Stochastic heat equation with white-noise drift

by

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*Supported by the DGICYT grant n° PB96–0087.

1 Introduction

The purpose of this paper is to establish the existence and uniqueness of a solution for anticipative stochastic evolution equations of the form

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} p(0,t,y,x)\, u_0(y)\, dy \\ &+ \int_{\mathbb{R}} \int_0^t p(s,t,y,x)\ F(s,y,u(s,y))\ dW_{s,y}\,, \end{aligned} \tag{1.1}$$

where $W = \{W(t,x),\ t \in [0,T],\ x \in \mathbb{R}\}$ is a zero mean Gaussian random field with covariance $\frac{1}{2}(s \wedge t)\,(|x|+|y|-|x-y|)$. We asume that $p(s,t,y,x)$ is a stochastic semigroup measurable with respect to the σ-field $\sigma\{W(r,x) - W(s,x),\ x \in \mathbb{R},\ r \in [s,t]\}$. The stochastic integral in Equation (1.1) is anticipative because the integrand is the product of an adapted factor $F(s,y,u(s,y))$ times $p(s,t,y,x)$, which is adapted to the future increments of the random field W. We interpret this integral in the Skorohod sense (see [15]) which coincides in this case with a two-sided stochastic integral (see [14]). The choice of this notion of stochastic integral is motivated by the concrete example handled in Section 5, where $p(s,t,y,x)$ is the backward heat kernel of the random operator $\frac{d^2}{dx^2} + \dot{v}(t,x)\ \frac{d}{dx}$, $\dot{v}(t,x)$ being a white-noise in time. In this case, $u(t,x)$ turns out to be (see Section 6) a weak solution of the stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{v}(t,x)\ \frac{\partial u}{\partial x} + F(t,x,u)\ \frac{\partial^2 W}{\partial t\, \partial x}\ . \tag{1.2}$$

A stochastic evolution equation of the form (1.1) on $\mathbb{R}^d$ perturbed by a noise of the form $W(ds,y)dy$, where W is a random field with covariance $(s \wedge t)\, Q(x,y)$, Q being a bounded function, has been studied in [13].Following the approach introduced in this paper we establish in Theorem 4.1 the existence and uniqueness of a solution to Equation (1.1) with values in $L^p_M(\mathbb{R})$. Here $L^p_M(\mathbb{R})$ means the space of real-valued functions f such that $\int_{\mathbb{R}} e^{-M(x)} |f(x)|^p\, dx < \infty$ where $M > 0$ and $p \geq 2$. This theorem is a consequence of the estimates of the moments of Skorohod integrals of the form

$$\int_{\mathbb{R}} \int_0^t p(s,t,y,x)\ \phi(s,y)\ dW_{s,y}\ ,$$

obtained in Section 3 by means of the techniques of the Malliavin calculus.

2 Preliminaries

For $s,t \in [0,T]$, $s \leq t$, we set $I^t = [0,t] \times \mathbb{R}$ and $I^t_s = [s,t] \times \mathbb{R}$. Consider a Gaussian family of random variables $W = \{W(A),\ A \in \mathcal{B}(I^T),\ \mu(A) < \infty\}$, defined on a complete probability space, with zero mean, and covariance function given by

$$E\left(W(A)\, W(B)\right) = \mu(A \cap B)\,,$$

where μ denotes the Lebesgue measure on I^T. We will assume that $\mathcal{F}$ is generated by W and the P-null sets. For each $s,t \in [0,T]$, $s \leq t$ we will denote by $\mathcal{F}_{s,t}$ the σ-algebra generated by $\{W(A),\ A \subset [s,t] \times \mathbb{R},\ \mu(A) < \infty\}$ and the P-null sets. We say that a stochastic process $u = \{u(t,x),\ (t,x) \in I^T\}$ is *adapted* if $u(t,x)$ is $\mathcal{F}_{0,t}$-measurable for each (t,x). Set $H = L^2(I^T,\ \mathcal{B}(I^T), \mu)$ and denote by $W(h) = \int_{I^T} h dW$ the Wiener integral of a deterministic function $h \in H$.

In the sequel we introduce the basic notation and results of the stochastic calculus of variations with respect to W. For a complete exposition we refer to [2, 11].

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f\left(W(h_1), \ldots, W(h_n)\right), \tag{2.1}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $h_1, \ldots, h_n \in H$. Given a random variable F of the form (2.1), we define its derivative as the stochastic process $\{D_{t,x} F,\ (t,x) \in I^T\}$ given by

$$D_{t,x} F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left(W(h_1), \ldots, W(h_n)\right)\, h_i(t,x), \quad (t,x) \in I^T.$$

More generally, we can define the iterated derivative operator on a cylindrical random variable F by setting

$$D^n_{t_1,x_1,\ldots,t_n,x_n} F = D_{t_1,x_1} \cdots D_{t_n,x_n} F\,.$$

The iterated derivative operator D^n is a closable unbounded operator from $L^2(\Omega)$ into $L^2((I^T)^n \times \Omega)$ for each $n \geq 1$. We denote by $\mathbb{D}^{n,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|^2_{n,2} = \|F\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \|D^i F\|^2_{L^2((I^T)^i \times \Omega)}\,.$$

If V is a real and separable Hilbert space we denote by $\mathbb{D}^{n,2}(V)$ the corresponding Sobolev space of V-valued random variables.

We denote by δ the adjoint of the derivative operator D. That is, the domain of δ (denoted by Dom δ) is the set of elements $u \in L^2(I^T \times \Omega)$ such that there exists a constant c satisfying

$$\left| E \int_{I^T} (D_{t,x} F)\ u(t,x)\, dt\, dx \right| \leq c\ \|F\|_{L^2(\Omega)}\,,$$

for all $F \in \mathcal{S}$. If $u \in \mathrm{Dom}\,\delta$, $\delta(u)$ is the element in $L^2(\Omega)$ characterized by

$$E\left(\delta(u)\, F\right) = E \int_{I^T} (D_{t,x} F)\ u(t,x)\, dt\, dx\,, \quad F \in \mathcal{S}\,.$$

The operator δ is an extension of the ItÙ integral (see Skorohod [15]), in the sense that the set $L^2_a(I^T \times \Omega)$ of square integrable and adapted processes is included in Dom δ and the operator δ restricted to $L^2_a(I^T \times \Omega)$ coincides with the ItÙ stochastic integral defined in [16]. We will make use of the notation $\delta(u) = \int_{I^T} u(t,x)\, dW_{t,x}$ for any $u \in \mathrm{Dom}\,\delta$.

We recall that $\mathbb{L}^{1,2} := L^2(I^T, \mathbb{D}^{1,2})$ is included in the domain of δ, and for a process $u \in \mathbb{L}^{1,2}$ we can compute the variance of the Skorohod integral of u as follows:

$$E\delta(u)^2 = E \int_{I^T} u^2(t,x)\, dt\, dx + E \int_{I^T} \int_{I^T} D_{s,y}\ , u(t,x)\, D_{t,x}\, u(s,y)\, dt\, dx\, ds\, dy\,.$$

We need the following results on the Skorohod integral:

Proposition 2.1 *Let $u \in \operatorname{Dom}\delta$ and consider a random variable $F \in \mathbb{D}^{1,2}$ such that $E\left(F^2 \int_{I^T} u(t,x)^2\, dt\, dx\right) < \infty$. Then*

$$\int_{I^T} Fu(t,x)\, dW_{t,x} = F \int_{I^T} u(t,x)\, dW_{t,x} - \int_{I^T} (D_{t,x} F)\ u(t,x)\, dt\, dx\,, \tag{2.2}$$

in the sense that $Fu \in \operatorname{Dom}\delta$ if and only if the right-hand side of (2.2) is square integrable.

Proposition 2.2 *Consider a process u in $\mathbb{L}^{1,2}$. Suppose that for almost all $(\theta, z) \in I^T$, the process $\{D_{\theta,z}\, u(s,y)\, \mathbb{1}_{[0,\theta]}(s),\ (s,y) \in I^T\}$ belongs to $\operatorname{Dom}\delta$ and, moreover,*

$$E \int_{I^T} \Big| \int_{I^\theta} D_{\theta,z}\ u(s,y)\, dW_{s,y} \Big|^2 d\theta\, dz < \infty\,.$$

Then u belongs to $\operatorname{Dom}\delta$ and we have the following expression for the variance of the Skorohod integral of u:

$$\begin{aligned} E\,\delta(u)^2 = E &\int_{I^T} u^2(s,y)\, ds\, dy \\ &+\ 2\,E \int_{I^T} u(\theta,z)\ \Big(\int_{I^\theta} D_{\theta,z}\ u(s,y)\, dW_{s,y}\Big) d\theta\, dz\,. \end{aligned} \tag{2.3}$$

We make use of the change-of-variables formula for the Skorohod integral:

Theorem 2.3 *Consider a process of the form $X_t = \int_{I^t} u(s,y)\, dW_{s,y}$, where*

(i) $u \in \mathbb{L}^{2,2}$,

(ii) $u \in L^\beta(I^T \times \Omega)$, *for some* $\beta > 2$,

(iii) $\int_{I^T} u^2(s,y)\, ds\, dy < N$,

for some positive constant N. Let $F : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that F'' is bounded. Then we have

$$\begin{aligned} F(X_t) = F(0) &+ \int_{I^t} F'(X_s)\ u(s,y)\, dW_{s,y} \\ &+ \frac{1}{2} \int_{I^t} F''(X_s)\ u^2(s,y)\, ds\, dy \\ &+ \int_{I^t} F''(X_s)\ u(s,y)\Big(\int_{I^s} D_{s,y}\, u(r,z)\, dW_{r,z}\Big) ds\, dy\,. \end{aligned} \tag{2.4}$$

Notice that under the assumptions of Theorem 2.3 the process X_t has a continuous version (see [2, 5]) and, moreover, $\{F'(X_s)\, u(s,y),\ (s,y) \in I^T\}$ belongs to $\operatorname{Dom}\delta$.

3 Estimates for the Skorohod integral

We denote by C a generic constant that can change from one formula to another one. Let $p(s,t,y,x)$ be a random measurable function defined on $\{0 \le s < t \le T,\ x,y \in \mathbb{R}\} \times \Omega$. We will assume that the following conditions hold:

(H1) For all $0 \le s < t \le T,\ x,y \in \mathbb{R},\ p(s,t,y,x)$ is $\mathcal{F}_{s,t}$-measurable.

(H2) $p(s,t,y,x) \ge 0$, for each $0 \le s < t \le T,\ x,y \in \mathbb{R}$.

(H3) For all $0 \le s < t \le T,\ x \in \mathbb{R},\ \int_{\mathbb{R}} p(s,t,y,x)\,dy = 1$.

(H4) For each $s \in [0,T],\ x,y \in \mathbb{R},\ p(s,t,y,\cdot)$ is continuous in $t \in (s,T]$ with values in $L^2(\mathbb{R})$.

(H5) For all $0 \le s < r < t \le T$, and $x,y \in \mathbb{R}$

$$p(s,t,y,x) = \int_{\mathbb{R}} p(s,r,y,z)\ p(r,t,z,x)\,dz\,.$$

(H6) For all $0 \le s < t \le T,\ x,y \in \mathbb{R},\ p(s,t,y,x) \in \mathbb{D}^{1,2}$ and $p(s,t,\cdot\,,x)$ belongs to $\mathbb{D}^{1,2}(L^2(\mathbb{R}))$. Moreover, there exists a version of the derivative such that the following limit exists in $L^2(\Omega;\,L^2(\mathbb{R}))$ for each s,z,t,x

$$D^-_{s,z}\,p(s,t,\cdot\,,x) = \lim_{\varepsilon\downarrow 0}\ D_{s,z}\ p(s-\varepsilon,t,\cdot\,,x)\,. \tag{3.1}$$

(H7) For all $0 \le s < t \le T,\ x,y \in \mathbb{R},\ p \ge 1$ there exists a nonnegative random variable $V_p(s,t,x)$ $\mathcal{F}_{s,t}$-measurable and $\delta_p > 0$ such that

$$p(s,t,y,x) \le V_p(s,t,x)\ \exp\Big(-\frac{|x-y|^2}{\delta_p(t-s)}\Big)$$

and satisfying that for all $p \le 1$, there exists a positive constant $C_{1,p}$ such that

$$\|V_p(s,t,x)\|_{L^p(\Omega)} \le C_{1,p}\,|t-s|^{-\frac{1}{2}}.$$

(H8) For all $0 \le s < t \le T,\ x,y,z \in \mathbb{R}$, and $p \ge 1$ there exists a nonnegative random variable $U_p(s,t,x)$ $\mathcal{F}_{s,t}$-measurable a constant $\gamma_p > 0$ and a nonnegative measurable deterministic function $f(y,z)$ such that

(i) $|D^-_{s,z}\ p(s,t,y,x)| \le U_p(s,t,x)\ \exp\Big(-\frac{|x-y|^2}{\gamma_p(t-s)}\Big)\,f(y,z)$,

(ii) $\sup_y\ \int_{\mathbb{R}}\ f^2(y,z)\ dz \le C_f$,

(iii) $\|U_p(s,t,x)\|_{L^p(\Omega)} \le C_{2,p}|t-s|^{-1}$,

for some positive constants $C_{2,p},\ C_f > 0$.

The following lemma is a straightforward consequence of the above hypotheses:

Lemma 3.1 *Under the above hypotheses we have that for all* $0 \leq r < s < t \leq T,\ x, y, z \in \mathbb{R}$,

$$D_{s,y}\ p(r,t,z,x) = \int_{\mathbb{R}} \left(D^{-}_{s,y}\ p(s,t,u,x)\right)\ p(r,s,z,u)\, du\,. \tag{3.2}$$

Proof. Taking into account the properties of the derivative operator and using hypotheses (H1), (H5) and (H6) we have that

$$\begin{aligned} D_{s,y}\, p(r,t,z,x) &= D_{s,y} \int_{\mathbb{R}} p(r, s-\varepsilon, z, u)\ p(s-\varepsilon, t, u, x)\, du \\ &= \int_{\mathbb{R}} p(r, s-\varepsilon, z, u)\ D_{s,y}\ p(s-\varepsilon, t, u, x)\, du\,. \end{aligned}$$

Now, letting ε tend to zero and using hypotheses (H1), (H4), (H6) and (H8) we can easily complete the proof. $\square$

We are now in a position to prove our estimates for the Skorohod integral. For all $M > 0$, we will denote by $L^p_M(I^T \times \Omega)$ the space of processes $\phi = \{\phi(s,y),\ (s,y) \in I^T\}$ such that

$$E \int_{I^T} e^{-M|y|}\ |\phi(s,y)|^p\ ds\, dy < \infty\,.$$

Theorem 3.2 *Fix* $p > 4$, $\alpha \in [0, \frac{p-4}{4p})$ *and* $M > 0$. *Let* $\phi = \{\phi(s,y),\ (s,y) \in I^T\}$ *be an adapted process in* $L^p_M(I^T \times \Omega)$. *Assume that* $p(s,t,y,x)$ *is a stochastic kernel satisfying hypotheses (H1) to (H8). Then, for almost all* $(t,x) \in I^T$, *the process*

$$\{\, (t-s)^{-\alpha}\ p(s,t,y,x)\ \phi(s,y)\, \mathbb{1}_{[0,t]}(s),\ (s,y) \in I^T\, \}$$

belongs to $\operatorname{Dom}\delta$, *and*

$$\begin{aligned} \int_{\mathbb{R}} e^{-M|x|}\, E\Big| \int_{I^t} (t-s)^{-\alpha}\, p(s,t,y,x)\ \phi(s,y)\, dW_{s,y} \Big|^p dx \\ \leq C \int_0^t (t-s)^{-\alpha - \frac{1}{4} - \frac{1}{p}} \left(\int_{\mathbb{R}} e^{-M|y|}\, E|\phi(s,y)|^p\, dy \right) ds\,, \end{aligned} \tag{3.3}$$

for some positive constant C *depending only on* α, p, T, M, δ_p, γ_p, $C_{1,p}$, $C_{2,p}$ *and* C_f.

Proof. Let us denote by $\mathcal{S}^a$ the space of simple and adapted processes of the form

$$\phi(s,y) = \sum_{i,j=0}^{m-1} F_{ij}\, \mathbb{1}_{(t_i, t_{i+1}]}(s)\, h_j(y)\,,$$

where $0 = t_0 < t_1 < \ldots < t_m = T$, $h_j \in \mathcal{C}^\infty_K(\mathbb{R})$ and the F_{ij} are $\mathcal{F}_{0,t_i}$-measurable functions in $\mathcal{S}$. Let ϕ be an adapted process in $L^p_M(I^T \times \Omega)$. We can find a sequence ϕ^n of processes in $\mathcal{S}^a$ such that

$$\lim_{n\to\infty} \int_0^T \Big(\int_{\mathbb{R}} e^{-M|y|}\ E\,|\phi^n(s,y) - \phi(s,y)|^p\, dy \Big) ds = 0\,.$$

We can easily check that this implies the existence of a subsequence n_k such that for almost all $t \in [0,T]$

$$\lim_{k\to\infty} \int_0^t (t-s)^{-\alpha-\frac{1}{4}-\frac{1}{p}} \Big(\int_{\mathbb{R}} e^{-M|y|} \, E\, |\phi^{n_k}(s,y) - \phi(s,y)|^p dy \Big) \, ds = 0 \,.$$

On the other hand, using the fact that $\alpha < \frac{1}{4}$ and hypothesis (H7) we have that

$$\begin{aligned}
A := \lim_{k\to\infty} E \int_0^T & \left(\int_{\mathbb{R}} e^{-M|x|} \Big(\int_{I^t} (t-s)^{-2\alpha} \, p^2(s,t,y,x) \right. \\
& \left. \times \, |\phi^{n_k}(s,y) - \phi(s,y)|^2 ds \, dy \Big) dx \right) dt \\
\le C_{1,2}^2 \lim_{k\to\infty} \int_0^T & \left(\int_{\mathbb{R}} e^{-M|x|} \Big(\int_{I^t} (t-s)^{-2\alpha-1} \, \exp\Big(- \frac{2|x-y|^2}{\delta_2(t-s)} \Big) \right. \\
& \left. \times \, E\, |\phi^{n_k}(s,y) - \phi(s,y)|^2 ds \, dy \Big) dx \right) dt \\
= C_{1,2}^2 \lim_{k\to\infty} \int_{I^T} & E\, |\phi^{n_k}(s,y) - \phi(s,y)|^2 \left(\int_{I_s^T} (t-s)^{-2\alpha-1} \right. \\
& \left. \times \exp\Big(- M|x| - \frac{2|x-y|^2}{\delta_2(t-s)} \Big) dt \, dx \right) ds \, dy \,.
\end{aligned}$$

Notice that

$$\begin{aligned}
\int_{\mathbb{R}} \exp\Big(- M|x| - \frac{2|x-y|^2}{\delta_2(t-s)} \Big) dx &= \int_{\mathbb{R}} \exp\Big(- M|x+y| - \frac{2\,x^2}{\delta_2(t-s)} \Big) dx \\
\le e^{-M\,|y|} \int_{\mathbb{R}} \exp\Big(M|x| - \frac{2\,x^2}{\delta_2(t-s)} \Big) dx &\le K_1 \, \sqrt{t-s} \; e^{-M\,|y|} \,,
\end{aligned}$$

where $K_1 = \sqrt{2\pi\delta_2} \; e^{\frac{M^2\delta_2 T}{8}}$. Then

$$A \le C_{1,2}^2 K_1 \lim_{k\to\infty} \int_{I^T} e^{-M\,|y|} \; E\, |\phi^{n_k}(s,y) - \phi(s,y)|^2 \, ds \, dy = 0 \,.$$

Then, choosing a subsequence (denoted again by n_k) we have that for almost all $(t,x) \in I^T$

$$\lim_{k\to\infty} E \int_{I^t} (t-s)^{-2\alpha} \; p^2(s,t,y,x) \; |\phi^{n_k}(s,y) - \phi(s,y)|^2 ds \, dy = 0 \,.$$

This allows us to suppose that $\phi \in \mathcal{S}^a$. Fix $t_0 > t_1$ in $[0,T]$ and define

$$\begin{aligned}
B_x(s,y) &= (t_0 - s)^{-\alpha} \; p(s,t_1,y,x) \; \phi(s,y) \\
X(t,x) &= \int_{I^t} B_x(s,y) \; dW_{s,y}, \quad t \in [0,t_1] \,.
\end{aligned}$$

Denote $F(x) = |x|^p$. Let F_N be the increasing sequence of functions defined by

$$F_N(x) = \int_0^{|x|} \int_0^y \Big(p(p-1)\, z^{p-2} \wedge N \Big) dz \, dy \,.$$

Suppose first that $p(s,t,y,x)$ is an elementary backward-adapted process of the form

$$\sum_{i,j,k=1}^{n} H_{ijk}\,\beta_j(y)\,\gamma_k(z)\,\mathbb{1}_{(s_i,s_{i+1}]}(s)\,,$$

where $H_{ijk} \in \mathcal{S}$, $\beta_j, \gamma_k \in \mathcal{C}_K^\infty(\mathbb{R})$, $0 = s_1 < \ldots < s_{n+1} = t$, and H_{ijk} is $\mathcal{F}_{s_{i+1},t_1}$-measurable. Then we can apply ItÙ's formula (see Theorem 2.3) for the function F_N and the process $B_x(s,y)$, obtaining that for all $t < t_1$,

$$\begin{aligned} E\Big(F_N(X(t,x))\Big) &= \frac{1}{2}\,E\int_{I^t} F_N''\,(X(s,x))\;B_x^2(s,y)\,ds\,dy \\ &+ E\int_{I^t} F_N''\,(X(s,x))\;B_x(s,y)\;\Big(\int_{I^s} D_{s,y}\,B_x(r,z)\;dW_{r,z}\Big)ds\,dy\,. \end{aligned} \tag{3.4}$$

Using hypotheses (H1), (H7) and (H8), Lemma 3.1 and the fact that ϕ is simple and adapted we can easily check that, for all $p(s,t,y,x)$ satisfying the hypotheses of the theorem, and for all $t < t_1$

(i) $E\displaystyle\int_{I^t} B_x^2(s,y)\,ds\,dy < \infty\,,$

(ii) $E\Big(\displaystyle\int_{I^t} B_x(s,y)\,dW_{s,y}\Big)^2 < \infty,$

(iii) $E\displaystyle\int_{I^t} \Big|\int_{I^s} D_{s,y}\;B_x(r,z)\,dW_{r,z}\Big|^2\,ds\,dy < \infty\,.$

This allows us to deduce that (3.4) still holds for every $p(s,t,y,x)$ satisfying hypotheses (H1) to (H8) and for all $t < t_1$. We can easily check that $F_N''(x) \le (2^{1+\frac{1}{p}} + \frac{1}{p(p-1)})(F_N(x))^{\frac{p-2}{p}}$. Then we have that

$$\begin{aligned} E\,F_N(X(t,x)) \le C_p\Bigg\{&\frac{1}{2}\,E\int_{I^t}\Big(F_N(X(s,x))\Big)^{\frac{p-2}{p}}\;B_x^2(s,y)\,ds\,dy \\ &+ E\int_{I^t}\Big(F_N\,(X(s,x))\Big)^{\frac{p-2}{p}}\;\Big|\int_{\mathbb{R}} B_x(s,y)\,\Big(\int_{I^s} D_{s,y}\;B_x(r,z)\,dW_{r,z}\Big)dy\Big|ds\Bigg\}. \end{aligned}$$

H^lder's inequality gives us that

$$\begin{aligned} E\,F_N(X(t,x)) \le C_p\Bigg\{&\int_0^t \frac{1}{2}\,\Big(E\,F_N(X(s,x))\Big)^{\frac{p-2}{p}}\;\Big(E\Big|\int_{\mathbb{R}} B_x^2(s,y)\,dy\Big|^{\frac{p}{2}}\Big)^{\frac{2}{p}}ds \\ &+\int_0^t \Big(E\,F_N(X(s,x))\Big)^{\frac{p-2}{p}}\Bigg(E\Big|\int_{\mathbb{R}} B_x(s,y)\Big(\int_{I^s} D_{s,y}B_x(r,z)\,dW_{r,z}\Big)dy\Big|^{\frac{p}{2}}\Bigg)^{\frac{2}{p}}ds\Bigg\}. \end{aligned}$$

Applying the lemma of [17], pg. 171 we obtain that

$$\begin{aligned} E\,F_N(X(t,x)) \le C_p\Bigg\{&\int_0^t \frac{1}{2}\Big(E\Big|\int_{\mathbb{R}} B_x^2(s,y)\,dy\Big|^{\frac{p}{2}}\Big)^{\frac{2}{p}}ds \\ &+\int_0^t\Bigg(E\Big|\int_{\mathbb{R}} B_x(s,y)\Big(\int_{I^s} D_{s,y}B_x(r,z)\,dW_{r,z}\Big)dy\Big|^{\frac{p}{2}}\Bigg)^{\frac{2}{p}}ds\Bigg\}^{\frac{p}{2}}. \end{aligned}$$

Fatou's lemma gives us that, letting N tend to infinity

$$
\begin{aligned}
E\,|X(t,x)|^p \le C_p\Big\{ \int_0^t \ \frac{1}{2}\Big(E\Big|\int_{\mathbb{R}} B_x^2(s,y)\,dy\Big|^{\frac{p}{2}}\Big)^{\frac{2}{p}} ds \\
+\int_0^t \left(E\Big|\int_{\mathbb{R}} B_x(s,y)\Big(\int_{I^s} D_{s,y} B_x(r,z)\,dW_{r,z}\Big)dy\Big|^{\frac{p}{2}}\right)^{\frac{2}{p}} ds\Big\}^{\frac{p}{2}} \\
=: C_p\Big(\frac{1}{2}I_1 + I_2\Big)^{\frac{p}{2}}.
\end{aligned}
\tag{3.5}
$$

We have that

$$
\begin{aligned}
I_1 &\le \int_0^t (t-s)^{-2\alpha}\Big(E\Big|\int_{\mathbb{R}} p^2(s,t_1,y,x)\,\phi^2(s,y)\,dy\Big|^{\frac{p}{2}}\Big)^{\frac{2}{p}} ds \\
&\le \int_0^t (t-s)^{-2\alpha}\left(E\Big|\int_{\mathbb{R}} \exp\Big(\frac{-2|x-y|^2}{\delta_p(t_1-s)}\Big)V_p^2(s,t_1,x)\,\phi^2(s,y)\,dy\Big|^{\frac{p}{2}}\right)^{\frac{2}{p}} ds \\
&\le C_{1,2}^2 \int_0^t (t-s)^{-2\alpha-1}\left(E\Big|\int_{\mathbb{R}} \exp\Big(\frac{-2|x-y|^2}{\delta_p(t_1-s)}\Big)\phi^2(s,y)\,dy\Big|^{\frac{p}{2}}\right)^{\frac{2}{p}} ds \\
&\le C \int_0^t (t-s)^{-2\alpha-\frac{1}{2}-\frac{1}{p}}\left(\int_{\mathbb{R}} \exp\Big(\frac{-2|x-y|^2}{\delta_p(t_1-s)}\Big)E\,|\phi\,(s,y)|^p\,dy\right)^{\frac{2}{p}} ds\,.
\end{aligned}
\tag{3.6}
$$

On the other hand, using Lemma 3.1 yields

$$
\begin{aligned}
&\int_{\mathbb{R}} B_x(s,y)\Big(\int_{I^s} D_{s,y}\ B_x(r,z)\,dW_{r,z}\Big)dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}} p(s,t_1,y,x)\,\phi(s,y) \\
&\qquad \times\Big(\int_{I^s}(t_0-r)^{-\alpha}[D_{s,y}\ p(r,t_1,z,x)]\,\phi(r,z)\,dW_{r,z}\Big)dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}} p(s,t_1,y,x)\,\phi(s,y) \\
&\qquad \times\Big(\int_{I^s}(t_0-r)^{-\alpha}\Big[\int_{\mathbb{R}} p(r,s,z,u)\,D^-_{s,y}\ p(s,t_1,u,x)\,du\Big]\phi(r,z)\,dW_{r,z}\Big)dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}} p(s,t_1,y,x)\,\phi(s,y)\Big[\int_{\mathbb{R}} D^-_{s,y}\ p(s,t_1,u,x) \\
&\qquad \times\Big(\int_{I^s}(t_0-r)^{-\alpha}p(r,s,z,u)\,\phi(r,z)\,dW_{r,z}\Big)du\Big]dy\,.
\end{aligned}
$$

Let us denote $Y(s,u) := \int_{I^s}(t_0-r)^{-\alpha}\,p(r,s,z,u)\,\phi(r,z)\,dW_{r,z}$. Notice that $X(t_1,x) = Y(t_1,x)$. We have proved that

$$
\begin{aligned}
&\int_{\mathbb{R}} B_x(s,y)\Big(\int_{I^s} D_{s,y}\ B_x(r,z)\,dW_{r,z}\Big)dy \\
&\quad = (t_0-s)^{-\alpha}\int_{\mathbb{R}}\ p(s,t_1,y,x)\,\phi(s,y)\ \Big[\int_{\mathbb{R}}\ D^-_{s,y}\ p(s,t_1,u,x)\ Y(s,u)\,du\Big]dy\,,
\end{aligned}
$$

and then

$$I_2 \le \int_0^t (t_0 - s)^{-\alpha} \Bigg(E \Big| \int_{\mathbb{R}} \; p(s, t_1, y, x) \; \phi(s, y)$$
$$\times \Big[\int_{\mathbb{R}} D^-_{s,y} \; p(s, t_1, u, x) \; Y(s, u) \, du \Big] dy \Big|^{\frac{p}{2}} \Bigg)^{\frac{2}{p}} ds \, .$$

We have that

$$E \Big| \int_{\mathbb{R}} \; p(s, t_1, y, x) \; \phi(s, y) \Big[\int_{\mathbb{R}} D^-_{s,y} \; p(s, t_1, u, x) \; Y(s, u) \, du \Big] dy \Big|^{\frac{p}{2}}$$
$$= E \Big| \int_{\mathbb{R}^2} \; p(s, t_1, y, x) \, D^-_{s,y} \; p(s, t_1, u, x) \, \phi(s, y) \, Y(s, u) \, du \, dy \Big|^{\frac{p}{2}}$$
$$\le E \Big| \int_{\mathbb{R}^2} \; V_p(s, t_1, x) \; U_p(s, t_1, x) \; f(u, y) \; \exp \Bigg(- \frac{|y - x|^2}{\delta_p(t_1 - s)} - \frac{|x - u|^2}{\gamma_p(t_1 - s)} igg)$$
$$\times |\phi(s, y) \; Y(s, u)| \; du \, dy \Big|^{\frac{p}{2}}$$
$$\le C \, |t_1 - s|^{-\frac{3p}{4}} \; E \Big| \int_{\mathbb{R}^2} \; f(u, y) \; \exp \Bigg(- \frac{|y - x|^2}{\delta_p(t_1 - s)} - \frac{|x - u|^2}{\gamma_p(t_1 - s)} \Bigg)$$
$$\times |\phi(s, y) \; Y(s, u)| \; du \, dy \Big|^{\frac{p}{2}} .$$

Applying Schwartz inequality we obtain

$$E \Big| \int_{\mathbb{R}} \; p(s, t_1, y, x) \; \phi(s, y) \Big[\int_{\mathbb{R}} D^-_{s,y} \; p(s, t_1, u, x) \; Y(s, u) \, du \Big] dy \Big|^{\frac{p}{2}}$$
$$\le C \, (t_1 - s)^{-\frac{3p}{4}} \; E \Bigg(\Big| \int_{\mathbb{R}^2} \; Y^2(s, u) \; f^2(u, y) \; e^{-\frac{|x-u|^2}{\gamma_p(t_1 - s)}} \; du \, dy \Big|^{\frac{p}{4}}$$
$$\times \Big| \int_{\mathbb{R}^2} \; \phi^2(s, y) \; e^{-\frac{|x-u|^2}{\gamma_p(t_1 - s)} - \frac{2\,|y-x|^2}{\delta_p(t_1 - s)}} \, du \, dy \Big|^{\frac{p}{4}} \Bigg)$$
$$\le C \, (t_1 - s)^{-\frac{5p}{8}} \; E \Bigg(\Big| \int_{\mathbb{R}} \; Y^2(s, u) \; e^{-\frac{|x-u|^2}{\gamma_p(t_1 - s)}} \, du \Big|^{\frac{p}{4}} \; \Big| \int_{\mathbb{R}} \; \phi^2(s, y) \; e^{-\frac{2\,|y-x|^2}{\delta(t_1 - s)}} \; dy \Big|^{\frac{p}{4}} \Bigg)$$
$$\le C \, (t_1 - s)^{-\frac{5p}{8}} \Bigg(E \Big| \int_{\mathbb{R}} \; Y^2(s, u) \; e^{-\frac{|x-u|^2}{\gamma_p(t_1 - s)}} \; du \Big|^{\frac{p}{2}}$$
$$+ E \Big| \int_{\mathbb{R}} \; \phi^2(s, y) \; e^{-\frac{2\,|y-x|^2}{\delta_p(t_1 - s)}} \, dy \Big|^{\frac{p}{2}} \Bigg)$$
$$\le C \, (t_1 - s)^{-\frac{3p}{8} - \frac{1}{2}} \; E \Bigg(\int_{\mathbb{R}} \; |Y(s, y)|^p \; e^{-\frac{|x-y|^2}{\gamma_p(t_1 - s)}} \, dy + \int_{\mathbb{R}} \; |\phi(s, y)|^p \; e^{-\frac{2\,|x-y|^2}{\delta_p(t_1 - s)}} \; dy \Bigg).$$

This yields

$$I_2 \le C \int_0^t (t_0-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}} \Bigg(\int_{\mathbb{R}} \exp\Big(-\frac{|x-y|^2}{(\frac{\delta_p}{2}\vee\gamma_p)(t_1-s)}\Big) \times E\Big(|\phi(s,y)|^p + |Y(s,y)|^p\Big) dy \Bigg)^{\frac{2}{p}} ds\,. \tag{3.7}$$

Putting (3.6) and (3.7) into (3.5) and using the fact that $\alpha < \frac{p-4}{4p}$ we obtain that

$$E\,|X(t,x)|^p \le C \int_0^t (t_0-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}} \Bigg(\int_{\mathbb{R}} \exp\Big(-\frac{|x-y|^2}{c\,(t_1-s)}\Big) \times E\Big(|\phi(s,y)|^p + |Y(s,y)|^p\Big) dy \Bigg) ds$$

where $c = \max(\delta_p, \gamma_p)$.

Now we make t tend to t_1 and use Fatou's lemma to obtain

$$E\,|Y(t_1,x)|^p \le C \int_0^{t_1} (t_1-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}} \Bigg(\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{c\,(t_1-s)}}\, E\Big(|\phi(s,y)|^p + |Y(s,y)|^p\Big) dy \Bigg) ds\,.$$

Using an iterative procedure we have that

$$E\,|Y(t,x)|^p \le C \int_0^t (t-s)^{-\alpha-\frac{3}{4}-\frac{1}{p}} \Big(\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{c\,(t-s)}}\, E\,|\phi(s,y)|^p dy \Big) ds\,,$$

for all $0 \le t \le t_0$. Finally for any fixed $t \in [0,T)$ letting the parameter t_0 in the definition of $Y(t,x)$ to converge to t and integrating with respect to the measure $e^{-M|x|}\,dx$ leads to the desired result. $\square$

Let us now consider the following additional condition over the stochastic kernel $p(s,t,y,x)$:

(H9)$_M$ There exists a constant $C_M > 0$ such that

$$\sup_{0\le r\le T} E\Bigg(\sup_{s\ge r} \int_{\mathbb{R}} e^{-M|x|}\, p(r,s,y,x)\,dx \Bigg) \le C_M e^{-M|y|}\,.$$

We will denote by $L^p_M(\mathbb{R})$ the space of functions $f:\mathbb{R}\to\mathbb{R}$ such that $\int_{\mathbb{R}} e^{-M|x|}|f(x)|^p dx < \infty$.

Theorem 3.3 *Fix $p > 8$ and $M > 0$. Let $\phi = \{\phi(s,y), (s,y) \in I^T\}$ be an adapted process in $L^p_M(I^T \times \Omega)$. Assume that $p(s,t,y,x)$ is a stochastic kernel satisfying conditions (H1) to (H8) and (H9)$_M$. Then for all $t \in [0,T]$, the process $\{p(s,t,y,x)\,\phi(s,y)\mathbb{1}_{[0,t]}(s), (s,y) \in I^T\}$ belongs to* Dom δ *for almost all $x \in \mathbb{R}$, and the stochastic process*

$$Z = \Big\{ Z_t = \int_{I^t} p(s,t,y,\cdot)\ \phi(s,y)\,dW_{s,y}, t \in [0,T] \Big\}$$

posseses a continuous version with values in $L^p_M(\mathbb{R})$. *Moreover,*

$$E\left(\sup_{0\le t\le T} \int_{\mathbb{R}} e^{-M|x|} \left| \int_{I^t} p(s,t,y,x)\ \phi(s,y)\ dW_{s,y}\right|^p dx\right) \le C \int_{I^T} e^{-M|y|}\ E\,|\phi(s,y)|^p\, ds\, dy\,, \tag{3.8}$$

for some positive constant C *depending only on* T, p, M, $C_{1,p}$, $C_{2,p}$, C_f, γ_p, δ_p *and* C_M.

Proof. Using the same arguments as in the proof of Theorem 3.2 we can assume that the process ϕ is simple. Fix $0<\alpha<\frac{p-4}{4p}$ and define

$$Y(r,u) = \int_{I^r} (r-s)^{-\alpha}\ p(s,r,y,u)\ \phi(s,y)\ dW_{s,y}\,.$$

As $p(s,t,y,x) = C_\alpha \int_{I_s^t} (t-r)^{\alpha-1}\,(r-s)^{-\alpha}\ p(s,r,y,u)\ p(r,t,u,x)\ dr\,du$, with $C_\alpha = \frac{\sin\pi\alpha}{\pi}$ it is easy to show that

$$Z_t(x) = C_\alpha \int_{I^t} (t-r)^{\alpha-1}\ p(r,t,u,x)\ Y(r,u)\,dr\,du\,.$$

Then we have that for any $t<t'$ and $\alpha\in\left(\frac{1}{p},\ \frac{p-4}{4p}\right)$

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-M|x|}\ |Z_{t'}(x)-Z_t(x)|^p\,dx\\
&\quad\le \int_{\mathbb{R}} e^{-M|x|} \left| \int_{I_t^{t'}} (t-r)^{\alpha-1}\ p(r,t',u,x)\ Y(r,u)\,dr\,du\right|^p dx\\
&\qquad + \int_{\mathbb{R}} e^{-M|x|} \left| \int_{I_t} (t-r)^{\alpha-1}\ [p(r,t',u,x)-p(r,t,u,x)]\ Y(r,u)\,dr\,du\right|^p dx\\
&\quad\le C_{\alpha,p}(t'-t)^{\alpha-\frac{1}{p}} \int_{I^{t'}} e^{-M|x|} \left| \int_{\mathbb{R}} p(r,t',u,x)\ Y(r,u)\,du\right|^p dr\,dx\\
&\qquad + C_{\alpha,p}\ t^{\alpha-\frac{1}{p}} \int_{I^t} e^{-M|x|} \left| \int_{\mathbb{R}} [p(r,t',u,x)-p(r,t,u,x)]\ Y(r,u)\,du\right|^p dr\,dx\\
&\quad\le C_{\alpha,p}(t'-t)^{\alpha-\frac{1}{p}} \int_{I^T} |Y(r,u)|^p \left(\sup_{s\ge r} \int_{\mathbb{R}} e^{-M|x|}\ p(r,s,u,x)\,dx\right) dr\,du\\
&\qquad + C \int_{I^T} e^{-M|x|}\,\mathbb{1}_{[0,t]}(r) \int_{\mathbb{R}} |p(r,t',u,x)-p(r,t,u,x)|\ |Y(r,u)|^p\,du\,dr\,dx\,.
\end{aligned}$$

By dominated convergence, and using hypothesis (H4) both summands in the above expression converges to zero as $|t'-t|\longrightarrow 0$. On the other hand, taking $t=0$ we obtain (3.8). □

We will also need the following L^2-estimate for the Skorohod integral.

Theorem 3.4 *Fix* $M>0$. *Let* $\phi=\{\phi(s,y),\ (s,y)\in I^T\}$ *be an adapted random field in* $L^2_M(I^T\times\Omega)$. *Assume that* $p(s,t,y,x)$ *is a stochastic kernel satisfying conditions (H1) to (H8). Then, for almost all* $(t,x)\in I^T$, *the process*

$$\{p(s,t,y,x)\ \phi(s,y)\,\mathbb{1}_{[0,t]}(s),\quad (s,y)\in I^T\}$$

belongs to $\operatorname{Dom}\delta$ *and we have that*

$$\begin{aligned}\int_{\mathbb{R}} e^{-M|x|}\, E\Big|\int_{I^t} p(s,t,y,x)\ \phi(s,y)\, dW_{s,y}\Big|^2\, dx \\ \le C \int_0^t (t-s)^{-\frac{3}{4}} \ \Big(\int_{\mathbb{R}} e^{-M|x|}\, E\,|\phi(s,y)|^2\, dy\Big)\, ds\,,\end{aligned} \tag{3.9}$$

for some positive constant C *depending only on* T, M, $C_{1,2}$, $C_{2,2}$, C_f, δ_2 *and* γ_2.

Proof. Using the same arguments as in the proof of Theorem 3.2 we can assume that $\phi \in \mathcal{S}^a$. Fix $(t,x) \in I^T$ and define

$$\begin{aligned}B_{t,x}(s,y) &= p(s,t,y,x)\ \phi(s,y)\, \mathbb{1}_{[0,t]}\,(s)\\ X(t,x) &= \int_{I^t} B_{t,x}\,(s,y)\, dW_{s,y}\,.\end{aligned}$$

By the isometry properties of the Skorohod integral (Proposition 2.2) we have that

$$\begin{aligned}\int_{\mathbb{R}} e^{-M|x|}\, E\,|X(t,x)|^2 dx &= \int_{\mathbb{R}} e^{-M|x|}\Big(\int_{I^t} E\,|B_{t,x}(s,y)|^2 ds\, dy\Big)dx\\ &+ 2\int_{\mathbb{R}} e^{-M|x|}\, E\Big[\int_{I^t} B_{t,x}(s,y)\,\Big(\int_{I^s} D_{s,y}\, B_{t,x}(r,z)\, dW_{r,z}\Big)\, ds\, dy\Big] dx\\ &= I_1 + 2\, I_2\,.\end{aligned} \tag{3.10}$$

By hypothesis (H7) we have that

$$\begin{aligned}I_1 \ &\le \textstyle\int_{\mathbb{R}} e^{-M|x|}\Bigg(\int_{I^t} E\,|V_2(s,t,x)|^2 \exp\Big(-\frac{2\ ,|x-y|^2}{\delta_2(t-s)}\Big)\ E\,|\phi(s,y)|^2 ds\, dy\Bigg)dx\\ &\le C_{1,2}^2 \textstyle\int_{I^t}(t-s)^{-1}\ E\,|\phi(s,y)|^2\Bigg(\int_{\mathbb{R}} \exp\Big(-M|x| - \frac{2\,|x-y|^2}{\delta_2(t-s)}\Big)\, dx\Bigg)ds\, dy\\ &\le C_{1,2}^2 K_1 \textstyle\int_{I^t}(t-s)^{-\frac{1}{2}}\ e^{-M|y|}\ E\,|\phi(s,y)|^2 ds\, dy.\end{aligned} \tag{3.11}$$

On the other hand, using the same arguments as in the proof of Theorem 3.2 it is easy to show that

$$\begin{aligned}I_2 \ &= E \textstyle\int_0^t \int_{\mathbb{R}^2} e^{-M|x|}\ p(s,t,y,x)\ \phi(s,y)\ \Big(\int_{\mathbb{R}} D^-_{s,y}\ p(s,t,u,x)\ X(s,u)\, du\Big)\, dx\, dy\, ds\\ &= E\ \textstyle\int_0^t \int_{\mathbb{R}^3} e^{-M|x|}\ \ p(s,t,y,x)\mathbb{D}^-_{s,y}\ p(s,t,u,x)\ \phi(s,y)\, X(s,u)\, dx\, dy\, du\, ds\\ &\le \textstyle\int_{I^t} e^{-M|x|}\ \Big(E\,|V_2(s,t,x)|^2\Big)^{\frac{1}{2}}\ \Big(E\,|U_2(s,t,x)|^2\Big)^{\frac{1}{2}}\\ &\quad\times\Big[\textstyle\int_{\mathbb{R}^2} \exp\Big(-\frac{|x-y|^2}{\delta_2(t-s)} - \frac{|x-u|^2}{\gamma_2(t-s)}\Big)\ f(u,y)\ E\,|\phi(s,y)\ X(s,u)|\, dy\, du\Big]\, ds\, dx\\ &\le C_{1,2}C_{2,2}\ \textstyle\int_{I^t}\ e^{-M|x|}(t-s)^{-\frac{3}{2}}\Bigg(\int_{\mathbb{R}^2}\ E|X(s,u)|^2\ f(u,y)^2\ e^{-\frac{|x-u|^2}{\gamma_2(t-s)}}\ du\, dy\Bigg)^{\frac{1}{2}}\\ &\quad\times\Bigg(\textstyle\int_{\mathbb{R}^2}\ E\,|\phi(s,y)|^2\ e^{-\frac{|x-u|^2}{\gamma_2(t-s)} - \frac{2\,|x-y|^2}{\delta_2(t-s)}}\ du\, dy\Bigg)^{\frac{1}{2}}\ ds\, dx\\ &\le C\ \textstyle\int_{I^t}\ e^{-M|x|}(t-s)^{-\frac{5}{4}}\Bigg(\int_{\mathbb{R}^2}\ e^{-\frac{|x-y|^2}{(\frac{\delta}{2}\vee\gamma_2)(t-s)}}\ \Big(E\,|X(s,y)|^2 + E\,|\phi(s,y)|^2\Big)dy\Bigg)ds\, dx\\ &\le C\ \textstyle\int_0^t(t-s)^{-\frac{3}{4}}\Bigg(\int_{\mathbb{R}}\ e^{-M|y|}\ \Big(E\,|X(s,y)|^2 + E\,|\phi(s,y)|^2\Big)dy\Bigg)ds\,.\end{aligned} \tag{3.12}$$

Now, substitutting (3.12) and (3.11) into (3.10) and using an iteration argument the result follows. □

Using the same arguments it is easy to show the following result.

Corollary 3.5 *Let* $\phi = \{\phi(s,y),\ (s,y) \in I^T\}$ *be an adapted process in* $L^2(I^T \times \Omega)$*. Assume that* $p(s,t,y,x)$ *is a random function satisfying hypotheses (H1) to (H8). Then, for almost all* $(t,x) \in I^T$*, the process*

$$\{p(s,t,y,x)\ \phi(s,y)\, \mathbb{1}_{[0,t]}(s),\ (s,y) \in I^T\}$$

belongs to $\operatorname{Dom}\delta$ *and*

$$\int_{\mathbb{R}} E\left|\int_{I^t} p(s,t,y,x)\ \phi(s,y)\ dW_{s,y}\right|^2 dx \le C\int_0^t (t-s)^{-\frac{3}{4}}\Big(\int_{\mathbb{R}} E\,|\phi(s,y)|^2 dy\Big)\, ds\,, \quad (3.13)$$

for some positive constant C *depending only on* T*,* $C_{1,2}$*,* $C_{2,2}$*,* C_f*,* δ_2 *and* γ_2*..*

4 Existence and uniqueness of solution for stochastic evolution equations with a random kernel

Our purpose in this section is to prove the existence and uniqueness of solution for the following anticipating stochastic evolution equation

$$u(t,x) = \int_{\mathbb{R}} p(0,t,y,x)\ u_0(y)\, dy + \int_{\mathbb{R}} p(s,t,y,x)\ F(s,y,u(s,y))\ dW_{s,y}\,, \qquad (4.1)$$

where $p(s,t,y,x)$ is a stochastic kernel satisfying conditions (H1) to (H8) and $(\mathrm{H9})_{\mathrm{M}}$, $u_0 : \mathbb{R} \to \mathbb{R}$ is the initial condition and $F : [0,T] \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$ is a stochastic random field. Let us consider the following hypotheses.

(F1) F is measurable with respect to the σ-field $\mathcal{B}([0,t] \times \mathbb{R}^2) \otimes \mathcal{F}_{0,t}$, when restricted to $[0,t] \times \mathbb{R}^2 \times \Omega$, for each $t \in [0,T]$.

(F2) For all $t \in [0,T],\ x,y,z \in \mathbb{R}$

$$|F(t,y,x) - F(t,y,z)| \le C\,|x-z|\,,$$

for some positive constant C.

$(\mathrm{F3})^{\mathrm{p}}_{\mathrm{M}}$ For all $t \in [0,T], x \in \mathbb{R}$,

$$|F(t,x,0)| \le h(x)\,,$$

for some $h \in L^p_M(\mathbb{R})$.

We are now in a position to prove the main result of this paper.

Theorem 4.1 *Fix* $M > 0$ *and* $p > 8$*. Let* u_0 *be a function in* $L^p_M(\mathbb{R})$*. Consider an adapted random field* $F(s,y,x)$ *satisfying conditions (F1) to* $(F3)^p_M$ *and a stochastic kernel* $p(s,t,y,x)$ *satisfying hypotheses (H1) to (H8) and* $(H9)_M$*. Then, there exists an unique adapted random field* $u = \{u(t,x),\ (t,x) \in I^T\}$ *in* $L^2_M(I^T \times \Omega)$ *that is solution of (4.1). Moreover,* *(i)* $\{u(t,\cdot),\ t\in[0,T]\}$ *is continuous a.s. as a process with values in* $L^p_M(\mathbb{R})$ *and*

$$E\Big(\sup_{0\le t\le T}\int_R e^{-M|x|}\,|u(t,x)|^p\,dx\Big)\le C, \tag{4.2}$$

for some positive constant C *depending only on* T, p, M, $C_{1,p}$, $C_{2,p}$, C_f, δ_p, γ_p *and* C_M.

(ii) If, moreover, u_0 *and* h *belong to* $L^2(\mathbb{R})$*, then* $u\in L^2(I^T\times\Omega)$.

Proof of existence and uniqueness. Suppose that u and v are two adapted solutions of (4.1) in $L^2_M(I^T\times\Omega)$, for some $M>0$. Then, for every $t\in[0,T]$ we can write

$$\begin{aligned}&\int_{\mathbb{R}} e^{-M|x|}\,E\,|u(t,x)-v(t,x)|^2\,dx\\&\quad=\int_{\mathbb{R}} e^{-M|x|}\,E\,\Big|\int_{I^t} p(s,t,y,x)\,\Big(F(s,y,u(s,y))-F(s,y,v(s,y))\Big)\,dW_{s,y}\Big|^2\,dx\,.\end{aligned}$$

By Theorem 3.4 and the Lipschitz condition on F we have that

$$\int_{\mathbb{R}} e^{-M|x|}\,E\,|u(t,x)-v(t,x)|^2\,dx\le\int_0^t(t-s)^{-\frac34}\Big(\int_{\mathbb{R}} e^{-M|y|}\,E\,|u(s,y)-v(s,y)|^2\,dy\Big)\,ds\,.$$

Appliying an iteration argument we obtain that

$$\int_{\mathbb{R}} e^{-M|x|}\,E\,|u(t,x)-v(t,x)|^2\,dx\le C\int_0^t\Big(\int_{\mathbb{R}} e^{-M|y|}\,E\,|u(s,y)-v(s,y)|^2\,dy\Big)\,ds\,,$$

from where we deduce that $\int_{\mathbb{R}} e^{-M|x|}\,E\,|u(t,x)-v(t,x)|^2\,dx=0$. Consider now the Picard approximations

$$\begin{cases}u^0(t,x)=\displaystyle\int_{\mathbb{R}} p(0,t,y,x)\,u_0(y)\,dy\\ u^n(t,x)=\displaystyle\int_{\mathbb{R}} p(0,t,y,x)\,u_0(y)\,dy+\int_{I^t} p(s,t,y,x)\,F(s,y,u^{n-1}(s,y))\,dW_{s,y}\,.\end{cases}$$

By hypothesis (H1), $u^0(t,x)$ is adapted. On the other hand, using hypotheses (H3) and $(\mathrm{H9})_{\mathrm{M}}$ we have that

$$\begin{aligned}&E\bigg(\int_{\mathbb{R}} e^{-M|x|}\Big|\int_{\mathbb{R}} p(0,t,y,x)\,u_0(y)\,dy\Big|^2\,dx\bigg)\\&\le E\bigg(\int_{\mathbb{R}}|u_0(y)|^2\Big(\int_{\mathbb{R}} e^{-M|x|}p(0,t,y,x)\,dx\Big)\,dy\bigg)\\&\le\int_{\mathbb{R}} e^{-M|y|}\,|u_0(y)|^2\,dy\,.\end{aligned}$$

Now, using induction on n and Theorem 3.4 it is easy to show that u^n is adapted and belongs to $L^2_M(I^T\times\Omega)$. Using a recurrence argument we can easily show that

$$\sum_{n=0}^{\infty}E\bigg(\int_{\mathbb{R}} e^{-M|x|}\,|u^{n+1}(t,x)-u^n(t,x)|^2\,dx\bigg)<\infty\,,$$

and the limit u of the sequence u^n provides the solution.

Proof of (i) Using the same arguments as in the proof of the existence we can see that the solution u belongs to $L^p_M(I^T \times \Omega)$. Now we have to show that the following two terms are a.s. continuous in $L^p_M(\mathbb{R})$:

$$\begin{aligned} A_1(t) &= \int_{\mathbb{R}} p(0,t,y,x)\ u_0(y)\,dy \\ A_2(t) &= \int_{I^t} p(s,t,y,x)\ F(s,y,u(s,y))\,dW_{s,y}\,. \end{aligned}$$

In order to prove the continuity of A, note that hypothesis $(\mathrm{H9})_{\mathrm{M}}$ implies that, for all φ and ϕ in $L^p_M(\mathbb{R})$

$$\begin{aligned} &E\Big(\sup_{0\le t\le T}\int_{\mathbb{R}} e^{-M|x|}\ \Big|\int_{\mathbb{R}} p(0,t,y,x)\ \big(\varphi(y)-\phi(y)\big)dy\Big|^p dx\Big) \\ &\qquad \le E\left(\sup_{0\le t\le T}\int_{\mathbb{R}} |\varphi(y)-\phi(y)|^p\ \Big(\int_{\mathbb{R}} e^{-M|x|}p(0,t,y,x)\,dx\Big)dy\right) \\ &\qquad \le \int_{\mathbb{R}} e^{-M|y|}\ |\varphi(y)-\phi(y)|^p\,dy\,. \end{aligned}$$

Hence, we can assume that u_0 is a smooth function with compact support. In this case

$$\begin{aligned} &\int_{\mathbb{R}} e^{-M|x|}\ \Big|\int_{\mathbb{R}} \Big(p(0,t+\varepsilon,y,x)-p(0,t,y,x)\Big)\ u_0(y)\,dy\Big|^p dx \\ &\qquad \le 2^{p-1}\,||u_0||_\infty \int_{\mathbb{R}\times K} e^{-M|x|}\ |p(0,t+\varepsilon,y,x)-p(0,t,y,x)|\,dx\,dy\,, \end{aligned}$$

which tends to zero by hypotheses (H4) and $(\mathrm{H9})_M$. The continuity of A_2 is an immediate consequence of Theorem 3.3. Finally, using a recurrence argument it is easy to prove that the Picard aproximations u^n satisfy that

$$\sum_{n=0}^{\infty} E\left(\sup_{0\le t\le T}\int_{\mathbb{R}} e^{-M|x|}\ |u^{n+1}(t,x)-u^n(t,x)|^p\,dx\right) < \infty\,,$$

from where (4.2) follows. □

Proof of existence in $L^2(I^T\times\Omega)$. Using hypothesis (H7) we have that

$$\begin{aligned} E\int_{I^T}\ \Big|\int_{\mathbb{R}} p(0,t,y,x)\ u_0(y)\,dy\Big|^2 dt\,dx &\le E\int_{I^T} |u_0(y)|^2\Big(\int_{\mathbb{R}} p(0,t,y,x)\,dx\Big)dt\,dy \\ &\le T\int_{\mathbb{R}} |u_0(y)|^2\,dy\,. \end{aligned}$$

Using now induction on n and Corollary 3.5 it is easy to show that $\int_{I^T} E|u^n(t,x)|^2\,dt\,dx < \infty$ and that u^n is a Cauchy sequence in $L^2(I^T\times\Omega)$. This implies that u belongs to $L^2(I^T\times\Omega)$.

For every $p\ge 1,\ p\ge\varepsilon>0$ and $K>0$ we denote by $W^{p,\varepsilon}(K)$ the set of continuous functions $f:[-K,K]\to\mathbb{R}$ such that

$$||f||^p_{p,\varepsilon,K} := \int_{[-K,K]^2} \frac{|f(x)-f(z)|^p}{|x-z|^{2+\varepsilon}}\,dx\,dz < \infty\,.$$

Notice that if $f \in W^{p,\varepsilon}(K)$, then f is Hˆlder continuous in $[-K,K]$ with order ε/p.

Now our purpose is to prove that, under some suitable hypotheses, the solution $u(t,\cdot)$ belongs to $W^{p,\varepsilon}(K)$, for some $p \geq 1,\ p \geq \varepsilon > 0$ and all $K > 0$.

Theorem 4.2 *Fix $p > 4$ and $M > 0$. Let u_0 be a function in $L^p_M(\mathbb{R})$. Consider an adapted random field $F(s,y,x)$ satisfying hypotheses (F1) to $(F3)^p_M$ and a stochastic kernel $p(s,t,y,x)$ satisfying (H1) to (H8) and $(H9)_M$. Then, the solution $u(t,x)$ constructed in Theorem 4.1 belongs a.s., as a function in x, to $W^{p,\varepsilon}(K)$, for all $p > 8,\ \varepsilon < \frac{p}{2} - 3$ and $K > 0$.*

Proof. We have to show that the following two terms belong to $W^{p,\varepsilon}(K)$:

$$B_1(x) = \int_{\mathbb{R}} p(0,t,y,x)\ u_0(y)\ dy$$
$$B_2(x) = \int_{I^t} p(s,t,y,x)\ F\Big(s,y,u(s,y)\Big)\, dW_{s,y}.$$

Using Minkowski's inequality we have that

$$\begin{aligned}
&E\int_{[-K,K]^2} \frac{|B_1(x)-B_1(z)|^p}{|x-z|^{2+\varepsilon}}\ dx\,dz \\
&= \int_{[-K,K]^2} |x-z|^{-2-\varepsilon}\ E\Big|\int_{\mathbb{R}} [p(0,t,y,x)-p(0,t,y,z)]\ u_0(y)\,dy\Big|^p dx\,dz\,, \\
&\leq \int_{[-K,K]^2} |x-z|^{-2-\varepsilon}\Big(\int_{\mathbb{R}} \|p(0,t,y,x)-p(0,t,y,z)\|_p\ |u_0(y)|\,dy\Big)^p dx\,dz\,.
\end{aligned}$$

Taking into account estimate (5.10) and the same arguments as in [4], pg. 17 it is easy to show that $\forall\, 0 \leq s < t \leq T\,,\ x,y,z \in \mathbb{R}\,,\ \beta \in [0,1]$ and $p \geq 1\,$,

$$\begin{aligned}
&\|p(s,t,y,x)-p(s,t,y,z)\|_p \\
&\quad \leq K\,|x-z|^{\beta}(t-s)^{-\frac{1}{2}(\beta+1)}\Big[\exp\Big(-\frac{|y-x|^2}{c(t-s)}\Big)+\exp\Big(-\frac{|y-z|^2}{c(t-s)}\Big)\Big]\,,
\end{aligned} \tag{4.3}$$

for some $K,c>0$. This gives us that, taking $\beta = 1$

$$\begin{aligned}
&E\int_{[-K,K]^2} \frac{|B_1(x)-B_1(z)|^p}{|x-z|^{2+\varepsilon}}\ dx\,dz \\
&\leq Ct^{-p}\int_{[-K,K]^2} |x-z|^{p-2-\varepsilon}\Bigg(\int_{\mathbb{R}} \exp\Big(-\frac{|y-x|^2}{c\,t}\Big)\ |u_0(y)|\,dy\Bigg)^p dx\,dz \\
&\leq Ct^{-p}\int_{[-K,K]} \Bigg(\int_{\mathbb{R}} \exp\Big(-\frac{|y-x|^2}{c\,t}\Big)\ |u_0(y)|\,dy\Bigg)^p dx \\
&\leq Ct^{-\frac{p}{2}-\frac{1}{2}}\int_{\mathbb{R}} e^{-M|y|}\ |u_0(y)|^p\,dy < \infty\,,
\end{aligned}$$

which gives us that $B_1(x)$ belongs a.s. to $W^{p,\varepsilon}(K)$. On the other hand, as in the proof of Theorem 3.3 we can write for $\alpha \in (0,\ \frac{p-4}{4p})$

$$B_2(x) = C_\alpha \int_{I^t} (t-r)^{\alpha-1} p(r,t,u,x)\ Y(r,u)\,dr\,du\,,$$

where

$$Y(r,u) := \int_{I^r} (r-s)^{-\alpha}\ p(s,r,y,u)\ F(s,y,u(s,y))\, dW_{s,y}\,.$$

This gives us that

$$\begin{aligned}
&E \int_{[-K,K]^2} \frac{|B_2(x)-B_2(z)|^p}{|x-z|^{2+\varepsilon}}\, dx\, dz \\
&= C\ E \int_{[-K,K]^2} |x-z|^{-2-\varepsilon} \Big| \int_{I^t} (t-r)^{\alpha-1} [p(r,t,u,x)-p(r,t,u,z)]\ Y(r,u)\, dr\, du \Big|^p dx\, dz\,.
\end{aligned}$$

Using Minkowski's inequality and the estimate (4.3) we obtain that

$$\begin{aligned}
&E \int_{[-K,K]^2} \frac{|B_2(x)-B_2(z)|^p}{|x-z|^{2+\varepsilon}}\, dx\, dz \\
&\le C \int_{[-K,K]^2} |x-z|^{\beta p-2-\varepsilon} \bigg(\int_{I^t} (t-r)^{\alpha-\frac{3}{2}-\frac{\beta}{2}}\ \exp\Big(-\frac{|u-x|^2}{c(t-r)}\Big)\ \|Y(r,u)\|_p\, dr\, du\bigg)^p dx\, dz \\
&\le C \int_{[-K,K]} \bigg[\int_0^t (t-r)^{\alpha-\frac{3}{2}-\frac{\beta}{2}} \Big(\int_{\mathbb{R}} \exp\Big(-\frac{|u-x|^2}{c(t-r)}\Big)\ \|Y(r,u)\|_p\, du\Big)\, dr\bigg]^p dx \\
&= C \bigg[\int_0^t (t-r)^{\alpha-\frac{3}{2}-\frac{\beta}{2}} \bigg(\int_{[-K,K]} \Big(\int_{\mathbb{R}} \exp\Big(-\frac{|u-x|^2}{c(t-r)}\Big)\ \|Y(r,u)\|_p\, du\Big)^p dx\bigg)^{\frac{1}{p}} dr\bigg]^p .
\end{aligned}$$

Using Holder's inequality we obtain

$$\begin{aligned}
&E \int_{[-K,K]^2} \frac{|B_2(x)-B_2(z)|^p}{|x-z|^{2+\varepsilon}}\, dx\, dz \\
&\le C \bigg(\int_0^t (t-r)^{\alpha-1-\frac{\beta}{2}} \Big(\int_{\mathbb{R}} e^{-M|u|}\ E|Y(r,u)|^p\, du\Big)^{\frac{1}{p}} dr\bigg)^p \\
&\le C\ t^{p\alpha-\beta\frac{p}{2}-1} \int_{I^t} e^{-M|u|}\ E\,|Y(r,u)|^p\, dr\, du\,,
\end{aligned}$$

provided $\alpha > \frac{1}{p}+\frac{\beta}{2}$. Finally, from the proof of Theorem 3.2 and the facts that $u_0 \in L^p(\mathbb{R})$ and $\alpha < \frac{p-4}{4p}$ it is easy to show that $\int_{I^t} e^{-M|u|}\ E\,|Y(r,u)|^p\, dr\, du < \infty$, which allows us to complete the proof. We have made use of the following conditions

$$p\,\beta > \varepsilon + 1, \quad \alpha > \frac{1}{p} + \frac{\beta}{2}\,, \quad \alpha < \frac{p-4}{4p}\,.$$

We can easily check that thanks to the fact that $p > 8$ we can take α and β such that these inequalities hold. □

5 Estimates for the heat kernel with white-noise drift

In this section, following the approach of [13] we construct and estimate the backward heat kernel of the random operator $\frac{d^2}{dx^2} + \dot{v}(t,x)\,\frac{d}{dx}$, where $v = \{v(t,x),\ t \in$

$[0,T],\ x\in\mathbb{R})\}$ is a zero mean Gaussian field which is Brownian in time. The differential $\dot v(t,x)dt := v(dt,x)$ is interpreted in the backward ItÙ sense. More precisely, we assume that v can be represented as

$$v(t,x)=\int_{I^t} g(x,y)\ dW_{s,y}\,, \tag{5.1}$$

where $g:\mathbb{R}^2\to\mathbb{R}$ is a measurable function, differentiable with respect to x, satisfying the following condition

$$\sup_x \int_{\mathbb{R}} \Big(g(x,y)^2+\frac{\partial g}{\partial x}(x,y)^2\Big)dy<\infty\,. \tag{5.2}$$

Set $G(x,y)=\int_{\mathbb{R}} g(x,z)\,g(y,z)\,dz$ and let us introduce the following *coercivity* condition:

(C1) $\quad \sum(x):=1-\frac{1}{2}\ G(x,x)\geq\varepsilon>0\,,$ for all $x\in\mathbb{R}$ and for some $\varepsilon>0$.

Let $b=\{b(t),\ t\in[0,T]\}$ be a Brownian motion with variance $2t$ defined on another probability space $(\mathcal{W},\mathcal{G},Q)$. Consider the following *backward stochastic differential equation* on the product probability space $(\Omega\times\mathcal{W},\ \mathcal{F}\times\mathcal{G},\ P\times Q)$:

$$\varphi_{t,s}(x)=x-\int_s^t\int_{\mathbb{R}} g(\varphi_{t,r}(x),y)\ dW_{r,y}+\int_s^t\ \sqrt{\Sigma(\varphi_{t,r}(x))}\ db_r\,. \tag{5.3}$$

Applying Theorems 3.4.1 and 4.5.1 in [7] one can prove that (5.3) has a solution $\varphi=\{\varphi_{t,s}(x),\ 0\leq s\leq t\leq T,\ x\in\mathbb{R}\}$ continuous in the three variables and verifying

$$\varphi_{r,s}\Big(\varphi_{t,r}(x)\Big)=\varphi_{t,s}(x)\,, \tag{5.4}$$

for all $s<r<t,\ x\in\mathbb{R}$.

Then we have the following result

Proposition 5.1 *Let v be a Gaussian random field of the form (5.1) where the function g satisfies the coercivity condition (C1) and assume that g is three times continuously differentiable in x and satisfies*

$$\sup_x\sum_{k=0}^{3}\int_{\mathbb{R}}\ |g^{(k)}\,(x,y)|^2\,dy<\infty\,.$$

Then there is a version of the density

$$p(s,t,y,x)=\frac{Q(\varphi_{t,s}(x)\in dy)}{dy}$$

which satisfies conditions (H1) to (H8) and $(H9)_M$ for each $M>0$.

Proof. Let us denote by δ^b and D^b the divergence and derivative operators with respect to the Brownian motion b. Applying the integration-by-parts formula of Malliavin calculus with respect to the Brownian motion b we obtain

$$p(s,t,y,x)=E_Q\Big(\mathbb{1}_{\{\varphi_{t,s}(x)>y\}}\ H_{t,s}(x)\Big)\,, \tag{5.5}$$

where

$$H_{t,s}(x) = \delta^b \left(\frac{D^b \varphi_{t,s}(x)}{\|D^b \varphi_{t,s}(x)\|^2} \right).$$

Hypothesis (H1) follows easily from the expression (5.5) because $\varphi_{t,s}(x)$ is $\mathcal{F}_{s,t}$-measurable. The fact that $y \mapsto p(s,t,y,x)$ is the probability density of $\varphi_{t,s}(x)$, which has a continuous version in all the variables $x, y \in \mathbb{R},\ 0 \le s < t \le T$, imply (H2), (H3) and (H4). Hypothesis (H5) is a consequence of the flow property (5.4).

Applying the derivative operator to (5.5) yields

$$\begin{aligned} D_{r,z}\ p(s,t,y,x) &= E_Q\Big(\mathbb{1}_{\{\varphi_{t,s}(x)>y\}}\ D_{r,z}\ H_{t,s}(x)\Big) \\ &\quad + E_Q\Big(\mathbb{1}_{\{\varphi_{t,s}(x)>y\}}\ \Psi_{t,s}(x)\Big), \end{aligned} \tag{5.6}$$

where

$$\Psi_{t,s}(x) = \delta^b \left(\frac{D^b\, \varphi_{t,s}(x)}{\|D^b\, \varphi_{t,s}(x)\|^2}\ D_{r,z}\ \varphi_{t,s}(x)\ H_{t,s}(x) \right).$$

Then hypothesis (H6) follows easily from Equation (5.6). Conditions (H7), (H8) and $(\mathrm{H9})_{\mathrm{M}}$ will be proved in the following lemmas. □

Lemma 5.2 *The stochastic kernel $p(s,t,y,x)$ satisfies condition (H7) with the constant $\delta_p = \frac{p}{K}$, for any $K < \frac{1}{4}$.*

Proof. By (5.5) the kernel p can be expressed as

$$p(s,t,y,x) = E_Q\Big(\mathbb{1}_{\{B_{t,s}(x)>y-x\}}\ H_{t,s}(x)\Big) \tag{5.7}$$

$$= E_Q\Big(\mathbb{1}_{\{-B_{t,s}>x-y\}}\ H_{t,s}(x)\Big), \tag{5.8}$$

where $B_{t,s}(x) = \varphi_{t,s}(x) - x$. Since B and $-B$ have the same distribution, it is sufficient to consider the expression in (5.7) and assume that $x \le y$. Using the trivial bound

$$\mathbb{1}_{\{B>a\}} \le \exp\ \frac{(K\,B^2)}{p(t-s)}\ \exp\ -\frac{(K\,a^2)}{p(t-s)}$$

for any $a \ge 0,\ k > 0$ we obtain

$$p(s,t,y,x) \le e^{-\frac{K\,|x-y|^2}{p(t-s)}}\ V_p(s,t,x),$$

where

$$V_p(s,t,x) = E_Q \left(\exp\ \frac{(K\ B_{t,s}(x)^2)}{p(t-s)}\ |H_{t,s}(x)| \right).$$

We only need to calculate $E\,|V_p(s,t,x)|^p$. By Schwartz's inequality

$$E\,|V_p(s,t,x)|^p \le \left(E\ \exp\ \frac{(2K\ B_{t,s}(x)^2)}{(t-s)}\ E\,|H_{t,s}(x)|^{2p} \right)^{\frac{1}{2}}.$$

Note that, if we fix t and let s vary, $B_{t,s}(x)$ becomes a backward martingale with quadratic variation

$$\langle B_{t,\cdot}(x)\rangle_s = \int_s^t \int_{\mathbb{R}} g^2(\varphi_{t,r}(x), y)\, dy\, dr + 2\int_s^t \Sigma\,(\varphi_{t,r}(x))dr$$
$$= (t-s).$$

This gives us that $B_{t,\cdot}(x)$ is a Brownian motion, and then, for any $K < \frac{1}{4}$

$$E\ \exp\left(\frac{2K}{(t-s)}\ B_{t,s}(x)^2\right) = \frac{1}{\sqrt{1-4K}}. \tag{5.9}$$

On the other hand, it is known (see [13], proof of Proposition 10, (5.5)) that

$$\left(E\ |H_{t,s}(x)|^{2p}\right)^{\frac{1}{2}} \le C_p(t-s)^{-\frac{p}{2}},$$

and now the proof is complete. □

Lemma 5.3 *The stochastic kernel* $p(s,t,y,x)$ *satisfies condition (H8) with the constant* $\gamma_p = \frac{p}{K}$, *for any* $K < \frac{1}{4}$.

Proof. We express $D^-_{s,z}\ p(s,t,y,x)$ as in [13] as

$$D^-_{s,z}\ p(s,t,y,x) = -\frac{\partial}{\partial y}\ \Big[p(s,t,y,x)\ g(y,z)\Big]$$
$$= -\frac{\partial p}{\partial y}\ (s,t,y,x)\ g(y,z) - p(s,t,y,x)\ \frac{\partial g}{\partial y}\ (y,z)\,.$$

Since g and $\frac{\partial g}{\partial y}$ satisfy condition (H8) (ii) and p satisfies the bound (H7), we only need to show

$$\Big|\frac{\partial p}{\partial y}\ (s,t,y,x)\Big| \le U_p(s,t,x)\ \exp\Big(-\frac{|x-y|^2}{\gamma_p(t-s)}\Big), \tag{5.10}$$

where $||U_p(s,t,x)||_{L^r(\Omega)} \le C_p(t-s)^{-1}$. Now taking the derivative $\frac{\partial}{\partial y}$ inside the formula (5.5) for p and integrating by parts we obtain

$$\frac{\partial p}{\partial y}\ (s,t,y,x) = E_Q\left(\mathbb{1}_{\{B_{t,s}(x)>y-x\}}\ H'_{t,s}(x)\right),$$

where

$$H'_{t,s}(x) = \delta^b\left(\frac{D^b\,\varphi_{b,s}(x)}{\|D^b\,\varphi_{t,s}(x)\|^2}\ H_{t,s}(x)\right).$$

The proof of Proposition 11 in [13] indicates that $\|H'_{b,s}(x)\|_q \le C_q(t-s)^{-1}$ for all $q \ge 1$. Therefore, the estimates on $E\ \exp\frac{2K\,B^2_{t,s}(x)}{(t-s)}$ from the proof of Lemma 5.2 yield the lemma. □

Lemma 5.4 *The stochastic kernel* $p(s,t,y,x)$ *satisfies condition* $(H9)_M$ *for all* $M > 0$.

Proof. By Equation (4.6) in [13] we know that

$$p(s,t,y,x)\,dx = q(s,t,y,x)\,dx + \int_{I_s^t} \Big[\int_{\mathbb{R}} g(z,y)\ \frac{\partial p}{\partial z}\ (s,r,y,z)\, q(r,t,z,x)\,dz \Big]\ dW_{r,y},$$

where $q(s,t,y,x) := \frac{1}{2\sqrt{\pi(t-s)}}\ \exp\Big(-\frac{|y-x|^2}{4\sqrt{t-s}}\Big)$. This gives us that

$$\int_{\mathbb{R}} e^{-M|x|}\ p(s,t,y,x)\,dx = \int_{\mathbb{R}} e^{-M|x|}\ q(s,t,y,x)\,dx$$
$$+ \int_{I_s^t} \Big[\int_{\mathbb{R}} g(z,y)\ \frac{\partial p}{\partial z}\ (s,r,y,z) \Big(\int_{\mathbb{R}} e^{-M|x|}\ q(r,t,z,x)\,dx \Big) dz \Big]\ dW_{r,y}$$
$$=: T_1 + T_2.$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}} e^{-M|x|}\ q(s,t,z,x)\,dx &= e^{-M|z|} + \int_r^t \Big(\int_{\mathbb{R}} e^{-M|x|}\ \frac{\partial q}{\partial \theta}\ (r,\theta,z,x)\,dx \Big) d\theta \\ &= e^{-M|z|} + \frac{M^2}{2}\ \int_r^t \Big(\int_{\mathbb{R}} e^{-M|x|}\ q(r,\theta,z,x)\,dx \Big) d\theta \\ &\quad - M \int_r^t q(r,\theta,z,0)\,d\theta\,.\end{aligned}$$

Fubini's stochastic theorem allows us then to write

$$\begin{aligned}T_2 &= \int_{I_s^t} \Bigg[\int_{\mathbb{R}} g(z,y)\ \frac{\partial p}{\partial z}\ (s,r,y,z)\ e^{-M|z|}\,dz \Bigg] dW_{dr,dy} \\ &+ \int_s^t \Bigg[\int_{I_s^\theta} \Bigg(\int_{\mathbb{R}} g(z,y)\ \frac{\partial p}{\partial z}\ (s,r,y,z) \Big(\int_{\mathbb{R}} e^{-M|x|}\ q(r,\theta,z,x)\,dz \Big) dW_{r,y} \Bigg) \Bigg] d\theta \\ &+ \int_s^t \Bigg[\int_{I_s^\theta} \Bigg(\int_{\mathbb{R}} g(z,y)\ \frac{\partial p}{\partial z}\ (s,r,y,z)\ q(r,\theta,z,0)\,dz \Bigg) dW_{r,y} \Bigg] d\theta\,.\end{aligned}$$

From (4.6) in [13] it follows that

$$\begin{aligned}T_2 &= \int_{I_s^t} \Big[\int_{\mathbb{R}} g(z,y)\ \frac{\partial p}{\partial z}\ (s,r,y,z)\ e^{-M|z|}\,dz \Big] dW_{r,y} \\ &+ \int_s^t \Big(\int_{\mathbb{R}} e^{-M|x|}\ \Big[p(s,\theta,y,x) - q(s,\theta,y,x) \Big]\,dx \Big)\,d\theta \\ &+ \int_s^t \Big[p(s,\theta,y,0) - q(s,\theta,y,0) \Big]\,d\theta\,.\end{aligned}$$

Using integration-by-parts formula it follows that

$$
\begin{aligned}
T_2 = &-M \int_{I_s^t} \Big[\int_{\mathbb{R}} g(z,y)\; p(s,r,y,z)\; e^{-M|z|}\; sg(z)\, dz \Big] dW_{r,y} \\
&+ \int_{I_s^t} \Big[\int_{\mathbb{R}} \frac{\partial g}{\partial z}\;(z,y)\; p(s,r,y,z)\; e^{-M|z|}\, dz \Big] dW_{r,y} \\
&+ \int_s^t \Big(\int_{\mathbb{R}} e^{-M|x|}\; \Big[p(s,\theta,y,x) - q(s,\theta,y,x) \Big]\, dx \Big)\, d\theta \\
&+ \int_s^t \Big[p(s,\theta,y,0) - q(s,\theta,y,0) \Big]\, d\theta\,.
\end{aligned}
$$

It is easy to show that for all $M > 0$,

$$
\begin{aligned}
&\int_s^T E\,|p(s,\theta,y,0)|\, d\theta + \int_s^T q(s,\theta,y,0)\, d\theta + \left(E \Big| \int_{\mathbb{R}} e^{-M|x|}\; p(s,\theta,y,x)\, dx \Big|^2 \right)^{\frac{1}{2}} \\
&\quad \le C_{M,T}\; e^{-M|y|}.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
E\left(\sup_{0\le t\le T} \int_{\mathbb{R}} e^{-M|x|}\; p(s,t,y,x)\, dx \right) &\le C_{M,T} \Bigg\{ e^{-M|y|} \\
&+ \left(E \int_{I_s^T} \Big(\int_{\mathbb{R}} \frac{\partial g}{\partial z}\;(z,y)\; p(s,r,y,z)\; e^{-M|z|}\, dz \Big)^2\, dr\, dy \right)^{\frac{1}{2}} \\
&+ \left(E \int_{I_s^T} \Big(\int_{\mathbb{R}} g(z,y)\; p(s,r,y,z)\; e^{-M|z|}\; sg(z)\, dz \Big)^2\, dr\, dy \right)^{\frac{1}{2}} \Bigg\} \\
&\le C_{M,T} \left\{ e^{-M|y|} + \Big(E \int_s^T \; \Big| \int_{\mathbb{R}} p(s,r,y,z)\; e^{-M|z|}\, dz \Big|^2\, dr \Big)^{\frac{1}{2}} \right\} \\
&\le C_{M,T}\; e^{-M|y|},
\end{aligned}
$$

which gives us $(\mathrm{H9})_{\mathrm{M}}$. Now the proof is complete. □

6 Equivalence of evolution and weak solutions

Assume the notations of Section 5. By (4.15) in [13] we know that $p(s,t,y,x)$ is the fundamental solution (in the variables t and x) of the equation

$$
du_t = \frac{\partial^2 u}{\partial x^2}\;(t,x)\, dt + v(dt,x)\; \frac{\partial u}{\partial x}\;(t,x)\,. \tag{6.1}
$$

Our purpose in this section is to study the following stochastic partial differential equation

$$
d\, u_t = \frac{\partial^2 u}{\partial x^2}\;(t,x)\, dt + v(dt,x)\; \frac{\partial u}{\partial x}\;(t,x) + F(t,x,u(t,x))\; \frac{\partial^2 W}{\partial t\, \partial x}\,, \tag{6.2}
$$

with initial condition $u_0 : \mathbb{R} \to \mathbb{R}$. Let us introduce the following definition.

Definition 6.1 *Let $u = \{u(t,x),\ (t,x) \in I^t\}$ be an adapted process. We say that u is a weak solution of (6.2) if for every $\psi \in \mathcal{C}_K^\infty(\mathbb{R})$ and $t \in [0,T]$ we have*

$$\begin{aligned}\int_{\mathbb{R}} \psi(x)\ u(t,x)\,dx = &\int_{\mathbb{R}} \psi(x)\ u_0(x)\,dx + \int_{I^t} \psi''(x)\ u(s,x)\,ds\,dx \\ &- \int_{\mathbb{R}} \psi(x) \Big(\int_{I^t} u(s,x)\ \frac{\partial g}{\partial x}\ (x,y)\ dW_{s,y} \Big)\,dx \\ &- \int_{\mathbb{R}} \psi'(x) \Big(\int_{I^t} u(s,x)\ g(x,y)\ dW_{s,y} \Big)\,dx \\ &+ \int_{I^t} \psi(x)\ u(s,x)\ dW_{s,x}\,.\end{aligned} \tag{6.3}$$

Now we have the following result.

Theorem 6.2 *Under the hypotheses of Theorem 4.1-ii), the solution $u = \{u(t,x),$ $(t,x) \in I^T\}$ of (1.1) is a weak solution of (6.2).*

Proof. Suppose that u is the solution of (1.1). Let $\{e_k,\ k \geq 1\}$ be a complete ortonormal system in $L^2(\mathbb{R})$. For all $m \geq 1$ and $(t,x) \in I^T$ we define

$$\begin{aligned}u^m(t,x) = &\int_{\mathbb{R}} p(0,t,y,x)\ u_0(y)\,dy \\ &+ \sum_{k=1}^m \int_{I^t} \Big(\int_{\mathbb{R}} p(s,t,z,x)\ F_s(z)\ e_k(z)\,dz \Big)\, e_k(y)\,dW_{s,y}\,,\end{aligned} \tag{6.4}$$

where $F_s(z) := F(s,z,u(s,z))$. The stochastic process $u^m(t,x)$ is well-defined because $\Big\{ \Big(\int_{\mathbb{R}}\ p(s,t,z,x)\,F_s(z)\,e_k(z)\,dz \Big) e_k(y)\ \mathbb{1}_{I^t}(s,y) \Big\}$ belongs to the domain of δ for each $k \geq 1$. This property can be proved by the arguments used in the proofs of Theorems 3.2 and 3.4. By (4.8) in [13] we know that for all $0 \leq s < t \leq T,\ x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$

$$\begin{aligned}\int_{\mathbb{R}} p(s,t,y,x)\ f(y)\,dy = f(x) &+ \int_s^t \Big(\int_{\mathbb{R}} \frac{\partial^2 p}{\partial x^2}\ (s,r,y,x)\ f(y)\,dy \Big)\,dr \\ &+ \int_s^t \Big(\int_{\mathbb{R}} \frac{\partial p}{\partial x}\ (s,r,y,x)\ f(y)\,dy \Big)\, v(dr,x)\,.\end{aligned}$$

This gives us that

$$\begin{aligned}u^m(t,x) = u_0(x) &+ \int_0^t \Big(\int_{\mathbb{R}} \frac{\partial^2 p}{\partial x^2}\ (0,r,y,x)\ u_0(y)\,dy \Big)\,dr \\ &+ \int_0^t \Big(\int_{\mathbb{R}} \frac{\partial p}{\partial x}\ (0,r,y,x)\ u_0(y)\,dy \Big)\, v(dr,x) \\ &+ \sum_{k=1}^m \int_{I^t} \phi(s,x)\ e_k(x)\ e_k(y)\ dW_{s,y} \\ &+ \sum_{k=1}^m \int_{I^t} \Big[\int_s^t \Big(\int_{\mathbb{R}} \frac{\partial^2 p}{\partial x^2}\ (s,r,z,x)\ F_s(z)\ e_k(z)\,dz \Big)\,dr \Big] e_k(y)\,dW_{s,y} \\ &+ \sum_{k=1}^m \int_{I^t} \Big[\int_s^t \Big(\int_{\mathbb{R}} \frac{\partial p}{\partial x}\ (s,r,z,x)\ F_s(z)\ e_k(z)\,dz \Big) v(dr,x) \Big] e_k(y)\,dW_{s,y}\,.\end{aligned}$$

Let ψ be a test function in $\mathcal{C}_K^\infty(\mathbb{R})$. Using integration by parts formula and Fubini's theorem it is easy to obtain that

$$
\begin{aligned}
&\int_{\mathbb{R}} u^m(t,x)\ \psi(x)\,dx = \int_{\mathbb{R}} \psi(x)\ u_0(x)\,dx \\
&\qquad + \int_{I^t} \psi''(x) \Big(\int_{\mathbb{R}} p(0,r,y,x)\ u_0(y)\,dy \Big)\,dr\,dx \\
&\qquad - \int_{I^t} \psi'(x) \Big(\int_{\mathbb{R}} p(0,r,y,x)\ u_0(y)\,dy \Big)\,v(dr,x)\,dx \\
&\qquad - \int_{I^t} \psi(x) \Big(\int_{\mathbb{R}} p(0,r,y,x)\ u_0(y)\,dy \Big)\,div\,v(dr,x)\,dx \\
&\qquad + \sum_{k=1}^m \int_{I^t} \Big(\int_{\mathbb{R}} \psi(x)\ F_s(x)\ e_k(x)\,dx \Big)\,e_k(y)\,dW_{s,y} \\
&\qquad + \sum_{k=1}^m \int_{\mathbb{R}} \psi''(x) \Bigg[\int_0^t \Bigg(\int_{I^r} \Big(\int_{\mathbb{R}} p(s,r,z,x)\ F_s(z)\ e_k(z)\,dz \Big)\,e_k(y)\,dW_{s,y} \Bigg)\,dr \Bigg]\,dx \\
&\qquad - \sum_{k=1}^m \int_{\mathbb{R}} \psi'(x) \Bigg[\int_0^t \Bigg(\int_{I^r} \Big(\int_{R} p(s,r,z,x)\ F_s(z)\ e_k(z)\,dz \Big)\,e_k(y)\,dW_{s,y} \Bigg)\,v(dr,x) \Bigg]\,dx \\
&\qquad - \sum_{k=1}^m \int_{\mathbb{R}} \psi(x) \Bigg[\int_0^t \Bigg(\int_{I^r} \Big(\int_{\mathbb{R}} p(s,r,z,x)\ F_s(z)\ e_k(z)\,dz \Big)\,e_k(y)\,dW_{s,y} \Bigg)\,div\,v(dr,x) \Bigg]\,dx\,.
\end{aligned}
$$

This gives us that

$$
\begin{aligned}
\int_{\mathbb{R}} u^m(t,x)\ \psi(x)\,dx &= \int_{\mathbb{R}} \psi(x)\ u_0(x)\,dx \\
&\quad + \sum_{k=1}^m \int_{I^t} \Big(\int_{\mathbb{R}} \psi(x)\ F_s(x)\ e_k(x)\,dx \Big)\,e_k(y)\,dW_{s,y} \\
&\quad + \int_{I^t} \psi''(x)\ u^m(r,x)\,dr\,dx \\
&\quad - \int_{\mathbb{R}} \psi'(x) \Big(\int_{I^t} u^m(r,x)\ g(x,y)\ dW_{r,y} \Big)\,dx \\
&\quad - \int_{\mathbb{R}} \psi(x) \Big(\int_{I^t} u^m(r,x)\ \frac{\partial g}{\partial x}\ (x,y)\ dW_{r,y} \Big)\,dx\,.
\end{aligned}
\tag{6.5}
$$

Notice that

$$
\begin{aligned}
&\lim_m\ E \Big| \sum_{k=1}^m \int_{I^t} \Big(\int_{\mathbb{R}} \psi(x)\ F_s(x)\ e_k(x)\,dx \Big)\,e_k(y)\ dW_{s,y} - \int_{I^t} \psi(y)\ F_{s,y}\ dW_{s,y} \Big|^2 \\
&\qquad = \lim_m\ E \int_0^t \sum_{k=m+1}^\infty \Big(\int_{\mathbb{R}} \psi(x)\ F_s(x)\ e_k(x)\,dx \Big)^2\,ds = 0\,.
\end{aligned}
$$

In order to complete the proof it suffices to show that for any smooth and cylindrical random variable $G \in \mathcal{S}$ we have

$$\lim_m E\Big(G \int_{\mathbb{R}} u^m(t,x)\ \psi(x)\,dx\Big) = E\Big(\dot{G} \int_{\mathbb{R}} u(t,x)\ \psi(x)\,dx\Big)\,,$$

$$\lim_m E\Big(G \int_{I^t} \psi''(x)\ u^m(r,x)\,dr\,dx\Big) = E\Big(G \int_{I^t} \psi''(x)\ u(r,x)\,dr\,dx\Big)\,,$$

$$\lim_m E\left(G \int_{\mathbb{R}} \psi'(x) \Big(\int_{I^t} u^m(r,x)\ g(x,y)\ dW_{r,y}\Big)\,dx\right)$$
$$= E\left(G \int_{\mathbb{R}} \psi'(x) \Big(\int_{I^t} u(r,x)\ g(x,y)\ dW_{r,y}\Big)\,dx\right),$$

and

$$\lim_m E\left(G \int_{\mathbb{R}} \psi(x) \Big(\int_{I^t} u^m(r,x)\ \frac{\partial g}{\partial x}\ (x,y)\ dW_{r,y}\Big)\,dx\right)$$
$$= E\left(G \int_{\mathbb{R}} \psi(x) \Big(\int_{I^t} u(r,x)\ \frac{\partial g}{\partial x}\ (x,y)\ dW_{r,y}\Big)\,dx\right).$$

These convergences are easily checked using the duality relationship between the Skorohod integral and the derivative operator. □

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