the approach developed by Kifer and Kunita in [10], based on the backward stochastic flow $\varphi_{t,s}(x)$ associated with $v(t,x)$. Section 5 is devoted to showing that this stochastic kernel satisfies the conditions $(\mathrm{v})_p$, $(\mathrm{vi})_p$ and $(\mathrm{vii})_p$, introduced in Section 3, provided the random field $v(t,x)$ satisfies some regularity and integrability condition in the variable $x$. As a consequence, this proves the existence and uniqueness of the solution to Equation (1.1) for this particular stochastic kernel. Finally, in Section 6 we show that the solution $u(t,x)$ to Equation (1.1) for this kernel is also a weak solution to Equation (1.4), thereby justifying the choice of the Skorohod formulation for seeking an adapted solution to the evolution equation (1.1).

## 2 Stochastic evolution equations with a stochastic kernel: Existence and uniqueness of a solution in $L^2(\mathbb{R}^d)$.

Fix a measurable space $(S, \mathcal{S})$ with a finite measure $\mu$ on it, as well as a time interval $[0,T]$ . Consider the product space $[0,T] \times S$ equipped with the product measure $\lambda \times \mu$, where $\lambda$ denotes the Lebesgue measure on $[0,T]$. Let $M = \{M(A), A \in \mathcal{B}([0,T]) \otimes \mathcal{S}\}$ be a centered Gaussian family of random variables, defined in some probability space $(\Omega, \mathcal{F}, P)$, with covariance function given by

$$E(M(A)M(B)) = (\lambda \times \mu)(A \cap B).$$

Suppose that $\mathcal{F}$ is generated by $M$. We will assume that the random field $W(t,y)$ appearing in Equation (1.1) is of the form

$$W(t,y) = \int_0^t \int_S a(\lambda, y) M(ds, d\lambda),$$

where $a$ is a deterministic measurable function verifying the following condition

$$C_a := \sup_{y \in \mathbb{R}^d} \int_S |a(\lambda, y)|^2 \mu(d\lambda) < \infty. \tag{2.1}$$

In this way, the covariance function of $W$ is

$$Q(x,y) = \int_S a(\lambda, x) \overline{a(\lambda, y)} \mu(d\lambda).$$

In principle, although $\mu$ is always a positive measure, $M$ and $a$ may be complex-valued. For notational simplicity, we will assume that $M$ and $a$ are real-valued. Condition (2.1) simply says that $Q(x,x) = EW(1,x)^2$ is a bounded function. This can be considered as a weak form of spatial subhomogeneity. For each $0 \le s < t \le T$ we will denote by $\mathcal{F}_{s,t}$ the $\sigma$ -field generated by the random variables $\{M(A), A \subset [s,t] \times S\}$. This definition coincides with the one given in the introduction in terms of the random field $W$.

We can develop a stochastic calculus of variations with respect to the Gaussian family $M$ following the lines of [18]. The reference Hilbert space is here $H = L^2([0,T] \times S, \lambda \times \mu)$. For an element $f \in H$ we can define the Gaussian random variable $M(f) := \int_{[0,T]\times S} f(s,\lambda) M(ds, d\lambda)$. Let $\mathcal{S}$ be the class of smooth random variables of the form

$$G = g(M(f_1), \ldots, M(f_n)),$$

where $g$ is an infinitely differentiable function with bounded derivatives of all orders. For a such a random variable we define its derivative as the random field on $[0, T] \times S$ given by

$$D_{s,\lambda} G = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} (M(f_1), \dots, M(f_n)) f_i(s, \lambda).$$

Iterated derivatives are defined in an obvious way. Then, for any integer $k \geq 1$ and any real number $p \geq 1$ the Sobolev space $\mathbb{D}^{k,p}$ is defined as the completion of $\mathcal{S}$ with respect to the seminorm

$$\|G\|_{k,p}^p := \sum_{j=0}^{k} E \left| \int_{([0,T]\times S)^j} |D_{s_1,\lambda_1} \dots D_{s_j,\lambda_j} G|^2 ds_1 \mu(d\lambda_1) \dots ds_j \mu(d\lambda_j) \right|^{\frac{p}{2}}.$$

For any real and separable Hilbert space $V$ we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of $V$-valued random variables. Note that the derivative operator preserves adaptedness and that if $F$ is $\mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}$ -measurable then $D_{r,\lambda} F = 0$ if $s \leq r \leq t$.

The Skorohod integral is defined as the adjoint of the derivative operator in $L^2(\Omega)$. That is, a square integrable random field $H_{s,\lambda}$ is Skorohod integrable if for any $G \in \mathcal{S}$ we have

$$\left| E \left[ \int_{[0,T]\times S} D_{s,\lambda} G H_{s,\lambda} ds \mu(d\lambda) \right] \right| \leq c \|G\|_{L^2(\Omega)}.$$

The Skorohod integral of $H$, denoted by $\delta(H)$, is then defined via the Riesz representation theorem by the duality relationship

$$E \left[ \int_{[0,T]\times S} D_{s,\lambda} G H_{s,\lambda} ds \mu(d\lambda) \right] = E \left[ G \delta(H) \right]. \tag{2.2}$$

We will also make use of the notation $\delta(H) = \int_{[0,T]\times S} H_{s,\lambda} M(ds, d\lambda)$. A random field $H_{s,\lambda}$ is said to be *adapted* if $H_{s,\lambda}$ is $\mathcal{F}_{0,s}$-measurable for each $(s, \lambda)$. It holds that square integrable adapted random fields are Skorohod integrable and the Skorohod integral with respect to $M$ coincides with the Itô stochastic integral, which can also be defined by means of the theory of martingale measures (see, for instance, [26]). On the other hand, processes in the space $\mathbb{D}^{1,2}(H)$ are also Skorohod integrable.

We will make use of the following formula for the $L^2$-norm of the Skorohod integral of an $L^2(\mathbb{R}^d)$-valued process. Henceforth we will denote by $\| \cdot \|_2$ the norm in $L^2(\mathbb{R}^d)$.

**Lemma 1** *Let $H = \{H_{s,\lambda}, s \in [0,T], \lambda \in S\}$ be an $L^2(\mathbb{R}^d)$-valued square integrable random field such that $H_{s,\lambda}(x)$ belongs to $\mathbb{D}^{1,2}$ for almost each $s, \lambda, x$, the process*

$$\left\{D_{s,\lambda} H_{r,\lambda'}(x) \mathbf{1}_{[0,s]}(r), (r,\lambda') \in [0,T] \times S\right\}$$

*is Skorohod integrable for almost each $s, \lambda, x$, and*

$$E \int_{\mathbb{R}^d} \int_{[0,T]\times S} \left| H_{s,\lambda}(x) \int_{[0,s]\times S} D_{s,\lambda} H_{r,\lambda'}(x) M(dr, d\lambda') \right| ds\mu(d\lambda)dx < \infty. \tag{2.3}$$

*Then $H$ is Skorohod integrable and*

$$\begin{aligned} & E \left\| \int_{[0,T]\times S} H_{s,\lambda} M(ds, d\lambda) \right\|_2^2 \\ = & \int_{[0,T]\times S} E \left\| H_{s,\lambda} \right\|_2^2 ds\mu(d\lambda) + 2E \int_{\mathbb{R}^d} \int_{[0,T]\times S} H_{s,\lambda}(x) \\ & \times \left( \int_{[0,s]\times S} D_{s,\lambda} H_{r,\lambda'}(x) M(dr, d\lambda') \right) ds\mu(d\lambda)dx. \end{aligned}$$

*Proof:* By an approximation argument we can assume that $H$ is an smooth elementary process. In this case the formula is a straightforward consequence of the duality relationship between the Skorohod integral and the derivative operator and the following isometry property:

$$\begin{aligned} & E \left\| \int_{[0,T]\times S} H_{s,\lambda} M(ds, d\lambda) \right\|_2^2 \\ = & \int_{[0,T]\times S} E \left\| H_{s,\lambda} \right\|_2^2 ds\mu(d\lambda) \\ & + 2E \int_S \int_0^T \left( \int_S \int_0^s \left\langle D_{s,\lambda} H_{r,\lambda'}, D_{r,\lambda'} H_{s,\lambda} \right\rangle_2 dr\mu(d\lambda') \right) ds\mu(d\lambda). \end{aligned}$$

□

We will also need the following Fubini theorem for the Skorohod integral, whose proof is an immediate consequence of the definition of the Skorohod integral.

**Lemma 2** *Let $\{H_{s,\lambda}(\theta), (s,\lambda) \in [0,T] \times S, \theta \in \Theta\}$ be a measurable random field parameterized by a measure space $(\Theta, \mathcal{O}, \nu)$ with finite measure $\nu$. Suppose that*

$$E \int_{[0,T]\times S} \left( \int_\Theta \left| H_{s,\lambda}(\theta) \right| \nu(d\theta) \right)^2 ds\mu(d\lambda) < \infty,$$

*$H_{s,\lambda}(\theta)$ is Skorohod integrable for $\nu$-almost all $\theta$, and $E\left(\int_\Theta |\delta H(\theta)|\,\nu(d\theta)\right)^2 < \infty$. Then $\int_\Theta H_{s,\lambda}(\theta)\nu(d\theta)$ is Skorohod integrable and*

$$\delta\left(\int_\Theta H(\theta)\nu(d\theta)\right) = \int_\Theta \delta H(\theta)\nu(d\theta).$$

Consider a measurable random field $\tilde{H}_{s,y}$ parameterized by $[0,T]\times\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |\tilde{H}_{s,y}a(\lambda,y)|dy < \infty$ a.s. The integral $\int_{[0,T]\times\mathbb{R}^d} \tilde{H}_{s,y}W(ds,y)dy$ is defined by

$$\int_{[0,T]\times\mathbb{R}^d} \tilde{H}_{s,y}W(ds,y)dy = \int_{[0,T]\times S}\left(\int_{\mathbb{R}^d} \tilde{H}_{s,y}a(\lambda,y)dy\right)M(ds,d\lambda),$$

provided the random field $\int_{\mathbb{R}^d} \tilde{H}_{s,y}a(\lambda,y)dy$ is Skorohod integrable with respect to $M$. With this definition, Equation (1.1) can be written in terms of $M$ as follows

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} p(0,t,y,x)\,u_0(y)\,dy \\ &\quad + \int_0^t\int_S\left(\int_{\mathbb{R}^d} p(s,t,y,x)F(s,y,u(s,y))a(\lambda,y)dy\right)M(ds,d\lambda). \end{aligned} \tag{2.4}$$

Let us now introduce the kind of stochastic kernels we are going to deal with.

**Definition 1** *A random function $p(s,t,y,x)$ defined for $0\le s<t\le T$, $x,y\in\mathbb{R}^d$ is called a* backward stochastic kernel *if the following conditions are satisfied:*

**(i)** *[Adaptedness] $p(s,t,y,x)$ is $\mathcal{F}_{s,t}$ -measurable,*

**(ii)** *$p(s,t,y,x)\ge 0$,*

**(iii)** *$\int_{\mathbb{R}^d} p(s,t,y,x)\,dy = 1$,*

**(iv)** *[Backward flow property] $\int_{\mathbb{R}^d} p(s,r,y,z)\,p(r,t,z,x)\,dz = p(s,t,y,x)$ for almost all $y\in\mathbb{R}^d$, and for all $x\in\mathbb{R}^d$, $0\le s<r<t\le T$.*

Consider the following additional conditions:

**(v)** $C_b := \sup_{s,t,y} E\int_{\mathbb{R}^d} p(s,t,y,x)dx < \infty$.

**(vi)** $p(s,t,y,x)$ belongs to the Sobolev space $\mathbb{D}^{1,2}$ for each $x, y \in \mathbb{R}^d$ and $0 \le s < t \le T$ and $p(s,t,\cdot,x)$ belongs to $\mathbb{D}^{1,2}(L^2(\mathbb{R}^d))$. Moreover, there exists a version of the derivative such that the following limit exists in $L^2(\Omega; L^2(\mathbb{R}^d))$ for each $s, \lambda, t, x$

$$D_{s,\lambda} p(s,t,\cdot,x) := \lim_{\epsilon \downarrow 0} D_{s,\lambda} p(s-\epsilon,t,\cdot,x).$$

Furthermore, $p(s,t,y,\cdot)$ is left continuous in $t \in (s,T]$ with values in $L^2(\Omega; L^2(\mathbb{R}^d))$.

**(vii)** There exist constants $c_1, c_2$, such that

$$\sup_z E \int_{S\times\mathbb{R}^d\times\mathbb{R}^d} |a(\lambda,y)| p(s,t,y,x) |D_{s,\lambda} p(s,t,z,x)| \mu(d\lambda) dx dy \le c_1 (t-s)^{-1/2},$$

and

$$\sup_y E \int_{S\times\mathbb{R}^d\times\mathbb{R}^d} |a(\lambda,y)| p(s,t,y,x) |D_{s,\lambda} p(s,t,z,x)| \mu(d\lambda) dz dx \le c_2 (t-s)^{-1/2}.$$

Taking into account the properties of the derivative operator and the fact that $p(s,t,\cdot,x)$ belongs to $\mathbb{D}^{1,2}(L^2(\mathbb{R}^d))$, we can write the following formula for $r < s < t$:

$$\begin{aligned} D_{s,\lambda} p(r,t,y,x) &= D_{s,\lambda} \int_{\mathbb{R}^d} p(r,s-\epsilon,y,z) p(s-\epsilon,t,z,x) dz \\ &= \int_{\mathbb{R}^d} p(r,s-\epsilon,y,z) D_{s,\lambda} p(s-\epsilon,t,z,x) dz, \end{aligned}$$

and, letting $\epsilon$ tend to zero and using hypothesis (vi) we deduce

$$D_{s,\lambda} p(r,t,y,x) = \int_{\mathbb{R}^d} p(r,s,y,z) D_{s,\lambda} p(s,t,z,x) dz. \tag{2.5}$$

We require the random function $F : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ to be progressively measurable and to satisfy the usual Lipschtiz and linear growth conditions in the variable $u$. That is, we assume that $F$ satisfies the following conditions:

**(a)** $F$ is measurable with respect to the $\sigma$-field $\mathcal{B}([0,t] \times \mathbb{R}^d \times \mathbb{R}) \otimes \mathcal{F}_{0,t}$ when restricted to $[0,t] \times \mathbb{R}^d \times \mathbb{R} \times \Omega$, for each $t \in [0,T]$.

**(b)** $|F(t,y,u) - F(t,y,v)| \leq K_T|u-v|$, for all $t \in [0,T]$, $y \in \mathbb{R}^d$, and $u, v \in \mathbb{R}$.

**(c)** $|F(t,y,u)| \leq K_T(1+|u|)$, for all $t \in [0,T]$, $y \in \mathbb{R}^d$, and $u \in \mathbb{R}$.

With these preliminaries we are able to state the main result of this section.

**Theorem 3** *Let $u_0$ be a function in $L^2(\mathbb{R}^d)$. Let $F(s,y,u)$ be a random function satisfying the above conditions (a), (b) and (c). Let $p(s,t,y,x)$ be a stochastic kernel in the sense of Definition 1 satisfying conditions (v), (vi) and (vii). Then there exists a unique adapted $L^2(\mathbb{R}^d)$-valued solution to Equation (2.4) such that $E\int_0^T \|u(s)\|_2^2\, ds < \infty$.*

For the proof of this theorem we need the following estimate of a Skorohod integral of the form $\int_0^t \int_{\mathbb{R}^d} p(s,t,y,x)\phi(s,y)W(ds,y)dy$, where $\phi(s,y)$ is an adapted square integrable process.

**Proposition 4** *Let $\phi = \{\phi(s,y), s \in [0,T], y \in \mathbb{R}^d\}$ be an adapted random field such that $E\int_0^T \|\phi(s)\|_2^2 ds < \infty$ . Let $p(s,t,y,x)$ be a stochastic kernel in the sense of Definition 1 satisfying conditions (v), (vi) and (vii). Then the $L^2(\mathbb{R}^d)$-valued random field*

$$\{\mathbf{1}_{[0,t]}(s)\int_{\mathbb{R}^d} p(s,t,y,\cdot)\phi(s,y)a(\lambda,y)dy, s \in [0,T], \lambda \in S\}$$

*is Skorohod integrable with respect to $M$ for almost all $t \in [0,T]$, and for some constant $C > 0$, which depends on $T$, $C_a$, $C_b$, $c_1$ and $c_2$ , it holds that*

$$E\left\|\int_0^t \int_S \left(\int_{\mathbb{R}^d} p(s,t,y,\cdot)\phi(s,y)a(\lambda,y)dy\right) M(ds,d\lambda)\right\|_2^2 \leq C\int_0^t (t-s)^{-1/2}E\|\phi(s)\|_2^2 ds.$$

*Proof:* Denote by $\mathcal{E}$ the class of smooth elementary adapted random fields of the form

$$\phi(s,y) = \sum_{i,k=1}^n G_{ik}b_k(y)\mathbf{1}_{(t_i,t_{i+1}]}(s), \tag{2.6}$$

where $G_{ik} \in \mathcal{S}$, $b_k \in \mathcal{C}_K^\infty(\mathbb{R}^d)$, $0 = t_1 < ... < t_{n+1} = T$, and $G_{ik}$ is $\mathcal{F}_{0,t_i}$-measurable. Let $\phi$ be an adapted random field such that $E\int_0^T \|\phi(s)\|_2^2 ds <$

$\infty$. We can find a sequence $\phi_n$ of smooth elementary adapted random fields in the class $\mathcal{E}$ satisfying

$$\lim_n \int_0^T E\left\|\phi_n(s)-\phi(s)\right\|_2^2 ds = 0.$$

Therefore the sequence of functions $t \to \int_0^t (t-s)^{-1/2} E\left\|\phi_n(s)-\phi(s)\right\|_2^2 ds$ converges to zero in $L^1([0,T])$. This implies the existence of a subsequence $n_i$ such that for all $t \in [0,T]$ out of a set of zero Lebesgue measure

$$\lim_i \int_0^t (t-s)^{-1/2} E\left\|\phi_{n_i}(s)-\phi(s)\right\|_2^2 ds = 0.$$

Recall that if a sequence $H_n$ of Skorohod integrable processes converges in $L^2(\Omega; H \otimes L^2(\mathbb{R}^d))$ to $H$, and $\delta(H_n)$ converges in $L^2(\Omega; L^2(\mathbb{R}^d))$, then $H$ is Skorohod integrable and $\delta(H)$ is the limit of $\delta(H_n)$. Hence, we can assume that $\phi$ is of the form (2.6). Set

$$H_{s,\lambda}(x) = \int_{\mathbb{R}^d} p(s,t,y,x)\phi(s,y)a(\lambda,y)dy,$$

and

$$\Phi_t(x) = \int_0^t \int_S H_{s,\lambda}(x) M(ds,d\lambda).$$

Notice that

$$\Phi_t(x) = \sum_{i,k=1}^n G_{ik} \int_{(t_i,t_{i+1}]\cap[0,t]\times S} \left(\int_{\mathbb{R}^d} p(s,t,y,x) b_k(y) a(\lambda,y)dy\right) M(ds,d\lambda).$$

Let us show that the process $H_{s,\lambda}(x)\mathbf{1}_{[0,t]}(s)$ verifies the assumptions of Lemma 1. By condition (v) we have

$$E\int_{[0,t]\times S} \left\|H_{s,\lambda}\right\|_2^2 ds\mu(d\lambda) \le C_a C_b \int_0^t E\left\|\phi_s\right\|_2^2 ds < \infty. \tag{2.7}$$

In order to check that the process $\left\{D_{s,\lambda}H_{r,\lambda'}(x)\mathbf{1}_{[0,s]}(r), (r,\lambda') \in [0,T]\times S\right\}$ is Skorohod integrable and condition (2.3) holds let us compute, using (2.5),

$$D_{s,\lambda}H_{r,\lambda'}(x) = D_{s,\lambda}\int_{\mathbb{R}^d} p(r,t,y,x)\phi(r,y)a(\lambda',y)dy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} D_{s,\lambda} p(r,t,y,x)\phi(r,y)a(\lambda',y)dy \\
&= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} p(r,s,y,z) D_{s,\lambda} p(s,t,z,x) dz \right) \phi(r,y) a(\lambda',y) dy \\
&= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} p(r,s,y,z) \phi(r,y) a(\lambda',y) dy \right) D_{s,\lambda} p(s,t,z,x) dz.
\end{aligned}
$$

Now we use the fact that the random variable $D_{s,\lambda} p(s,t,\cdot,x)$ is $\mathcal{F}_{s,t}$-measurable and belongs to the space $L^2(\Omega; L^2(\mathbb{R}^d))$. This fact, together with the particular form of the process $\phi$ imply that $D_{s,\lambda} H_{r,\lambda'}(x)$ is Skorohod integrable in $[0,s] \times S$, and $D_{s,\lambda} p(s,t,z,x)$ can be factorized out of the Skorohod integral, obtaining

$$
\int_{[0,s]\times S} D_{s,\lambda} H_{r,\lambda'}(x) M(dr, d\lambda') = \int_{\mathbb{R}^d} D_{s,\lambda} p(s,t,z,x) \Phi_s(z) dz.
$$

As a consequence,

$$
\begin{aligned}
& E \int_{\mathbb{R}^d} \int_{[0,t]\times S} \left| H_{s,\lambda}(x) \int_{[0,s]\times S} D_{s,\lambda} H_{r,\lambda'}(x) M(dr, d\lambda') \right| ds \mu(d\lambda) dx \\
= \; & E \int_{\mathbb{R}^d} \int_{[0,t]\times S} |H_{s,\lambda}(x)| \int_{\mathbb{R}^d} |D_{s,\lambda} p(s,t,z,x) \Phi_s(z)| \, dz ds \mu(d\lambda) dx \\
\leq \; & \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_S E \left( \int_{\mathbb{R}^d} p(s,t,y,x) \left| D_{s,\lambda} p(s,t,z,x) \right| dx \right) |a(\lambda,y)| \, \mu(d\lambda) \\
& \times \left[ E\phi(s,y)^2 + E\Phi_s(z)^2 \right] dz dy ds \\
\leq \; & \frac{1}{2} \left( c_2 \int_0^t (t-s)^{-1/2} E \left\| \phi_s \right\|_2^2 ds + c_1 \int_0^t (t-s)^{-1/2} E \left\| \Phi_s \right\|_2^2 ds \right). \quad (2.8)
\end{aligned}
$$

The right-hand side of Equation (2.8) is finite because $\phi$ is smooth and elementary and, hence, condition (2.3) holds. As a consequence, Lemma 1 yields

$$
\begin{aligned}
E \left\| \Phi_t \right\|_2^2 \; = \; & E \int_{[0,t]\times S} \left\| H_{s,\lambda} \right\|_2^2 ds \mu(d\lambda) \\
& + 2E \int_{\mathbb{R}^d} \int_{[0,t]\times S} H_{s,\lambda}(x) \\
& \times \left( \int_{[0,s]\times S} D_{s,\lambda} H_{r,\lambda'}(x) M(dr, d\lambda') \right) ds \mu(d\lambda) dx. \quad (2.9)
\end{aligned}
$$

Substituting (2.7) and (2.8) into (2.9) yields

$$\begin{aligned} E\left\|\Phi_t\right\|_2^2 &\le C_a C_b \int_0^t E\left\|\phi_s\right\|_2^2 ds \\ &\quad + c_2 \int_0^t (t-s)^{-1/2} E\left\|\phi_s\right\|_2^2 ds + c_1 \int_0^t (t-s)^{-1/2} E\left\|\Phi_s\right\|_2^2 ds. \end{aligned}$$

This expression easily implies the desired result by a suitable generalization of Gronwall's lemma. □

*Proof of Theorem 3:* Suppose that $u$ and $v$ are two adapted solutions to Eq. (2.4), such that $E\int_0^T \|u(s)\|_2^2\, ds < \infty$ and $E\int_0^T \|v(s)\|_2^2\, ds < \infty$. Set $F_{s,y}(u) = F(s,y,u(s,y))$. From Proposition 4 and the Lipschitz property of the function $F$, we obtain

$$\begin{aligned} & E\left\|u_t - v_t\right\|_2^2 \\ = \ & E\left\| \int_0^t \int_S \left( \int_{\mathbb{R}^d} p(s,t,y,z)\left[F_{s,y}(u) - F_{s,y}(v)\right] a(\lambda,y)dy \right) M(ds,d\lambda) \right\|_2^2 \\ \le \ & C\int_0^t (t-s)^{-1/2} E\|F_s(u) - F_s(v)\|_2^2 ds \\ \le \ & CK_T^2 \int_0^t (t-s)^{-1/2} E\|u_s - v_s\|_2^2 ds, \end{aligned}$$

and this implies that $u = v$. The proof of the existence can be done by the usual Picard iteration procedure. That is, we define recursively

$$u^0(t,x) = \int_{\mathbb{R}^d} p(0,t,y,x)\, u_0(y)\, dy,$$

and

$$\begin{aligned} u^{n+1}(t,x) &= \int_{\mathbb{R}^d} p(0,t,y,x)\, u_0(y)\, dy \\ &\quad + \int_0^t \int_S \left( \int_{\mathbb{R}^d} p(s,t,y,x) F(s,y,u^n(s,y)) a(\lambda,y) dy \right) M(ds,d\lambda), \end{aligned}$$

for all $n \ge 0$. Using $u_0 \in L^2\left(\mathbb{R}^d\right)$ and condition (v), we obtain $E\left\|u_t^0\right\|_2^2 < \infty$. It follows by induction, using Proposition 4, that $E\sum_{n=0}^\infty \left\|u_t^{n+1} - u_t^n\right\|_2^2 < \infty$, and the limit of the sequence $u^n$ provides the solution. □

Under the condition that the initial Picard approximation $u^0(t)$ is a continuous function from $[0,T]$ into $L^2\left(\Omega \times \mathbb{R}^d\right)$, we can use the previous proofs to show that the solution to Equation (2.4) shares the same continuity property. A sufficient condition for $u^0$'s continuity in $L^2\left(\Omega\times\mathbb{R}^d\right)$ is:

$$\lim_{h\to 0}\sup_y E\int_{\mathbb{R}^d}\left|p\left(0,t+h,y,x\right)-p\left(0,t,y,x\right)\right|dx=0.$$

If $u^0$ happens to be almost-surely continuous from $[0,T]$ into $L^2\left(\mathbb{R}^d\right)$, one might ask whether the solution to Equation (2.4) still shares the same property. This is true under additional conditions on the stochastic kernel, as the results of the next section show.

# 3 Stochastic evolution equations with a stochastic kernel: Continuity of the solution.

In this section we will show that the solution $u(t)$ of Equation (2.4) obtained in the last section is continuous in time. For this we need some additional integrability conditions on the kernel $p(s,t,y,x)$. Fix $p\geq 2$ and consider the following hypotheses:

**(v)**$_p$

$$C_{p,1}:=\sup_{s,y}E\left(\sup_{t\in[s,T]}\int_{\mathbb{R}^d}p(s,t,y,x)dx\right)^{\frac{p}{2}}<\infty.$$

**(vi)**$_p$ Condition (vi) holds and for each $K>0$, and $0<\delta<t$ , we have

$$\int_0^{t-\delta}E\left|\int_0^s\int_{\mathbb{R}^d\times B_K\times S}\left|D_{s,\lambda}p(s,t,y,x)\right|^2dxdy\mu(d\lambda)dr\right|^{p/2}ds<\infty,$$

where $B_K=\left\{y\in\mathbb{R}^d,|y|\leq K\right\}$. Moreover, for every compact set $K\subset\mathbb{R}^d$ and for all $s$, $t\to p\left(s,t,\cdot,\cdot\right)$ is continuous in $(s,T]$ with values in $L^1\left(\mathbb{R}^d\times B_K\right)$.

**(vii)**$_p$ There exists a constant $C_{p,2}$ such that for all $y,z\in\mathbb{R}^d$

$$\sup_y E\left|\int_{S\times\mathbb{R}^d\times\mathbb{R}^d}|D_{s,\lambda}p(s,t,z,x)D_{s\lambda}p(s,t,y,x)|\mu(d\lambda)dzdx\right|^{\frac{p}{2}}$$
$$\leq C_{p,2}(t-s)^{-\frac{p}{2}}\ .$$

The main ingredient in the proof of the continuity of the solution to the stochastic evolution equation (2.4) are the estimates for the Skorohod integral established in the next two theorems. These theorems are analogous to Theorem 3.2 and Theorem 3.3 in [15] for the case of a random semigroup $T_{t,s}$ which is $\mathcal{F}_{0,t}$-measurable. For a given random field $\phi = \{\phi(s,y), s \in [0,T], y \in \mathbb{R}^d\}$ we will define the operator

$$(S(s,t)\phi)(\lambda, x) = \int_{\mathbb{R}^d} p(s,t,y,x)\phi(s,y)a(\lambda,y)dy.$$

Using condition (2.1) we have

$$\|S(s,t)\phi\|^2_{L^2(S\times\mathbb{R}^d)} \le C_a \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(s,t,y,x)\phi(s,y)^2 dydx. \tag{3.1}$$

**Proposition 5** *Fix* $2 \le p < 4$ *and* $\alpha \in [0, \frac{1}{2})$. *Let* $\phi = \{\phi(s,y), s \in [0,T], y \in \mathbb{R}^d\}$ *be an adapted random field such that* $E\int_0^T \|\phi(s)\|_2^p ds < \infty$. *Let* $p(s,t,y,x)$ *be a stochastic kernel in the sense of Definition 1 satisfying conditions (v)*$_p$*, (vi)*$_p$ *and (vii)*$_p$*. Then the* $L^2(\mathbb{R}^d)$*-valued random field*

$$\{(t-s)_+^{-\alpha}\left(S(s,t)\phi\right)(\lambda,\cdot), s\in[0,T], \lambda \in S\}$$

*is Skorohod integrable with respect to* $M$ *for almost all* $t \in [0,T]$, *and for some constant* $C_1 > 0$, *which depends on* $T$, $p$, $\alpha$, $C_a$, $C_{p,1}$ *and* $C_{p,2}$, *it holds that*

$$E\left\|\int_0^t\int_S (t-s)^{-\alpha}\left(\int_{\mathbb{R}^d} p(s,t,y,\cdot)\phi(s,y)a(\lambda,y)dy\right)M(ds,d\lambda)\right\|_2^p$$
$$\le C_1\int_0^t (t-s)^{-2\alpha+\frac{p}{4}(2\alpha-1)}E\|\phi(s)\|_2^p ds. \tag{3.2}$$

*Proof:* As in the proof of Proposition 4, we can assume that $\phi$ is of the form (2.6). Fix $t_0 > t_1$ in $[0,T]$, and define

$$B_{s,\lambda}(x) = \mathbf{1}_{[0,t_1]}(s)(t_0-s)^{-\alpha}(S(s,t_1)\phi)(\lambda,x),$$

and

$$X_t(x) = \int_{[0,t]\times S} B_{s,\lambda}(x)M(ds,d\lambda),$$

for $t \in [0,t_1]$ . Suppose first that $p(s,t_1,y,x)$ is an elementary backward-adapted process of the form

$$\sum_{i,j,k=1}^n H_{ijk}\beta_j(y)\gamma_k(x)\mathbf{1}_{(s_i,s_{i+1}]}(s), \tag{3.3}$$

where $H_{ijk} \in \mathcal{S}$, $\beta_j, \gamma_k \in \mathcal{C}_K^\infty(\mathbb{R}^d)$, $0 = s_1 < ... < s_{n+1} = t_1$, and $H_{ijk}$ is $\mathcal{F}_{s_{i+1},t_1}$-measurable. Applying Itô's formula for Hilbert-valued Skorohod integrals (see, for instance, Proposition 2.9 in [15]), taking the mathematical expectation, and using Cauchy-Schwarz inequality as in the proof of Theorem 3.2 in [15] we obtain

$$\begin{aligned} E\|X_t\|_2^p &\leq \frac{p(p-1)}{2} E\int_0^t \|X_s\|_2^{p-2} \Big(\|B_s\|^2_{L^2(S\times\mathbb{R}^d)} + 2\,\|B_s\|_{L^2(S\times\mathbb{R}^d)} \\ &\qquad \times \left\| \int_{[0,s]\times S} D_{s,\cdot} B_{r,\lambda'}(\cdot) M(dr, d\lambda') \right\|_{L^2(S\times\mathbb{R}^d)} \Bigg) ds. \end{aligned}$$

Unlike the proof of Theorem 3.2 in [15] here we keep together the factors $\|B_s\|_{L^2(S\times\mathbb{R}^d)}$ and $\left\|\int_{[0,s]\times S} D_{s,\cdot} B_{r,\lambda'}(\cdot) M(dr, d\lambda')\right\|_{L^2(S\times\mathbb{R}^d)}$. Then, Hölder's inequality leads to the following estimate

$$E\|X_t\|_2^p \leq \int_0^t (E\|X_s\|_2^p)^{\frac{p-2}{p}} A_s ds,$$

where

$$\begin{aligned} A_s &= p(p-1)(t_0-s)^{-2\alpha}\frac{1}{2}\Bigg\{\left(E\,\|(t_0-s)^\alpha B_s\|^p_{L^2(S\times\mathbb{R}^d)}\right)^{2/p} \\ &\quad + \left(E\,\|(t_0-s)^{2\alpha} B_s\|^p_{L^2(S\times\mathbb{R}^d)}\right)^{1/p} \\ &\quad \times \left(E\left\|\int_{[0,s]\times S} D_{s,\cdot} B_{r,\lambda'}(\cdot) M(dr, d\lambda')\right\|^p_{L^2(S\times\mathbb{R}^d)}\right)^{1/p}\Bigg\}. \end{aligned}$$

Because $\phi$ is a simple process, it is easily seen that by condition $(\text{v})_p$ we have $E\|X_t\|_2^p < \infty$. Then the first lemma in [27] implies that

$$E\|X_t\|_2^p \leq \left(\frac{2}{p}\int_0^t A_s ds\right)^{p/2},$$

that is,

$$\begin{aligned} E\|X_t\|_2^p &\leq C(T,p,\alpha)\Bigg(\int_0^t (t_0-s)^{-2\alpha}\Lambda_1(s) ds \\ &\qquad + \int_0^t (t_0-s)^{-2\alpha+\frac{\alpha p}{2}} \sqrt{\Lambda_1(s)\Lambda_2(s)} ds\Bigg), \end{aligned} \tag{3.4}$$

where

$$\Lambda_1(s) = E \left\| S(s,t_1)\phi \right\|^p_{L^2(S\times\mathbb{R}^d)},$$

and

$$\Lambda_2(s) = E \left\| \int_{[0,s]\times S} D_{s,\cdot} B_{r,\lambda'} M(dr, d\lambda') \right\|^p_{L^2(S\times\mathbb{R}^d)}.$$

Conditions $(\mathrm{v})_p$ and $(\mathrm{vi})_p$ guarantee that the right-hand side of (3.4) is finite. As a consequence, we can approximate $p(s,t_1,y,x)$ by elementary backward-adapted processes of the form (3.3) in such a way that (3.4) still holds. The term $\Lambda_1(s)$ can be estimated as follows using (3.1) and assumption $(\mathrm{v})_p$:

$$\begin{aligned}
\Lambda_1(s) &\leq C_a^{\frac{p}{2}} E \left( \int_{\mathbb{R}^d\times\mathbb{R}^d} p(s,t_1,y,x)\phi(s,y)^2 dy dx \right)^{\frac{p}{2}} \\
&\leq C_a^{\frac{p}{2}} \int_{\mathbb{R}^d} E\left[ \|\phi(s)\|_2^{p-2} \phi(s,y)^2 \right] E\left( \int_{\mathbb{R}^d} p(s,t_1,y,x) dx \right)^{\frac{p}{2}} dy \\
&\leq C_a^{\frac{p}{2}} C_{p,1} E\|\phi(s)\|_2^p. \qquad (3.5)
\end{aligned}$$

In order to estimate the term $\Lambda_2(s)$ we first write, using (2.5)

$$\begin{aligned}
D_{s,\lambda} B_{r,\lambda'}(x) &= (t_0 - r)^{-\alpha} \int_{\mathbb{R}^d} D_{s,\lambda} p(r,t_1,y,x)\phi(r,y) a(\lambda', y) dy \\
&= (t_0 - r)^{-\alpha} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} p(r,s,y,z) D_{s,\lambda} p(s,t_1,z,x) dz \right) \phi(r,y) a(\lambda', y) dy \\
&= (t_0 - r)^{-\alpha} \int_{\mathbb{R}^d} (S(r,s)\phi)(\lambda', z) D_{s,\lambda} p(s,t_1,z,x) dz.
\end{aligned}$$

Now we use the fact that the random variable $D_{s,\lambda} p(s,t_1,z,x)$ is $\mathcal{F}_{s,t_1}$-measurable. This implies that it can be factorized out of the Skorohod integral, and we obtain

$$\begin{aligned}
&E \left\| \int_{[0,s]\times S} D_{s,\cdot} B_{r,\lambda'}(\cdot) M(dr, d\lambda') \right\|^p_{L^2(S\times\mathbb{R}^d)} \\
&= E \left\| \int_{\mathbb{R}^d} D_{s,\cdot} p(s,t_1,z,\cdot) Y(s,z) dz \right\|^p_{L^2(S\times\mathbb{R}^d)}, \qquad (3.6)
\end{aligned}$$

where

$$Y(s,z) = \int_{[0,s]\times S} (t_0 - r)^{-\alpha} (S(r,s)\phi)(\lambda', z) M(dr, d\lambda').$$

Applying condition (vii)$_p$ we get

$$\Lambda_2(s) \le C_{p,2}(t_1 - s)^{-\frac{p}{2}} E\|Y(s)\|_2^p. \qquad (3.7)$$

Substituting (3.5) and (3.7) into (3.4) yields

$$\begin{aligned} E\|X_t\|_2^p &\le C \left( \int_0^t (t_0 - s)^{-2\alpha} E\|\phi(s)\|_2^p ds \right. \\ &\quad \left. + \int_0^t (t_0 - s)^{-2\alpha + \frac{p\alpha}{2}} (t_1 - s)^{-\frac{p}{4}} \left( E\|\phi(s)\|_2^p E\|Y(s)\|_2^p \right)^{\frac{1}{2}} ds \right), \end{aligned}$$

for some constant $C$ which depends on $T$, $p$, $\alpha$, $C_a$, $C_{p,1}$ and $C_{p,2}$. If we take $t = t_1$, we have $X_t = Y(t)$ and we obtain

$$\begin{aligned} E\|X_t\|_2^p &\le C \left( \int_0^t (t - s)^{-2\alpha + \frac{p}{4}(2\alpha - 1)} E\|\phi(s)\|_2^p ds \right. \\ &\quad \left. + \int_0^t (t - s)^{-2\alpha + \frac{p}{4}(2\alpha - 1)} E\|X(s)\|_2^p ds \right). \end{aligned}$$

Applying Gronwall's lemma we get the desired result. □

Note that in the proof of Proposition 5 we have only used, instead of (v)$_p$, the weaker estimate:

$$\sup_{s,t,y} E \left( \int_{\mathbb{R}^d} p(s, t, y, x) dx \right)^{\frac{p}{2}} < \infty.$$

Condition (v)$_p$ is required in the proof of the maximal inequality.

**Proposition 6** *Fix $2 < p < 4$. Let $\phi = \{\phi(s, y), s \in [0, T], y \in \mathbb{R}^d\}$ be an adapted random field such that $E \int_0^T \|\phi(s)\|_2^p ds < \infty$. Let $p(s, t, y, x)$ be a stochastic kernel in the sense of Definition 1 satisfying conditions (v)$_p$, (vi)$_p$ and (vii)$_p$. Then the $L^2(\mathbb{R}^d)$-valued random field*

$$\{\mathbf{1}_{[0,t]}(s) \int_{\mathbb{R}^d} p(s, t, y, x)\phi(s, y)a(\lambda, y)dy, s \in [0, T], \lambda \in S\}$$

*is Skorohod integrable with respect to $M$ and there exists a constant $C_2 > 0$, which depends on $T$, $p$, $\alpha$, $C_a$ and $C_{p,1}$ and $C_{p,2}$ such that*

$$E \left( \sup_{0 \le t \le T} \left\| \int_{[0,t] \times S} \left( \int_{\mathbb{R}^d} p(s, t, y, \cdot)\phi(s, y)a(\lambda, y)dy \right) M(ds, d\lambda) \right\|_2^p \right) \le C_2 \int_0^t E\|\phi(s)\|_2^p ds.$$

*Proof:* We will make use of the factorization method in order to handle the supremum in $t$. Fix $\alpha \in (1/p, 1/2)$. We can write

$$p(s,t,y,x) = C_\alpha \int_{\mathbb{R}^d} \int_s^t p(s,r,y,z)(r-s)^{-\alpha} p(r,t,z,x)(t-r)^{\alpha-1} dr dz, \quad (3.8)$$

where $C_\alpha = \frac{\sin \pi\alpha}{\pi}$. By Proposition 5 for all $r \in [0,T]$ a.e. the process $(r-s)_+^{-\alpha}(S(s,r)\phi)(\lambda,z)$ is Skorohod integrable. Using (3.8) and Fubini's theorem for the Skorohod integral (see Lemma 2) yields

$$\begin{aligned}
& \int_{[0,t]\times S} (S(s,t)\phi)(\lambda,x) M(ds,d\lambda) \\
= \ & C_\alpha \int_{[0,t]\times S} \Big( \int_{\mathbb{R}^d} \int_s^t (r-s)^{-\alpha} (S(s,r)\phi)(\lambda,z) \\
& \times p(r,t,z,x)(t-r)^{\alpha-1} dr dz \Big) M(ds,d\lambda) \\
= \ & C_\alpha \int_0^t \int_{\mathbb{R}^d} \left( \int_{[0,r]\times S} (r-s)^{-\alpha} (S(s,r)\phi)(\lambda,z) p(r,t,z,x) M(ds,d\lambda) \right) \\
& \times (t-r)^{\alpha-1} dr dz.
\end{aligned}$$

The term $p(r,t,z,x)$ can be factorized out of the Skorohod integral and we obtain

$$\begin{aligned}
& \int_{[0,t]\times S} (S(s,t)\phi)(x,\lambda) M(ds,d\lambda) \\
= \ & C_\alpha \int_0^t \int_{\mathbb{R}^d} Y_r(z) p(r,t,z,x)(t-r)^{\alpha-1} dr dz, \qquad (3.9)
\end{aligned}$$

where

$$Y_r(z) = \int_{[0,r]\times S} (r-s)^{-\alpha} (S(s,r)\phi)(\lambda,z) M(ds,d\lambda).$$

Applying condition (v)$_p$ and Hölder's inequality we obtain

$$\begin{aligned}
& E\left(\sup_{0\le t\le T}\left\|\int_{[0,t]\times S}(S(s,t)\phi)(\lambda,\cdot)M(ds,d\lambda)\right\|_2^p\right) \\
\le\ & \frac{1}{\pi^p}E\left(\sup_{0\le t\le T}\int_0^t (t-r)^{\alpha-1}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} Y_r(z)^2 p(r,t,z,x)dxdz\right)^{1/2}dr\right)^p \\
\le\ & C(p,\alpha,T)E\sup_{0\le t\le T}\int_0^T\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} Y_r(z)^2 p(r,t,z,x)dxdz\right)^{p/2}dr \\
\le\ & C(p,\alpha,T)E\int_0^T\int_{\mathbb{R}^d}\|Y_r\|_2^{p-2}Y_r(z)^2\sup_{t\in[r,T]}\left(\int_{\mathbb{R}^d}p(r,t,z,x)dx\right)^{p/2}dzdr \\
\le\ & C(p,\alpha,T)C_{p,1}\int_0^T E\|Y_r\|_2^p\,dr. \qquad (3.10)
\end{aligned}$$

Finally using the estimate (3.2) yields

$$\begin{aligned}
& E\left(\sup_{0\le t\le T}\left\|\int_{[0,t]\times S}(S(s,t)\phi)(\lambda,\cdot)M(ds,d\lambda)\right\|_2^p\right) \\
\le\ & C(p,\alpha,T)C_{p,1}C_1\int_0^T\int_0^r (r-s)^{-2\alpha+\frac{p}{4}(2\alpha-1)}E\|\phi(s)\|_2^p\,dsdr \\
\le\ & C_2\int_0^T E\|\phi(s)\|_2^p\,ds.
\end{aligned}$$

□

We are ready to state and prove our main result.

**Theorem 7** *Let $u_0$ be a function in $L^2(\mathbb{R}^d)$. Let $F(s,y,u)$ be a random function satisfying the above conditions (a), (b) and (c). Let $p(s,t,y,x)$ be a stochastic kernel in the sense of Definition 1 satisfying conditions (v)$_p$, (vi)$_p$ and (vii)$_p$ for some $p>2$. Then the $L^2(\mathbb{R}^d)$ -valued solution to Equation (2.4) has a continuous version and satisfies*

$$E\left(\sup_{0\le t\le T}\|u(t)\|_2^p\right)<\infty. \qquad (3.11)$$

*Proof:* We first have to show that the following two terms satisfy estimate (3.11):

$$A_1(t)=\int_{\mathbb{R}^d}p\left(0,t,y,x\right)u_0\left(y\right)dy,$$

and

$$A_2(t) = \int_0^t \int_S \left( \int_{\mathbb{R}^d} p(s,t,y,x) F(s,y,u(s,y)) a(\lambda, y) dy \right) M(ds, d\lambda).$$

The first estimate follows from condition $(\mathrm{vi})_p$, and the second from Proposition (6) and theorem 3. On the other hand, we have to show that $A_1$ and $A_2$ are continuous. First notice that if $Z_r(y)$ is a bounded random field with compact support $K$ in $y$, we have

$$\begin{aligned} & \left\| \int_{\mathbb{R}^d} Z_r(y) \left[ p(r, t+\delta, y, \cdot) - p(r,t,y,\cdot) \right] dy \right\|_2^2 \\ \leq \; & 2 \|Z\|_\infty^2 \int_{\mathbb{R}^d} \int_K \left| p(r, t+\delta, y, x) - p(r,t,y,x) \right| dy dx, \end{aligned}$$

which tends to zero almost surely for all $r$ by condition $(\mathrm{vi})_p$.

Taking into account condition $(\mathrm{v})_p$, in proving the continuity of $A_1$ we can assume that $u_0$ is a smooth function with compact support, and in this case the continuity follows from property $(\mathrm{vi})_p$ by letting $Z_0(y) = u_0(y)$ above. In order to show the continuity of $A_2$ we write, using (3.9)

$$\begin{aligned} & \|A_2(t+\delta) - A_2(t)\|_2 \\ \leq \; & \frac{1}{\pi} \left\| \int_0^t \int_{\mathbb{R}^d} Y_r(y) \left[ p(r,t+\delta,y,\cdot) - p(r,t,y,\cdot) \right] \; (t-r)^{\alpha-1} dr dy \right\|_2 \\ & + \frac{1}{\pi} \left\| \int_t^{t+\delta} \int_{\mathbb{R}^d} Y_r(y) p(r,t+\delta,y,\cdot) \; (t-r)^{\alpha-1} dr dy \right\|_2 \\ : \; & = \Gamma_1 + \Gamma_2. \end{aligned}$$

By the maximal inequality (3.10) we can assume that $Y_r(x)$ is a smooth elementary adapted process. In this case, letting $Z_r(y) = Y_r(y)$ above, it is clear from property $(\mathrm{vi})_p$ that the term

$$\int_{\mathbb{R}^d} Y_r(y) \left[ p(r,t+\delta,y,\cdot) - p(r,t,y,\cdot) \right] dy$$

converges to zero in $L^2(\mathbb{R}^d)$ as $\delta$ tends to zero for each fixed $r$ and $\omega$. Then the convergence of $\Gamma_1$ to zero follows by the dominated convergence theorem and property $(\mathrm{v})_p$. Finally property $(\mathrm{v})_p$ also implies that $\Gamma_2$ converges to zero as $\delta$ tends to zero. □

# 4 Stochastic semigroups generated by random partial differential operators

Suppose that $p(s,t,y,x)$ is a stochastic kernel in the sense of Definition 1. Set $T_{t,s}f(x) = \int_{\mathbb{R}^d} p(s,t,y,x)f(y)dy$, where $f$ is a bounded Borel function on $\mathbb{R}^d$. Then $T_{t,s}$ defines a stochastic semigroup of positive operators. Let us recall, following [10], the definition and some properties of this type of stochastic semigroups.

Let $\mathcal{C}_b$ be the Banach space of bounded continuous functions on $\mathbb{R}^d$.

**Definition 2** *A family* $\{T_{t,s}, 0 \le s < t \le T\}$ *of random linear operators on* $\mathcal{C}_b$ *is called a* stochastic semigroup *if it satisfies the following conditions:*

**(i)** $T_{t,s}f \ge 0$ *for any* $f \ge 0$, $T_{t,s}1 = 1$, *and* $T_{t,s}f_n \downarrow 0$ *for any sequence* $f_n \downarrow 0$ *in* $\mathcal{C}_b$,

**(ii)** $T_{t,u}T_{u,s} = T_{t,s}$ *for any* $s < u < t$,

**(iii)** $T_{t,s}f$ *is an* $\mathcal{F}_{s,t}$*-measurable random variable for each* $f \in \mathcal{C}_b$ *and each* $s < t$.

Then a stochastic kernel $p(s,t,y,x)$ in the sense of Definition 1 gives rise to a stochastic semigroup, provided $T_{t,s}f$ is continuous whenever $f$ is continuous and bounded. Conversely, if a stochastic semigroup is such that the probability measure induced by $T_{t,s}$ is absolutely continuous with respect to the Lebesgue measure, then we can find a version of its density and this will produce a stochastic kernel.

In this section we construct a stochastic semigroup whose infinitesimal generator is the random operator $\Delta + \dot{v} \cdot \nabla$, where $v(t,x)$ is a $d$-dimensional Gaussian field that is Brownian in time, and the differential $\dot{v}(t,x)\,dt := v(dt,x)$ is interpreted in the backward Itô sense. Assume that $v(t,x)$ can be represented as

$$v(t,x) = \int_0^t \int_S g(\lambda,x)\, M(ds,d\lambda),$$

where $g : S \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a measurable function, differentiable with respect to the variable $x$, and satisfying the following condition:

$$C_g := \sup_x \int_S \left( |g(\lambda,x)|^2 + |\nabla_x g(\lambda,x)|^2 \right) \mu(d\lambda) < \infty. \tag{4.1}$$

This condition means that both $v(1,x)$ and its derivative $\nabla_x v(1,x)$, which only needs to exists in the $L^2(\Omega)$ sense, have variances that are bounded in $x$. Set $G_{ij}(x,y) = \int_S g_i(\lambda,x)g_j(\lambda,y)\mu(d\lambda)$ and $G(x) = G(x,x)$. Let us introduce the following condition:

**(H1)** $\Sigma(x) = I - \frac{1}{2}G(x) \geq \epsilon I$ for each $x \in \mathbb{R}^d$, and for some $\epsilon > 0$.

This is know as the *coercivity* condition. Let $\sigma$ be a matrix such that $\sigma\sigma^* = \Sigma$.

Let $b(t)$ be a $d$-dimensional standard Brownian motion with variance $2t$ defined on another probability space $(\mathcal{W}, \mathcal{G}, Q)$. Consider the following *backward stochastic differential equation* on the product probability space $(\Omega \times \mathcal{W}, \mathcal{F} \times \mathcal{G}, P \times Q)$

$$\varphi_{t,s}(x) = x - \int_s^t v\left(dr, \varphi_{t,r}(x)\right) + \int_s^t \sigma\left(\varphi_{t,r}(x)\right) b(dr). \tag{4.2}$$

Applying Theorems 3.4.1 and 4.5.1 in [13] one can prove that Equation (4.2) has a solution $\varphi = \left\{\varphi_{t,s}(x), 0 \leq s \leq t \leq T, x \in \mathbb{R}^d\right\}$ which is a stochastic flow of homeomorphisms. This means that

$$\varphi_{r,s}\left(\varphi_{t,r}(x)\right) = \varphi_{t,s}(x),$$

for all $s < r < t, x$, a.s. Moreover $\varphi_{t,s}(x)$ is continuous in the three variables. Equation (4.2) can also be written as

$$\varphi_{t,s}(x) = x - \int_s^t \int_S g(\lambda, \varphi_{t,r}(x)) M(dr, d\lambda) + \int_s^t \sigma\left(\varphi_{t,r}(x)\right) b(dr). \tag{4.3}$$

For each $0 \leq s \leq t \leq T$ we introduce the random operator $T_{t,s}$ defined by

$$T_{t,s} f(x) = E_Q(f(\varphi_{t,s}(x)),$$

where $f$ belongs to $\mathcal{C}_b$. In the sequel we will denote by $E$ the mathematical expectation with respect to the probabilities $P$ and $P \times Q$, and by $E_Q$ the expectation with respect to $Q$.

**Proposition 8** *The operators $T_{t,s}$ form a stochastic semigroup in the sense of Definition 2.*

*Proof:* It is not difficult to show that $T_{t,s}f$ belongs to $\mathcal{C}_b$ for any $f$ in $\mathcal{C}_b$. Properties (i) and (iii) are obvious. Property (ii) follows from the flow property:

$$\begin{aligned} T_{t,s}f(x) &= E_Q(f(\varphi_{t,s}(x)) = E_Q(f(\varphi_{u,s}\left(\varphi_{t,u}(x)\right)) \\ &= E_Q\left(E_Q(f(\varphi_{u,s}(z))_{|z=\varphi_{t,u}(x)}\right) = T_{t,u}T_{u,s}f(x). \end{aligned}$$

□

Let $\mathcal{C}^2$ be the class of functions which are twice continuously differentiable, and let $\mathcal{C}_b^2$ be the space of functions in $\mathcal{C}^2$ and which are bounded and have bounded partial derivatives up to the second order. The stochastic semigroup satisfies the following *forward Kolmogorov equation* (although time flows backward in this equation, it must be called the forward equation because it flows in the same direction as that used to define the flow $\varphi$):

$$T_{t,s}f(x) = f(x) + \int_s^t T_{t,r}(v(dr,\cdot)\nabla f)(x) + \int_s^t T_{t,r}(\Delta f)(x)dr, \qquad (4.4)$$

for any function $f$ in the space $\mathcal{C}_b^2$. The second summand in the right-hand side of the above equation has to be understood as the expectation with respect to the probability $Q$ of a backward stochastic integral:

$$\int_s^t T_{t,r}(v(dr,\cdot)\nabla f)(x) = E_Q \int_s^t v(dr,\varphi_{t,r}(x))(\nabla f)(\varphi_{t,r}(x)). \qquad (4.5)$$

Equation (4.4) follows easily from Itô's formula:

$$\begin{aligned} f(\varphi_{t,s}(x)) &= f(x) + \int_s^t (\nabla f)(\varphi_{t,r}(x))v(dr,\varphi_{t,r}(x)) \\ &\quad + \int_s^t (\nabla f)(\varphi_{t,r}(x))\sigma\left(\varphi_{t,r}(x)\right)b(dr) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^d \int_s^t \frac{\partial^2 f}{\partial x_i \partial x_j}(\varphi_{t,r}(x))G_{ij}(\varphi_{t,r}(x))dr \\ &\quad + \sum_{i,j=1}^d \int_s^t \frac{\partial^2 f}{\partial x_i \partial x_j}(\varphi_{t,r}(x))(\sigma\sigma^*)_{ij}(\varphi_{t,r}(x))dr. \end{aligned}$$

As a consequence, integrating with respect to the probability $Q$ we obtain

$$\begin{aligned} T_{t,s}f(x) &= f(x) + E_Q \int_s^t v(dr,\varphi_{t,r}(x))(\nabla f)(\varphi_{t,r}(x)) \\ &\quad + E_Q \int_s^t (\Delta f)(\varphi_{t,r}(x))dr, \end{aligned}$$

which is equation (4.4).

The following conditions (stronger that (4.1)) imply that the random operators $T_{t,s}$ map $\mathcal{C}_b^2$ into $\mathcal{C}^2 \cap \mathcal{C}_b$:

$$\sup_x \int_S \left|\partial_\alpha g\left(\lambda, x\right)\right|^2 \mu\left(d\lambda\right) < \infty, \tag{4.6}$$

$$\int_S \left|\partial_\alpha g\left(\lambda, x\right) - \partial_\alpha g\left(\lambda, y\right)\right|^2 \mu\left(d\lambda\right) \leq C\left|x-y\right|^\delta, \tag{4.7}$$

for some $\delta > 0$ and for any multiindex $\alpha = (\alpha_1, ..., \alpha_d)$ such that $|\alpha| = \alpha_1 + ... + \alpha_d \leq 2$. Indeed, under these conditions the mappings $\varphi_{s,r}(x)$ are twice continuously differentiable in $x$, with almost-surely Hölder-continuous second derivative (see [12], chapters 3 and 4). Henceforth we will assume conditions (4.6) and (4.7). Under these conditions, the stochastic semigroup $T_{t,s}$ also satisfies the following *backward Kolmogorov equation:*

$$T_{t,s}f(x) = f(x) + \int_s^t \nabla T_{r,s}f(x)v(dr,x) + \int_s^t \Delta T_{r,s}f(x)dr, \tag{4.8}$$

for any $f \in \mathcal{C}_K^\infty(\mathbb{R}^d)$. Here the stochastic integral is an ordinary Itô integral.

*Proof of the backward Kolmogorov equation:* Let $s = t_0 < t_1 < ... < t_n = t$ be a subdivision of the time interval $[s,t]$. Using the semigroup property we can write

$$T_{t,s}f(x) - f(x) = \sum_{i=0}^{n-1} \left(T_{t_{i+1},t_i} - I\right) T_{t_i,s}f(x).$$

By using a standard localization argument, one can show that Equation (4.4) actually holds for any test function $g \in \mathcal{C}^2 \cap \mathcal{C}_b$ satisfying the conditions

$$\int_s^t E\left|\nabla g(\varphi_{t,r}(x))\right|^2 dr \quad < \quad \infty, \tag{4.9}$$

$$\int_s^t E\left|\Delta g(\varphi_{t,r}(x))\right| dr \quad < \quad \infty. \tag{4.10}$$

We wish to use Equation (4.4) on the inteval $[t_i,t_{i+1}]$ with the test function $g = T_{t_i,s}f$ . This is legitimate: as $T_{t_i,s}f$ is independent of $\mathcal{F}_{t_i,t_{i+1}}$, it may be considered as deterministic; moreover, $g$ is in $\mathcal{C}^2 \cap \mathcal{C}_b$ and satisfies bounds (4.9) and (4.10), which is proved by exploiting the following bound on the derivatives of $\varphi$: for $j = 1, 2$

$$\sup_{s \leq t \leq T} E\left|\nabla^j \varphi_{t,s}(x)\right|^2 < \infty. \tag{4.11}$$

This last fact is proved by using Itô's formula on the SDEs satisfied by $s \to \nabla^j \varphi_{t,s}(x)$. Therefore,

$$\begin{aligned} T_{t,s}f(x) - f(x) &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} T_{t_{i+1},r} \left( (v(dr,\cdot)\nabla T_{t_i,s} f \right)(x) \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} T_{t_{i+1},r} \left( \Delta T_{t_i,s} f \right)(x) dr \\ &= A_1 + A_2 + A_3 + A_4, \end{aligned}$$

where,

$$\begin{aligned} A_1 &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( T_{t_{i+1},r} - I \right) \left( (v(dr,\cdot)\nabla T_{t_i,s} f \right)(x), \\ A_2 &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( T_{t_{i+1},r} - I \right) \left( \Delta T_{t_i,s} f \right)(x) dr, \\ A_3 &= \sum_{i=0}^{n-1} (v(t_{i+1},x) - v(t_i,x)) \nabla T_{t_i,s} f(x), \\ A_4 &= \sum_{i=0}^{n-1} \Delta T_{t_i,s} f(x)(t_{i+1} - t_i). \end{aligned}$$

The terms $A_1$ and $A_2$ converge to zero in $L^2(\Omega)$ and the terms $A_3$ and $A_4$ converge respectively to the last two summands in the right-hand side of (4.8), as the mesh of the partition tends to zero. Let us first prove the convergence of $A_1$. Using the expression (4.5 ) we can write

$$\begin{aligned} A_1 &= \sum_{i=0}^{n-1} \left( E_Q \int_{t_i}^{t_{i+1}} v(dr, \varphi_{t_{i+1},r}(x)) \left( \nabla T_{t_i,s} f \right)(\varphi_{t_{i+1},r}(x)) \right. \\ &\quad \left. - \int_{t_i}^{t_{i+1}} v(dr,x) \left( \nabla T_{t_i,s} f \right)(x) \right) \\ &= A_{11} + A_{12}, \end{aligned}$$

where

$$A_{11} = \sum_{i=0}^{n-1} E_Q \int_{t_i}^{t_{i+1}} v(dr, \varphi_{t_{i+1},r}(x)) \left[ \left( \nabla T_{t_i,s} f \right)(\varphi_{t_{i+1},r}(x)) - \left( \nabla T_{t_i,s} f \right)(x) \right],$$

and

$$A_{12}=\sum_{i=0}^{n-1}E_Q\int_{t_i}^{t_{i+1}}\Big[v(dr,\varphi_{t_{i+1},r}(x))\left(\nabla T_{t_i,s}f\right)(x)-v(dr,x)\left(\nabla T_{t_i,s}f\right)(x)\Big].$$

Then we have

$$\begin{aligned}
E(A_{11}^2) &= E\sum_{i=0}^{n-1}\sum_{k,l=1}^{d}\int_{t_i}^{t_{i+1}}G_{kl}(\varphi_{t_{i+1},r}(x))\\
&\quad\times\Big[\left(\nabla_k T_{t_i,s}f\right)(\varphi_{t_{i+1},r}(x))-\left(\nabla_k T_{t_i,s}f\right)(x)\Big]\\
&\quad\times\Big[\left(\nabla_l T_{t_i,s}f\right)(\varphi_{t_{i+1},r}(x))-\left(\nabla_l T_{t_i,s}f\right)(x)\Big]dr\\
&\le \sup_x|G(x)|\sup_{t,s,x}E\left|\nabla^2T_{t,s}f\,(x)\right|^2\sup_{|t-r|<|\pi|}E\left|\varphi_{t,r}(x)-x\right|^2,
\end{aligned}$$

where $|\pi|=\sup_i(t_{i+1}-t_i)$. Note that, by (4.11),

$$\sup_{t,s,x}E\left|\nabla T_{t,s}f(x)\right|^2\le\|\nabla f\|_\infty^2\sup_{t,s,x}E\left|\nabla\varphi_{t,s}(x)\right|^2<\infty,$$

and, similarly $\sup_{t,s}E\left|\nabla^2T_{t,s}f\right|^2<\infty$. Hence, $E(A_{11}^2)$ converges to zero as $|\pi|\downarrow 0$. For the term $A_{12}$ we can write

$$\begin{aligned}
E(A_{12}^2) &= E\sum_{i=0}^{n-1}\sum_{k,l=1}^{d}\int_{t_i}^{t_{i+1}}\Big[G_{kl}(\varphi_{t_{i+1},r}(x))-2G_{kl}(\varphi_{t_{i+1},r}(x),x)+G_{kl}(x)\Big]\\
&\quad\times\left(\nabla_k T_{t_i,s}f\right)(x)\left(\nabla_l T_{t_i,s}f\right)(x)dr\\
&\le \sup_{t,s,x}E\left|\nabla T_{t,s}f(x)\right|^2\left\|\frac{\partial^2G}{\partial x\partial y}\right\|_\infty\sup_{|t-r|<|\pi|}E\left|\varphi_{t,r}(x)-x\right|^2,
\end{aligned}$$

and again this converges to zero as $|\pi|\downarrow 0$. The convergence of $A_2$ would follow by the same arguments. Let us show that $A_3$ converges to the second summand of the right-hand side of (4.8).

$$
\begin{aligned}
& E\left|A_3-\int_s^t \nabla T_{r,s} f(x) v(dr,x)\right|^2 \\
= \; & E\left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} v(dr,x)\left[\nabla T_{t_i,s} f(x)-\nabla T_{r,s} f(x)\right]\right|^2 \\
= \; & E\sum_{i=0}^{n-1}\sum_{k,l=1}^{d}\int_{t_i}^{t_{i+1}} G_{kl}(x)\left[\nabla_k T_{t_i,s} f(x)-\nabla_k T_{r,s} f(x)\right] \\
& \times\left[\nabla_l T_{t_i,s} f(x)-\nabla_l T_{r,s} f(x)\right] dr \\
\leq \; & T\|G\|_\infty \sup_{|r-u|\leq|\pi|,r,u\geq s} E\left|\nabla T_{u,s} f(x)-\nabla T_{r,s} f(x)\right|^2 \\
\leq \; & T\|G\|_\infty \sup_{|r-u|\leq|\pi|,r,u\geq s} E\left|\nabla\left[f(\varphi_{u,s}(x))\right]-\nabla\left[f(\varphi_{r,s}(x))\right]\right|^2 .
\end{aligned}
$$

To show that this converges to zero as $|\pi|\downarrow 0$, we write

$$
\begin{aligned}
& E\left|\nabla\left[f(\varphi_{u,s}(x))\right]-\nabla\left[f(\varphi_{r,s}(x))\right]\right|^2 \\
\leq \; & 2E\left|\left(\nabla f\left(\varphi_{u,s}(x)\right)-\nabla f\left(\varphi_{r,s}(x)\right)\right)\nabla\varphi_{u,s}(x)\right|^2 \\
& +2E\left|\left(\nabla\varphi_{u,s}(x)-\nabla\varphi_{r,s}(x)\right)\nabla f\left(\varphi_{r,s}(x)\right)\right|^2 \\
\leq \; & 2\left\|\nabla^2 f\right\|_\infty^2\left[E\left|\nabla\varphi_{u,s}(x)\right|^4\right]^{\frac{1}{2}}\left[E\left|\nabla\varphi_{u,s}(x)-\nabla\varphi_{r,s}(x)\right|^4\right]^{\frac{1}{2}} \\
& +2\left\|\nabla f\right\|_\infty^2 E\left|\nabla\varphi_{u,s}(x)-\nabla\varphi_{r,s}(x)\right|^2 .
\end{aligned}
$$

Therefore, we only need to show that $E\left|\nabla\varphi_{u,s}(x)-\nabla\varphi_{r,s}(x)\right|^4$ tends to zero when $u\downarrow r$, uniformly in $s$, and this follows easily using Itô's formula. In a similar way one can show that $A_4$ converges in $L^2$ to the last summand in the right-hand side of (4.8). □

The next proposition shows that the marginal probability density $p\left(s,t,y,x\right)=Q\left[\varphi_{t,s}\left(x\right)\in dy\right]/dy$ exists and satisfies the conditions given in Definition 1. This result has been proved by Kunita in [13] (see also [10], Theorem 2.4) under the condition that the random field $v(t,x)$ can be represented as a finite linear combination of independent ordinary Brownian motions multiplied by $\mathcal{C}^\infty$-vector fields on $\mathbb{R}^d$. Our proof, which only requires condition (4.1), is based on the criterion of absolute continuity proved by Bouleau and Hirsch [4], and uses the techniques of the partial Malliavin calculus (see [20]).

**Proposition 9** *Suppose that $g$ satisfies condition (4.1) and the coercivity condition (H1). Let $\varphi_{t,s}(x)$ be the stochastic flow solution of the backward stochastic differential equation (4.3). Then, there is a version of the the marginal density $p(s,t,y,x) = Q\left[\varphi_{t,s}(x) \in dy\right]/dy$ which satisfies conditions (i) to (iv) of Definition 1.*

*Proof:* One can show that the random variables $\varphi^j_{t,s}(x)$, $0 \le s < t \le T$, $x \in \mathbb{R}^d$, $j = 1, \ldots, d$, belong to the Sobolev space $\mathbb{D}^{1,2}$, with respect to the product measure $P \times Q$. Let us denote by $D^k_\theta$ the derivative operator with respect to the Brownian motion $b$. We have the following linear equation for the derivative of $\varphi_{t,s}(x)$, for $\theta \in [s,t]$, $1 \le i,k \le d$,

$$\begin{aligned} D^k_\theta \varphi^i_{t,s}(x) &= \sigma_{ik}(\varphi_{t,\theta}(x)) - \int_s^\theta \int_S \nabla g_i(\lambda, \varphi_{t,r}(x)) D^k_\theta \varphi_{t,r}(x) M(dr, d\lambda) \\ &\quad + \sum_{l=1}^d \int_s^\theta \nabla \sigma_{il}(\lambda, \varphi_{t,r}(x)) D^k_\theta \varphi_{t,r}(x) b_l(dr). \qquad (4.12) \end{aligned}$$

Let us denote by $\gamma_{\varphi_{t,s}(x)}$ the Malliavin matrix of the random vector $\varphi_{t,s}(x)$, that is,

$$\gamma^{ij}_{\varphi_{t,s}(x)} = \sum_{k=1}^d \int_s^t D^k_\theta \varphi^i_{t,s}(x) D^k_\theta \varphi^j_{t,s}(x) d\theta \ .$$

By means of the techniques of the partial Malliavin calculus it follows that (see Theorem 4.2 in [20]) almost surely the marginal law of $\varphi_{t,s}(x)$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$ for all $0 \le s < t \le T$, and $x \in \mathbb{R}^d$, if almost surely we have

$$\det \gamma_{\varphi_{t,s}(x)} > 0, \qquad \forall \quad 0 \le s < t \le T, x \in \mathbb{R}^d. \qquad (4.13)$$

Condition (4.13) is an immediate consequence of (4.12). In fact, notice first that we can choose a version of the derivative $D_\theta \varphi_{t,s}(x)$ which is continuous in $s \le \theta \le t$, and in $x$. Then, let $v$ be a unit vector in $\mathbb{R}^d$ such that

$$v^t \gamma_{\varphi_{t,s}(x)} v = \sum_{k=1}^d \int_s^t (\sum_{i=1}^d D^k_\theta \varphi^i_{t,s}(x) v_i)^2 d\theta = 0.$$

This implies $\sum_{i=1}^d D^k_\theta \varphi^i_{t,s}(x) v_i = 0$ for each $k$ and $\theta$, and choosing $\theta = s$ we get $\sum_{i=1}^d \sigma_{ik}(\varphi_{t,s}(x)) v_i = 0$ which implies $v = 0$. Hence, (4.13) holds.

Properties (i) to (iv) of Definition 1 hold trivially due to the fact that $T_{t,s}$ is a stochastic semigroup. □

The forward and backward Kolmogorov equations can be written as follows in terms of the stochastic kernel $p(s,t,y,x)$:

$$\begin{aligned}\int_{\mathbb{R}^d} p(s,t,y,x)f(y)dy &= f(x) + \int_s^t \int_{\mathbb{R}^d} p(r,t,y,x)v(dr,y)\nabla f(y)dy \\ &\quad + \int_s^t \int_{\mathbb{R}^d} p(r,t,y,x)\Delta f(y)dydr, \qquad (4.14)\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^d} p(s,t,y,x)f(y)dy &= f(x) + \int_s^t v(dr,x)\int_{\mathbb{R}^d} \nabla_x p(s,r,y,x)f(y)dy \\ &\quad + \int_s^t \int_{\mathbb{R}^d} \Delta_x p(s,r,y,x)f(y)dydr. \qquad (4.15)\end{aligned}$$

Equation (4.14) is the weak formulation of the following stochastic partial differential equation where the stochastic integral is interpreted as an Itô backward stochastic integral:

$$p(ds,y) = \Delta_y p(s,y)ds + p(s,y)v(ds,y)\cdot\nabla_y,$$

(see, for example, [12, Chapter 6]). Formally, as announced in the introduction, we can say that $p(s,t,y,x)$ is the heat kernel of the operator $\Delta + \dot{v}\cdot\nabla$. On the other hand, Equation (4.15) leads to the following evolution equation

$$p(s,t,y,x) = q(s,t,y,x) + \int_s^t \int_{\mathbb{R}^d} v(dr,z)\nabla_z\left[p(s,r,y,z)\right]q(r,t,z,x)dz, \qquad (4.16)$$

where $q(s,t,y,x)$ denotes the heat kernel, that is,

$$q(s,t,y,x) = (4\pi(t-s))^{-d/2}\exp\left(-\frac{|x-y|^2}{4(t-s)}\right).$$

# 5 Estimates of a stochastic heat kernel

Let $\varphi_{t,s}(x)$ be the stochastic flow solution of Equation (4.3), and denote by $p(s,t,y,x)$ its associated stochastic kernel. In this section we will show that, under suitable assumptions, the stochastic kernel $p(s,t,y,x)$ also satisfies conditions $(\mathrm{v})_p$, $(\mathrm{vi})_p$ and $(\mathrm{vii})_p$ for all $p\geq 2$. Condition $(\mathrm{v})_p$ will be a consequence of the backward Kolmogorov evolution equation (4.16). On the other hand, we will make use of the techniques of the Malliavin calculus in order to provide a priori estimates for the integral of the stochastic kernel $p(s,t,y,x)$ in $x$, and to show conditions $(\mathrm{vi})_p$ and $(\mathrm{vii})_p$.

**Proposition 10** *We assume the coercivity condition (H1). Suppose that $g$ is $d+1$ times continuously differentiable in the variable $x$, and the following integrability condition holds*

$$\sup_x \int_S \left| \frac{\partial^{|\alpha|} g(\lambda, x)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^d)^{\alpha_d}} \right|^2 \mu(d\lambda) < \infty, \tag{5.1}$$

*for any multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| \leq d+1$, where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Then the stochastic kernel $p(s,t,y,x)$ satisfies*

$$\sup_{t \in [s,T], y \in \mathbb{R}^d} E\left( \int_{\mathbb{R}^d} p(s,t,y,x) dx \right)^p < \infty,$$

*for all $p \geq 1$.*

*Proof:* We will denote by $\delta$ the divergence operator with respect to the Brownian motion $b$. Applying the integration-by-parts formula of Malliavin calculus with respect to the Brownian motion $b$ we deduce the following expression for the stochastic kernel $p(s,t,y,x)$

$$p(s,t,y,x) = E_Q\left( \mathbf{1}_{\{\varphi_{t,s}(x) > y\}} H_{t,s}(x) \right), \tag{5.2}$$

where $\varphi_{t,s}(x)y$ means $\varphi^i_{t,s}(x)y^i$ for each coordinate $i = 1, \ldots, d$, and $H_{t,s}(x)$ is a random variable given by

$$H_{t,s}(x) = \delta\Bigg( \left(\gamma^{-1}_{\varphi_{t,s}(x)} D\varphi_{t,s}(x)\right)^d \delta\Bigg( \left(\gamma^{-1}_{\varphi_{t,s}(x)} D\varphi_{t,s}(x)\right)^{d-1} \delta\Bigg( \cdots \delta\left( \left(\gamma^{-1}_{\varphi_{t,s}(x)} D\varphi_{t,s}(x)\right)^1 \right) \Bigg) \cdots \Bigg).$$

Equation (5.2) follows from the duality relationship between the derivative and divergence operators and from the fact that

$$\frac{\partial^d}{\partial z^1 \cdots \partial z^d} \mathbf{1}_{\{z > y\}} = \delta_y(z).$$

Let $\sigma$ be a subset of the set of indexes $\{1, \ldots, d\}$. Clearly

$$\frac{\partial^d}{\partial z^1 \cdots \partial z^d} \mathbf{1}_{\{z^i < y^i, i \in \sigma, z^i > y^i, i \notin \sigma\}} = (-1)^{|\sigma|} \delta_y(z),$$

where $|\sigma|$ is the cardinality of $\sigma$. Hence, for every $\sigma$ we can write the following alternative formula for the kernel $p(s,t,y,x)$

$$p(s,t,y,x) = (-1)^{|\sigma|} E_Q \left( \mathbf{1}_{\{\varphi^i_{t,s}(x)<y^i, i\in\sigma, \varphi^i_{t,s}(x)>y^i, i\notin\sigma\}} H_{t,s}(x) \right). \tag{5.3}$$

Consider the following decomposition

$$\int_{\mathbb{R}^d} p(s,t,y,x)dx = \sum_{\sigma\subset\{1,\dots,d\}} \int_{Q_\sigma} p(s,t,y,x)dx,$$

where $Q_\sigma$ denotes the region

$$Q_\sigma = \{x\in\mathbb{R}^d : x^i < y^i, i\in\sigma, x^i > y^i, i\notin\sigma\}.$$

Set

$$B_{t,s} = -\int_s^t \int_S g(\lambda, \varphi_{t,r}(x)) M(dr, d\lambda) + \int_s^t \sigma(\varphi_{t,r}(x)) b(dr). \tag{5.4}$$

Then, $\varphi_{t,s}(x) = x + B_{t,s}$. If $x \in Q_\sigma$ we use the expression for $p(s,t,y,x)$ given in (5.3). In this way we obtain for $x\in Q_\sigma$

$$\begin{aligned} p(s,t,y,x) &\le E_Q \left( \mathbf{1}_{\{\varphi^i_{t,s}(x)>y^i, i\in\sigma, \varphi^i_{t,s}(x)<y^i, i\notin\sigma\}} |H_{t,s}(x)| \right) \\ &\le E_Q \left( \mathbf{1}_{\{x^i-y^i>-B^i_{t,s}, i\in\sigma, x^i-y^i<-B^i_{t,s}, i\notin\sigma\}} |H_{t,s}(x)| \right) \\ &\le E_Q \left( \mathbf{1}_{\{e^{|x^i-y^i|}<e^{B^i_{t,s}}, i\in\sigma, e^{|x^i-y^i|}<e^{-B^i_{t,s}}, i\notin\sigma\}} |H_{t,s}(x)| \right) \\ &\le e^{-\sum_{i=1}^d \frac{|x^i-y^i|}{\sqrt{t-s}}} E_Q \left( \exp\left( \frac{1}{\sqrt{t-s}} \left( \sum_{i\in\sigma} B^i_{t,s} - \sum_{i\notin\sigma} B^i_{t,s} \right)\right) |H_{t,s}(x)| \right). \end{aligned}$$

Integrating the above expression over $Q_\sigma$, summing over $\sigma$, taking the mathematical expectation of the power $p$ and using Cauchy-Schwartz inequality yields

$$
\begin{aligned}
& E\left(\int_{\mathbb{R}^d} p(s,t,y,x)dx\right)^p \\
\leq\; & 2^{d(p-1)}(t-s)^{\frac{d}{2}(p-1)} \\
& \times \sum_{\sigma\subset\{1,\dots,d\}} \left(\int_{Q_\sigma} e^{-\sum_{i=1}^d \frac{|x^i-y^i|}{\sqrt{t-s}}} dx\right) \\
& \times \sup_{x\in Q_\sigma} E\left(\exp\left(\frac{p}{\sqrt{t-s}}\left(-\sum_{i\in\sigma} B^i_{t,s} + \sum_{i\notin\sigma} B^i_{t,s}\right)\right)|H_{t,s}(x)|^p\right) \\
\leq\; & 2^{d(p-1)}(t-s)^{\frac{dp}{2}} \\
& \times \sum_{\sigma\subset\{1,\dots,d\}} \sup_{x\in Q_\sigma} \left(E\exp\left(\frac{2p}{\sqrt{t-s}}\left(-\sum_{i\in\sigma} B^i_{t,s} + \sum_{i\notin\sigma} B^i_{t,s}\right)\right)\right)^{\frac{1}{2}} \\
& \times \|H_{t,s}(x)\|_{2p}^p.
\end{aligned}
$$

Note that if $M_t = -\sum_{i\in\sigma} B^i_{t,s} + \sum_{i\notin\sigma} B^i_{t,s}$, then the quadratic variation of this martingale is

$$
\begin{aligned}
\langle M\rangle_t &= \int_s^t \int_S \left|\sum_{i\in\sigma} g_i(\lambda,\varphi_{t,r}(x)) - \sum_{i\notin\sigma} g_i(\lambda,\varphi_{t,r}(x))\right|^2 \mu(d\lambda)dr \\
&\quad +2\sum_{k=1}^d \int_s^t \left|\sum_{i\in\sigma} \sigma_{ik}(\varphi_{t,r}(x)) - \sum_{i\notin\sigma} \sigma_{ik}(\varphi_{t,r}(x))\right|^2 dr \\
&= 2d(t-s),
\end{aligned}
$$

because $\sigma\sigma^* = \Sigma = I - \frac{1}{2}G$. As a consequence, we obtain

$$
E\exp\left(\frac{2p}{\sqrt{t-s}}\left(-\sum_{i\in\sigma} B^i_{t,s} + \sum_{i\notin\sigma} B^i_{t,s}\right)\right) \leq \exp(4p^2 d),
$$

and

$$
E\left(\int_{\mathbb{R}^d} p(s,t,y,x)dx\right)^p \leq e^{4p^2 d} 2^{pd}(t-s)^{\frac{dp}{2}} \sup_{x\in\mathbb{R}^d} \|H_{t,s}(x)\|_{2p}^p.
$$

We claim that for all $p \geq 2$ we have

$$
\sup_{s\leq t, x\in\mathbb{R}^d} \|H_{t,s}(x)\|_p \leq C(t-s)^{-\frac{d}{2}}, \tag{5.5}
$$

for some constant $C > 0$. This inequality would complete the proof. In order to show (5.5) we proceed as follows. Set for $j = 1, \ldots, d$

$$H_j = \delta\left(\left(\gamma^{-1}_{\varphi_{t,s}(x)} D\varphi_{t,s}(x)\right)^j H_{j-1}\right),$$

and $H_0 = 1$. Then $H_d = H_{t,s}(x)$. With this notation we can write for $j = 1, \ldots, d$

$$\begin{aligned} H_j &= \left(\gamma^{-1}_{\varphi_{t,s}(x)} \delta D\varphi_{t,s}(x)\right)^j H_{j-1} - \left(\left\langle D\left(\gamma^{-1}_{\varphi_{t,s}(x)} H_{j-1}\right), D\varphi_{t,s}(x)\right\rangle\right)^j \\ &= A_j H_{j-1} + \langle DH_{j-1}, B_j\rangle, \end{aligned}$$

where for $j = 1, \ldots, d$,

$$A_j = \left(\gamma^{-1}_{\varphi_{t,s}(x)} \delta D\varphi_{t,s}(x)\right)^j - \left(\left\langle D\gamma^{-1}_{\varphi_{t,s}(x)}, D\varphi_{t,s}(x)\right\rangle\right)^j, \tag{5.6}$$

and

$$B_j = -\left(\gamma^{-1}_{\varphi_{t,s}(x)} D\varphi_{t,s}(x)\right)^j. \tag{5.7}$$

By Hölder's inequality for the Sobolev norms in the Wiener space, for each integer $k \geq 0$ and each real $p1$ we can write for $j = 1, \ldots, d$

$$\|H_j\|_{k,p} \leq C_{k,p}\left(\|A_j\|_{k,2p}\|H_{j-1}\|_{k,2p} + \|B_j\|_{k,2p}\|H_{j-1}\|_{k+1,2p}\right).$$

As a consequence in order to prove the inequality (5.5) it suffices to show that for $j = 1, \ldots, d$

$$\|A_j\|_{d-j,2^{d-j+1}p} \leq c_1(t-s)^{-\frac{1}{2}}, \tag{5.8}$$

and for $j = 2, \ldots, d$

$$\|B_j\|_{d-j,2^{d-j+1}p} \leq c_1(t-s)^{-\frac{1}{2}}. \tag{5.9}$$

From formulas (5.6) and (5.7) and using Hölder's inequality it follows that, for each integer $k \geq 0$, each real $p1$ and each index $j = 1, \ldots, d$, we have

$$\begin{aligned} \|A_j\|_{k,p} &\leq C_{k,p}\|\gamma^{-1}_{\varphi_{t,s}(x)}\|_{k+1,2p}\|D\varphi_{t,s}(x)\|_{k+1,2p} \\ \|B_j\|_{k,p} &\leq C_{k,p}\|\gamma^{-1}_{\varphi_{t,s}(x)}\|_{k,2p}\|D\varphi_{t,s}(x)\|_{k,2p}. \end{aligned}$$

Hence, in order to show (5.8) and (5.9) it suffices to check that for each $p1$ and $k = 1, ..., d+1$,

$$\|\gamma^{-1}_{\varphi_{t,s}(x)}\|_{k,p} \leq c_{k,p}(t-s)^{-1} \tag{5.10}$$

and

$$\|D\varphi_{t,s}(x)\|_{k,p} \le c_{k,p}(t-s)^{\frac{1}{2}}. \tag{5.11}$$

Property (5.11) follows from the estimate

$$\sup_{x,0\le s<t\le T,\theta_1,\dots\theta_k\in[s,t],1\le j_1,\dots,j_k\le d} E\left(|D^{j_1}_{\theta_1}\cdots D^{j_k}_{\theta_k}\varphi_{t,s}(x)|^p\right) < \infty. \tag{5.12}$$

This estimate can be easily checked using Burkholder's inequality and condition (5.1). In fact, condition (5.1) together with the coercivity hypothesis (H1) imply that $G$ and $\sigma$ have $d+1$ bounded derivatives. In order to show the estimate (5.10), taking into account (5.11) and the formula for the derivative of the inverse of a matrix, it suffices to show that for any $p \ge 2$ we have

$$\|\gamma^{-1}_{\varphi_{t,s}(x)}\|_p \le c(t-s)^{-1}, \tag{5.13}$$

and this follows from

$$\|\left(\det\gamma_{\varphi_{t,s}(x)}\right)^{-1}\|_p \le c(t-s)^{-d}. \tag{5.14}$$

The proof of the inequality (5.14) requires some computations. Set

$$\Gamma = \inf_{|v|=1} v^t\gamma_{\varphi_{t,s}(x)}v.$$

We can write

$$\begin{aligned} E\left(\left(\det\gamma_{\varphi_{t,s}(x)}\right)^{-p}\right) &= \int_0^\infty py^{p-1}P\left(\det\gamma_{\varphi_{t,s}(x)} < \frac{1}{y}\right)dy \\ &\le \int_0^\infty py^{p-1}P\left(\Gamma^d < \frac{1}{y}\right)dy. \end{aligned}$$

We have for any unit vector $v$

$$\begin{aligned} v^t\gamma_{\varphi_{t,s}(x)}v &= \sum_{i=1}^d\int_s^t\left|\sum_{j=1}^d v^jD^i_\theta\varphi^j_{t,s}(x)\right|^2 d\theta \\ &\ge \sum_{i=1}^d\int_s^{(t-s)h+s}\left|\sum_{j=1}^d v^jD^i_\theta\varphi^j_{t,s}(x)\right|^2 d\theta \\ &\ge \frac{\epsilon}{2}(t-s)h - \sum_{i=1}^d\int_s^{(t-s)h+s}\left|\sum_{j=1}^d v^jG^{i,j}_{\theta,s}\right|^2 d\theta \\ &\ge \frac{\epsilon}{2}(t-s)h - \int_s^{(t-s)h+s}|G_{\theta,s}|^2d\theta, \end{aligned}$$

where $h \in [0,1]$ and

$$
\begin{aligned}
G^{i,j}_{\theta,s} &= -\int_s^\theta \int_S \nabla g_j(\lambda, \varphi_{t,r}(x)) D^i_\theta \varphi_{t,r}(x) M(dr, d\lambda) \\
&\quad + \int_s^\theta \nabla \sigma_{jk}(\lambda, \varphi_{t,r}(x)) D^i_\theta \varphi_{t,r}(x) b(dr).
\end{aligned}
$$

We decompose the previous integral as follows

$$
\begin{aligned}
&\int_0^\infty p y^{p-1} P\left(\Gamma^d < \frac{1}{y}\right) dy \\
&= \int_0^{\left(\frac{4}{\varepsilon(t-s)}\right)^d} p y^{p-1} P\left(\Gamma^d < \frac{1}{y}\right) dy \\
&\quad + \int_{\left(\frac{4}{\varepsilon(t-s)}\right)^d}^\infty p y^{p-1} P\left(\frac{\epsilon}{2}(t-s)h - \int_s^{(t-s)h+s} |G_{\theta,s}|^2 d\theta < \frac{1}{y^{1/d}}\right) dy \\
&\le \left(\frac{4}{\varepsilon(t-s)}\right)^{pd} + \Phi.
\end{aligned}
$$

In order to estimate the integral $\Phi$ we will take a value of $h$ depending on $y$, that is, $h = \frac{4}{\epsilon(t-s)} y^{-1/d}$. Notice that for $y \ge \left(\frac{4}{\varepsilon(t-s)}\right)^d$ we have $h \le 1$. With this value of $h$ we obtain for $q > pd$

$$
\begin{aligned}
\Phi &\le \int_{\left(\frac{4}{\varepsilon(t-s)}\right)^d}^\infty p y^{p-1} P\left(\int_s^{(t-s)h+s} |G_{\theta,s}|^2 d\theta \frac{1}{y^{1/d}}\right) dy \\
&\le \int_{\left(\frac{4}{\varepsilon(t-s)}\right)^d}^\infty p y^{p-1+\frac{q}{d}} E\left(\left|\int_s^{(t-s)h+s} |G_{\theta,s}|^2 d\theta\right|^q\right) dy \\
&\le C \int_{\left(\frac{4}{\varepsilon(t-s)}\right)^d}^\infty p y^{p-1-\frac{q}{d}} dy \\
&= C'(t-s)^{-d(p-\frac{q}{d})}.
\end{aligned}
$$

This completes the proof of the proposition. □

In a similar way we can check conditions $(\mathrm{vi})_p$ and $(\mathrm{vii})_p$.

**Proposition 11** *Suppose that $g$ is $d+3$ times continuously differentiable in the variable $x$, and the following integrability condition holds*

$$
\sup_x \int_S \left| \frac{\partial^{|\alpha|} g(\lambda, x)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^d)^{\alpha_d}} \right|^2 \mu(d\lambda) < \infty, \tag{5.15}
$$

*for any multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| \le d+3$, where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Then the stochastic kernel $p(s,t,y,x)$ satisfies condition $(vi)_p$ and*

$$\sup_y E\left|\int_{S\times\mathbb{R}^d\times\mathbb{R}^d} |D_{s,\lambda}p(s,t,x,z)D_{s,\lambda}p(s,t,y,z)|\mu(d\lambda)dzdx\right|^p \le C_{2p,2}(t-s)^{-p}, \tag{5.16}$$

*for all $p \ge 1$.*

*Proof:* We recall that $D_{s,\lambda}$ denotes the derivative with respect to the Gaussian measure $M$. From the expression (5.2) for the density $p(s,t,y,x)$ it follows easily that $p(s,t,y,x)$ belongs to the Sobolev space $\mathbb{D}^{1,2}$, and its derivative for $\theta \in [s,t]$ is given by

$$\begin{aligned} D_{\theta,\lambda}p(s,t,y,x) &= E_Q\left(\mathbf{1}_{\{\varphi_{t,s}(x)y\}}D_{\theta,\lambda}H_{t,s}(x)\right) \\ &\quad + E_Q\left(\mathbf{1}_{\{\varphi_{t,s}(x)y\}}\Psi_{t,s}(x)\right), \end{aligned} \tag{5.17}$$

where $\Psi_{t,s}(x)$ is the random variable

$$\Psi_{t,s}(x) = \sum_{j=1}^d \delta\left(\left(\left(\gamma_{\varphi_{t,s}(x)}\right)^{-1} D\varphi_{t,s}(x)\right)^j D_{\theta,\lambda}\varphi^j_{t,s}(x)H_{t,s}(x)\right).$$

From Equation (5.17) it is not difficult to show that property $(\mathrm{vi})_p$ holds.

On the other hand, applying the operator $D_{s,\lambda}$ to Equation(4.14) yields

$$\begin{aligned} \int_{\mathbb{R}^d} D_{s,\lambda}p(s-\varepsilon,t,y,x)f(y)dy &= \int_{\mathbb{R}^d} p(s,t,y,x)\nabla f(y)g(\lambda,y)dy \\ &\quad + \int_{s-\varepsilon}^s \int_{\mathbb{R}^d} D_{s,\lambda}p(r,t,y,x)v(dr,y)\nabla f(y)dy \\ &\quad + \int_{s-\varepsilon}^s \int_{\mathbb{R}^d} D_{s,\lambda}p(r,t,y,x)\Delta f(y)dydr, \end{aligned}$$

and letting $\varepsilon$ tend to zero we obtain

$$D_{s,\lambda}p(s,t,y,x) = -\mathrm{div}\left[p(s,t,y,x)g(\lambda,y)\right].$$

Hence, in order to show (5.16) we have to estimate the following quantities:

$$A_1 = E\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla_x p(s,t,x,z)|\,|\nabla_y p(s,t,y,z)|\,dzdx\right|^p,$$

$$A_2 = E\left|\int_{\mathbb{R}^d} |\nabla_y p(s,t,y,z)|\, dz\right|^p,$$

$$A_3 = E\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla_y p(s,t,x,z)|\, p(s,t,y,z) dz dx\right|^p,$$

The most difficult term is the first one. We will give some ideas about the estimation of this term and for the others one can use a similar procedure. Using the integration-by-parts formula of Malliavin calculus yields

$$\frac{\partial}{\partial x^j} p(s,t,x,z) = E_Q\left(\mathbf{1}_{\{\varphi_{t,s}(z)x\}} H^j_{t,s}(z)\right),$$

where $H^j_{t,s}(z)$ is the random variable

$$H^j_{t,s}(z) = \delta\left(\left(\gamma^{-1}_{\varphi_{t,s}(z)} D\varphi_{t,s}(z)\right)^j H_{t,s}(z)\right).$$

Given two subsets $\sigma, \tau$ of $\{1,...,d\}$ define

$$\begin{aligned} Q_{\sigma,\tau} &= \{(x,z)\in\mathbb{R}^d\times\mathbb{R}^d : x^i < z^i, i\in\sigma, x^i > z^i, i\notin\sigma; \\ z^j &< y^j, j\in\tau, z^j > y^j, j\notin\tau\}. \end{aligned}$$

If $(x,z)\in Q_{\sigma,\tau}$ we will write

$$\begin{aligned} &\left|\frac{\partial}{\partial x^j} p(s,t,x,z)\frac{\partial}{\partial y^k} p(s,t,y,z)\right| \\ \leq\ & E_Q\left(\mathbf{1}_{\{\varphi^i_{t,s}(z)<x^i, i\in\sigma, \varphi^i_{t,s}(z)>x^i, i\notin\sigma\}} |H^j_{t,s}(z)|\right) \\ & \times E_Q\left(\mathbf{1}_{\{\varphi^i_{t,s}(z)>y^i, i\in\tau, \varphi^i_{t,s}(z)<y^i, i\notin\tau\}} |H^k_{t,s}(z)|\right) \\ \leq\ & E_Q\left(\mathbf{1}_{\{z^i-x^i<-B^i_{t,s}(z), i\in\sigma, x^i-z^i<B^i_{t,s}(z), i\notin\sigma\}} |H^j_{t,s}(z)|\right) \\ & \times E_Q\left(\mathbf{1}_{\{y^i-z^i<B^i_{t,s}, i\in\tau, z^i-y^i<-B^i_{t,s}, i\notin\tau\}} |H^k_{t,s}(z)|\right) \\ \leq\ & e^{-\sum_{i=1}^d \frac{|z^i-x^i|+|y^i-z^i|}{\sqrt{t-s}}} E_Q\left(e^{M_{t,s}(z)} |H^j_{t,s}(z)|\right) E_Q\left(e^{N_{t,s}(z)} |H^k_{t,s}(z)|\right), \end{aligned}$$

where $B_{t,s}(z)$ has been defined in (5.4),

$$M_{t,s}(z) = \frac{1}{\sqrt{t-s}}\left(-\sum_{i\in\sigma} B^i_{t,s}(z) + \sum_{i\notin\sigma} B^i_{t,s}(z)\right),$$

and

$$N_{t,s}(z) = \frac{1}{\sqrt{t-s}} \left( \sum_{i \in \tau} B^i_{t,s}(z) - \sum_{i \notin \tau} B^i_{t,s}(z) \right).$$

Integrating on $Q_{\sigma,\tau}$, summing with respect to the sets $\sigma, \tau$, taking the expectation of the power $p$, and using Hölder's inequality we obtain, as in the proof of Proposition 10,

$$\begin{aligned} & E\left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \frac{\partial}{\partial x^j} p(s,t,x,z) \frac{\partial}{\partial y^k} p(s,t,y,z) \right| dz dx \right)^p \\ \leq \quad & C(t-s)^{dp} \sup_z \left[ E \left| H^j_{t,s}(z) \right|^{4p} E \left| H^k_{t,s}(z) \right|^{4p} \right]^{1/4}. \end{aligned}$$

It holds that $\left\| H^j_{t,s}(z) \right\|_{4p} \leq C(t-s)^{-d/2-1/2}$, and, as a consequence we obtain $A_1 \leq C(t-s)^{-p}$. □

Let us show condition $(\mathrm{v})_p$, using the backward Kolmogorov equation.

**Proposition 12** *Suppose that $g$ satisfies condition (5.1) and the coercivity condition (H1). Then for all $p \geq 2$*

$$\sup_{s,y} E \left( \sup_{t \in [s,T]} \int_{\mathbb{R}^d} p(s,t,y,x) dx \right)^p < \infty.$$

*Proof:* Integrating with respect to $x$ all terms in Equation (4.16) and using the integration by parts formula yields

$$\begin{aligned} \int_{\mathbb{R}^d} p(s,t,y,x) dx &= 1 + \int_s^t \int_{\mathbb{R}^d} v(dr,z) \nabla_z p(s,r,y,z) dz \\ &= 1 - \int_s^t \int_{\mathbb{R}^d} \mathrm{div} v(dr,z) p(s,r,y,z) dz \\ &= 1 - \int_s^t \int_S \left( \int_{\mathbb{R}^d} \mathrm{div} g(\lambda,z) p(s,r,y,z) dz \right) M(dr,d\lambda). \end{aligned}$$

Applying Doob's maximal inequality Burkholder's inequality we obtain for any $t_1 > s$

$$
\begin{aligned}
& E\left(\sup_{t\in[s,t_1]}\int_{\mathbb{R}^d} p(s,t,y,x)dx\right)^p \\
\leq\ & C_p\left(1+E\left|\int_s^{t_1}\int_S\left(\int_{\mathbb{R}^d}\operatorname{div}g(\lambda,z)p(s,r,y,z)dz\right)^2\mu(d\lambda)dr\right|^{p/2}\right) \\
\leq\ & C_p\left(1+kE\left|\int_s^{t_1}\left(\int_{\mathbb{R}^d}p(s,r,y,z)dz\right)^2 dr\right|^{p/2}\right),
\end{aligned}
$$

where

$$
k=\sup_z\left|\int_S\left|\operatorname{div}g(\lambda,z)\right|^2\mu(d\lambda)\right|^{p/2}.
$$

Set $\Phi(t_1)=E\left(\sup_{t\in[s,t_1]}\int_{\mathbb{R}^d}p(s,t,y,x)dx\right)^p$. We have proved that

$$
\Phi(t_1)\leq C_p\left(1+k\int_s^{t_1}\Phi(r)dr\right).
$$

Finally, Gronwall's lemma allows to conclude the proof. Notice that we need to assume that $\sup_{t\in[s,T]}E\left(\int_{\mathbb{R}^d}p(s,t,y,x)dx\right)^p$ is finite, and this property follows from Proposition 10. $\square$

# 6 Equivalence of Evolution and Weak equations

Suppose that $p(s,t,y,x)$ is the stochastic kernel introduced in Proposition 9. That is, $p(s,t,y,x)$ is the marginal probability density $Q\left[\varphi_{t,s}(x)\in dy\right]/dy$ of the backward stochastic flow $\varphi$ criven by the vector field $-v+\sigma b$. By Equation (4.15) we know that $p(s,t,y,x)$ is the fundamental solution (in the variables $x,t$) of the equation

$$
u\left(dt,x\right)=\Delta_x u\left(t,x\right)dt+v\left(dt,x\right)\cdot\nabla_x u\left(t,x\right).
$$

The purpose of this section is to show that the evolution solution to Equation (1.1) obtained in Theorem 7 is a *weak* solution to the following stochastic partial differential equation

$$
du_t=\Delta_x u\left(t,x\right)dt+v\left(dt,x\right)\cdot\nabla_x u\left(t,x\right)+F(t,x,u\left(t,x\right))W(dt,x). \quad (6.1)
$$

Let us first introduce the notion of weak solution:

**Definition 3** *Let* $u = \left\{u(t,x), t \in [0,T], x \in \mathbb{R}^d\right\}$ *be an adapted random field such that* $E\int_0^T \|u(s)\|_2^2\, ds < \infty$ *. Suppose that the Gaussian random field* $v(t,x)$ *and* $W(t,x)$ *satisfy conditions (4.1) and (2.1) respectively. We say that* $u$ *is a weak solution to (6.1) if for every* $\phi \in \mathcal{C}_k^\infty\left(\mathbb{R}^d\right)$ *we have*

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(x)\, u(t,x)\, dx = \int_{\mathbb{R}^d} \phi(x)\, u_0(x)\, dx + \int_0^t \int_{\mathbb{R}^d} u(t,x)\, \Delta\phi(x)\, dx \\ -\int_0^t \int_{\mathbb{R}^d} u(s,x)\, \mathrm{div}\left[\phi(x)\, v(ds,x)\right] dx \\ +\int_0^t \int_{\mathbb{R}^d} u(s,x)\, W(ds,x)\, \phi(x)\, dx. \quad (6.2)\end{aligned}$$

Under condition (2.1) the second stochastic integral in Equation (6.2) is well-defined. Moreover, under condition (4.1) the first stochastic integral in equation (6.2) is also well-defined as:

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} u(t,x)\, \mathrm{div}\left[\phi(x)\, v(ds,x)\right] dx \\ =\ &\int_0^t \int_S \int_{\mathbb{R}^d} u(s,x)\left[\phi(x)\, \mathrm{div} g(\lambda,x) + \nabla\phi(x)\, \nabla g(\lambda,x)\right] M(ds,d\lambda)\, dx.\end{aligned}$$

**Theorem 13** *Assume the velocity field* $v$ *satisfies the coercivity condition (H1), and condition (5.15). If* $u$ *is the evolution solution of Equation (1.4), then it is a weak solution of (6.1) in the sense of Definition 3.*

*Proof :* Assume that $u$ is the evolution solution to Equation (1.4). The evolution equation (1.4) can be written in the following way using the stochastic semigroup $T_{t,s}$:

$$u(t,x) = T_{t,0}u_0(x) + \int_{[0,t]\times S} T_{t,s}\left[F_s(u)\, a(\lambda)\right](x) M(ds,d\lambda), \qquad (6.3)$$

where $F_s(u)$ denotes the random function $F(s,y,u(s,y))$. We are going to use the fact that if $t \geq s$, then $T_{t,s}$ satisfies the backward Kolmogorov equation (4.8). Notice that this equation holds for any function $f$ in $L^2(\mathbb{R}^d)$. Indeed, using the fact that the kernel $p$ is in $C^2\left(\mathbb{R}^d\right)$, we have that $\nabla_x^2 T_{r,s} f(x) = \nabla_x^2 \int p(s,r,y,x)\, f(y)\, dy = \int dy f(y)\, \nabla_x^2 p(s,r,y,x)$. As a consequence, we obtain:

$$
\begin{aligned}
u\left(t,x\right) &= u_0(x)+\int_0^t \nabla T_{r,0}u_0(x)v(dr,x)+\int_0^t \Delta T_{r,0}u_0(x)dr \\
&+\int_{[0,t]\times S} F\left(s,x,u\left(s,x\right)\right)a(\lambda,x)M(ds,d\lambda) \\
&+\int_{[0,t]\times S}\left(\int_s^t \nabla T_{r,s}\left[F_s(u)a(\lambda)\right](x)v(dr,x)\right)M(ds,d\lambda) \\
&+\int_{[0,t]\times S}\left(\int_s^t \Delta T_{r,s}\left[F_s\left(u\right)a(\lambda)\right](x)dr\right)M(ds,d\lambda).
\end{aligned}
$$

Fix a test function $\phi$ in $\mathcal{C}_k^\infty\left(\mathbb{R}^d\right)$. Multiplying the above equation by $\phi$, integrating with respect to $x$ and using integration by parts yields

$$
\begin{aligned}
\left(u\left(t\right),\phi\right) &= \left(u_0,\phi\right) \\
&-\int_{\mathbb{R}^d}\int_0^t \nabla\phi(x)T_{r,0}u_0(x)v(dr,x)dx \\
&-\int_{\mathbb{R}^d}\int_0^t \phi(x)T_{r,0}u_0(x)\mathrm{div}v(dr,x)dx+\int_0^t\left(\Delta\phi,T_{r,0}u_0\right)dr \\
&+\int_{\mathbb{R}^d}\int_{[0,t]\times S}\phi(x)F\left(s,x,u\left(s,x\right)\right)a(\lambda,x)M(ds,d\lambda)dx \\
&-\int_{\mathbb{R}^d}\int_{[0,t]\times S}\left(\int_s^t \nabla\phi(x)T_{r,s}\left[F_s\left(u\right)a(\lambda)\right](x)v(dr,x)\right)M(ds,d\lambda)dx \\
&-\int_{\mathbb{R}^d}\int_{[0,t]\times S}\left(\int_s^t \phi(x)T_{r,s}\left[F_s\left(u\right)a(\lambda)\right](x)\mathrm{div}v(dr,x)\right)M(ds,d\lambda)dx \\
&+\int_{[0,t]\times S}\left(\int_s^t\left(\Delta\phi,T_{r,s}\left[F_s\left(u\right)a(\lambda)\right]\right)dr\right)M(ds,d\lambda),
\end{aligned}
$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbb{R}^d)$. Now we apply Fubini's theorem and we obtain

$$
\begin{aligned}
(u(t),\phi) &= (u_0,\phi) \\
&-\int_{\mathbb{R}^d}\int_0^t \nabla\phi(x)T_{r,0}u_0(x)v(dr,x)dx \\
&-\int_{\mathbb{R}^d}\int_0^t \phi(x)T_{r,0}u_0(x)\mathrm{div}v(dr,x)dx+\int_0^t (\Delta\phi,T_{r,0}u_0)\,dr \\
&+\int_{\mathbb{R}^d}\int_{[0,t]\times S}\phi(x)F(s,x,u(s,x))\,a(\lambda,x)M(ds,d\lambda)dx \\
&-\int_{\mathbb{R}^d}\nabla\phi(x)\int_0^t\left(\int_{[0,r]\times S}T_{r,s}\left[F_s(u)\,a(\lambda)\right](x)M(ds,d\lambda)\right)v(dr,x)dx \\
&-\int_{\mathbb{R}^d}\phi(x)\int_0^t\left(\int_{[0,r]\times S}T_{r,s}\left[F_s(u)\,a(\lambda)\right](x)M(ds,d\lambda)\right)\mathrm{div}v(dr,x)dx \\
&+\int_{\mathbb{R}^d}\Delta\phi(x)\int_0^t\left(\int_{[0,r]\times S}T_{r,s}\left[F_s(u)\,a(\lambda)\right](x)M(ds,d\lambda)\right)drdx.
\end{aligned}
$$

Using (6.3) yields

$$
\begin{aligned}
(u(t),\phi) &= (u_0,\phi)+\int_{\mathbb{R}^d}\int_{[0,t]\times S}\phi(x)F(s,x,u(s,x))\,a(\lambda,x)M(ds,d\lambda)dx \\
&-\int_0^t\int_{\mathbb{R}^d}\nabla\phi(x)u(r,x)v(dr,x)dx \\
&-\int_0^t\int_{\mathbb{R}^d}\phi(x)u(r,x)\mathrm{div}v(dr,x)dx \\
&+\int_0^t\int_{\mathbb{R}^d}\Delta\phi(x)u(r,x)drdx.
\end{aligned}
$$

which is exactly Equation (6.2). $\square$

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