Convergence of a branching and interacting particle system to the solution of a nonlinear stochastic PDE

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Abstract

The solution of a nonlinear parabolic SPDE on the circle, with multiplicative Gaussian noise that is white-noise in time and a bonafide function in space, is approximated by a system of branching and interacting particles. Convergence of the system is established in the space of continuous-function-valued càdlàg processes via a mollification procedure.

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1 Introduction

Let S be the one-dimensional torus. We consider the following equation on $[0,1] \times S$:

$$v(dt,x) = \frac{1}{2}\Delta v(t,x)dt + F(v(t,x))v(t,x)W(dt,x), \tag{1}$$

where, Δ is the Laplace operator on S, F is a Lipshitz function, and W is a Gaussian noise, white in time, with a bonafide spatial correlation. The goal is to approximate this parabolic stochastic partial differential equation (SPDE) by a system of branching and interacting particles that will be described in Section 3.

Branching particle systems have been found to be useful in biologically motivated applications in which, generally speaking, they converge, after a proper rescaling procedure, to a SPDE having a noise term of the form $v^{1/2}\dot{W}$ or $(v(1-v))^{1/2}\dot{W}$ (we refer to [10] and the references therein for an account on the topic), where $\dot{W}$ is space-time white noise. The procedure leading to this convergence does not allow for "color" in the noise term $\dot{W}$. On the other hand, a newly discovered application of branching particle systems is the representation of the solution to Zakai's equation in non-linear stochastic filtering (see [2]); this SPDE has a noise term of the type $v\dot{W}$ where now $\dot{W}$ is merely a one-dimensional white noise in time. Our point of view here is different from both approaches: we are not bound by a specific type of application; we approximate a general class of nonlinear parabolic SPDE by a particle system with both branching and interactions. The multiplicative noise term is not restricted to being one-dimensional white noise or space-time white noise, and most importantly, an arbitrary Lipshitz nonlinearity is allowable. We expect to report in a future publication that our particle system representation will provide a numerical approximation method as an alternative to [7], whose proof seems to rely heavily on the fact that the stochastic heat equation is under consideration. In our case, due to the branching mechanism, our noise term is conveniently written in the form $F(v)v\dot{W}$. For simplicity, we have chosen to work with functions F that are bounded and uniformly Lipschitz, but our method seems to apply to a wide number of cases. In a subsequent communication, the boundedness and uniformity will be removed, the operator Δ and the underlying space will be generalized. Note that we consider noise terms W with arbitrary functional dependence on x.

The work in [9], a particle representation formula for a very general class of SPDEs, differs from our point of view in several ways: [9] use directly an infinite system of interacting particles, the particles do not branch, they have weights, and most importantly, the SPDE is understood in the measure-valued sense, and the function F, defined on the space of measures, is assumed to be Lipshitz in the weak topology. As such, F is not allowed to depend on pointwise values of the measure's density, i.e. the SPDE in [9] does not encompass terms of the form $F(v(t,x))$. In the simple case $F = 1$, the work of [9] applies to the Zakai equation, and the distinction between our class of equations and their disappears.

However, even in this case, the system we propose is still distinct from that of [9]: for the Zakai equation, our system coincides with the branching particle system proposed in [2], which is proved therein to be of practical numerical value. More evidence in favor of such systems exists in the discrete time genetic algorithm for Zakai's equation in [4], and its generalization in [5] beyond standard nonlinear filtering. These discrete-time algorithms include branching as well as interactions, the latter being responsible for the nonlinearity. Our method can be viewed as a non-linear and colored extension of [2] rather than a continuous-time extension of [4], [5]. As in these works, our nonlinearity is due to particle interactions, but these do not appear to be comparable to the interactions in [4], [5]. At a given level n of the approximation, we mollify the particle system, seen as an empirical measure, by the heat kernel H_ε on S, for a given small positive scale constant ε. We then plug that mollification into the function F. We show that as $n \to \infty$, for fixed ε, the mollification converges in $D([0,1]; C(S))$, the space of càdlàg continuous-function-valued processes, to a limit process v_ε, which solves a mollified version of (1), namely

$$v_\varepsilon(dt,x) = \frac{1}{2}\Delta v_\varepsilon(t,x)dt + F([v_\varepsilon * H_\varepsilon](t,x))v_\varepsilon(t,x)W(dt,dx). \tag{2}$$

It is then easily seen that v_ε tends to the solution of (1). Convergence to (2) in the space of continuous functions requires significantly more work than convergence in the space of measures. This is another aspect that sets our results apart from previous work, such as in [2], [9], and [10], where only weak convergence of measures is considered.

Our paper is organized as follows: Section 2 is devoted to the definition of our SPDE, Section 3 reveals the approximating particle system under consideration. Then we prove tightness of our sequence of approximations in Section 4, and identify its limit in Section 5.

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several minor errors and rightly insisted on more detailed proofs in several instances.

2 Stochastic PDE on the circle

$S = S^1$ is identified with $[0, 2\pi)$, equipped with the Haar (linear Lebesgue) measure, the angular metric and the Laplacian Δ (with periodic boundary conditions) induced by $\mathbf{R}^2$. Let $\{e_l; l \geq 1\}$ be the eigenfunctions of Δ, an orthonormal basis of $L^2(S)$. These are bounded by a common universal constant K_u. On a complete probability space $(\Omega, \mathcal{F}, P)$, we assume that W is a Gaussian field on $\mathbf{R}_+ \times S$ defined by its covariance

$$E\left[W\left(t,x\right)W\left(s,y\right)\right] = s \wedge tQ\left(x,y\right)$$

for some spatial covariance Q on $S \times S$. The main assumption is

(H) There exists $M > 0$ such that for all $x \in S$, $Q\left(x\right) := Q\left(x,x\right) \leq M$.

Note that this condition implies that Q is bounded by M since $Q\left(x,y\right)^2 \leq Q\left(x,x\right)Q\left(y,y\right)$; (H) says that $W\left(t,x\right)$ is a square-integrable random variable for all x. One can show that (H) is equivalent to saying that the Gaussian field $W\left(t,\cdot\right)$ is P-almost-surely in $L^2\left(S\right)$. The following is an example of how to construct such a W, with the further property of spatial homogeneity: let $\{W^l; l \geq 1\}$ a family of independent Brownian motions and $\{\beta_l; l \geq 1\}$ a collection of positive coefficients. Define a cylindrical noise formally as

$$W(s,x) = \sum_{l=0}^{\infty} \beta_l^{1/2} e_l(x) W^l(s), \tag{3}$$

Then its spatial covariance is $Q\left(x,y\right) = \sum_{l\geq 1} \beta_l e_l(x) e_l\left(y\right)$, and condition (H) can be shown to be equivalent to the summability condition $\sum_{l\geq 0} \beta_l < \infty$.

In this paper, we will concern ourselves with the weak form of equation (1), where $F : \mathbf{R} \to \mathbf{R}$ is a function in C_b^1, that is bounded and Lipshitz, and $v_0 \in C(S)$:

$$\begin{aligned} \int \varphi\left(x\right) v(t,x) dx = & \int \varphi\left(x\right) v_0(x) dx + \frac{1}{2} \int_0^t \int \Delta\left[\varphi\right]\left(x\right) v(s,x) ds dx \\ & + \int_0^t \int \varphi\left(x\right) F(v(s,x)) v(s,x) W(ds,x) dx, \end{aligned} \tag{4}$$

P-almost-surely, for all φ in an appropriate class of smooth test functions. Under assumption (H), equation (4) is known to have a unique solution in $C([0,1] \times S)$, as shown e.g. in [3].

3 The branching and interacting particle system

We will approximate the solution v to (4), considered as a measure-valued process, by a branching and interacting particle system that can be described as follows: with $\{H_t(z); t > 0, z \in S\}$ the family of heat kernels of $\frac{1}{2}\Delta$, set $H_t^x = H_t(x - \cdot)$. Let $\mathcal{M}(S)$ be the space of finite measures on S. The empirical measure of our particle system at time $t \in [0,1]$, an $\mathcal{M}(S)$-valued r.v., will be denoted by $V_{\varepsilon,n}(t)$, for $\varepsilon > 0$ and $n \geq 1$. Then, extending the ideas of [2], we define

1. At time $t = 0$, $V_{\varepsilon,n}(0)$ is independent of ε and given by $V_n(0) = \frac{1}{n}\sum_{j=1}^{n}\delta_{x_j^n}$,where the particles' initial positions $x_j^n \in S$ for all $n, j \geq 1$, are such that $V_n(0)$ converges weakly to v_0 as $n \to \infty$.

2. Suppose our particle system's empirical measure $V_{\varepsilon,n}$ is given at time $\frac{i}{n}$ for $0 \leq i \leq n-1$. Let $\kappa_n(t)$ the number of particles alive at time t (we will write $\kappa_n(i)$ for $\kappa_n(\frac{i}{n})$), and $\mathcal{F}_t$ the natural filtration associated to $V_{\varepsilon,n}$. Then, on the interval $[\frac{i}{n}, \frac{i+1}{n})$, each particle moves independently of the others according to a Brownian motion path, which is carried by the probability space $(\Omega, \mathcal{F}, P)$, and is independent of W. Let us call b^j the path of the j-th particle, so that $V_{\varepsilon,n}(t) = \frac{1}{n}\sum_{j=1}^{\kappa_n(i)}\delta_{b^j(t)}$

3. At time $\frac{i+1}{n}$, each particle branches according to a probability law depending on the approximate density of the system on $[\frac{i}{n}, \frac{i+1}{n})$: for $x \in S$, set

$$v_{\varepsilon,n}(t,x) = (V_{\varepsilon,n}(t), H_\varepsilon^x) = \int V_{\varepsilon,n}(t)\,(dy)\,H_\varepsilon^x\,(y) = \frac{1}{n}\sum_{j=1}^{\kappa_n(i)} H_\varepsilon^x\left(b^j(t)\right)$$

and $\bar{\mathcal{F}}_i = \vee_{s<\frac{i}{n}}\mathcal{F}_s = \mathcal{F}_{\frac{i}{n}-}$. Given $\bar{\mathcal{F}}_{i+1}$, each particle branches independently of the others. It gives birth to a number $q_n(i+1,j)$ of offsprings, where $q_n(i+1,j)$ is a random non-negative integer with mean

$$\mu_n(i+1,j) = \exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} F\left(v_{\varepsilon,n}(s,b_s^j)\right)W(ds,b_s^j) - \frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} Q(b_s^j)F^2(\left(v_{\varepsilon,n}(s,b_s^j)\right))ds\right),$$

with law concentrated at the two integers $m, m+1$ nearest to its mean μ, and with minimum variance $\sigma_n^2(i+1,j)$. This means this law is equal to $(m+1-\mu)\,\delta_m +$

$(\mu - m)\,\delta_{m+1}$. The randomness of q_n, still supported by $(\Omega, \mathcal{F}, P)$, is independent of everything else, including, of course, the randomness that is used in μ, the parameter used to define the law of q_n itself.

The particle systems that lead to the Dawson-Watanabe (DW) superprocesses (see [10]), differs from our particle system (and the one in [2]) in the intensity of the noise in the branching mechanism. In [10], the variance of the number of offsprings is taken to be just large enough to ensure the apparition of a white noise term in the limiting process (the familiar $\sqrt{v\,(t,x)}W\,(dt,dx)$). The noise thus results from the spread in the number of offsprings, while the mean number of offsprings is non-random given the past history. In contrast, we decide to force the variance in the number of offsprings to be as small as possible, much smaller than what would lead to a DW superprocess. Introducing randomness in our mean number of offsprings is what produces the noise term in the limiting process. Moreover, our method goes beyond both DW superprocesses and [2] as it allows to plug in the particle system's (mollified) empirical distribution into a factor in the mean number of particles, resulting in an arbitrary nonlinearity F in the limiting SPDE.

4 Tightness of the sequence

Lemma 4.1 *There exists a constant $c > 0$ such that, for all $n \geq 1$, $t \in [0,1]$, $0 \leq i \leq n-1$, and $1 \leq j \leq \kappa_n(i)$,*

1. $E[\kappa_n(t)] = n$,
2. $\sigma_n^2(i,j) \leq 1/4$,
3. $E[\sum_{j=1}^{\kappa_n(i)} \sigma_n^2(i,j)] \leq cn^{1/2}$, *and*
4. $E[\kappa_n(i)^4] \leq cn^4$.

Proof: The first three estimates can be proved as in [2, Lemma 5.6 and Proposition 3.1]. To prove the last statement, the analogue statements for the powers 2 and 3 are easier, and can be used to establish the statement for the power 4, by induction on i. Thus fix i and assume that for $0 \leq k \leq i$, $E[\kappa_n(k)^4] \leq d_i n^4$ for some constant $d_i > 0$ that will be chosen below. Note we have

$$E\left[(\kappa_n(i+1))^4\right] = E\left[\left(\sum_{j=1}^{\kappa_n(i)} q_n(i+1,j)\right)^4\right] = \sum_{p=1}^{5} A_p,$$

with A_p being a sum of products of the form $q_n(i+1,j_1)q_n(i+1,j_2)q_n(i+1,j_3)q_n(i+1,j_4)$ where, for A_5, all indices j_l's are different, in A_4 exactly two of them are the same, in A_3 there are two identical pairs, in A_2, exactly three are the same, and in A_1 they are all the same. The quantity A_1 can be written as:

$$A_1 = E\left[\sum_{j=1}^{\kappa_n(i)} E\left[(q_n(i+1,j))^4 \middle| \bar{\mathcal{F}}_{i+1}\right]\right].$$

Recalling the conditional law of $q_n(i+1,j)$ from point 3 in Section 3 and the bound $\sigma_n^2 \leq 1/4$, it is easily seen that

$$E\left[(q_n(i+1,j))^4 \middle| \bar{\mathcal{F}}_{i+1}\right] \leq \mu_n^4(i+1,j) + \frac{3}{2}\mu_n^2(i+1,j) + \mu_n(i+1,j) + \frac{1}{4}. \tag{5}$$

Furthermore, $\mu_n(i+1,j)$ being the endpoint of an exponential martingale on $[i/n;(i+1)/n]$ yields

$$E\left[\sum_{j=1}^{\kappa_n(i)} \mu_n^4(i+1,j)\right] \leq \exp\left(\frac{6M\|F\|^2}{n}\right) E\left[\kappa_n(i)\right] \leq cn.$$

We get the same kinds of estimates for the other terms of the right hand side of (5), which yields

$$A_1 \leq \left(\exp\left(\frac{6M\|F\|^2}{n}\right) + \frac{3}{2}\exp\left(\frac{M\|F\|^2}{n}\right) + \frac{5}{4}\right) n = cn.$$

The computations for A_2 are slightly different:

$$\begin{aligned} A_2 &= 2E\left[\sum_{j_1,j_2\in S_2} E\left[q_n^3(i+1,j_1)\middle|\bar{\mathcal{F}}_{i+1}\right] E\left[q_n(i+1,j_2)\middle|\bar{\mathcal{F}}_{i+1}\right]\right] \qquad (6)\\ &\leq 2E\left[\sum_{j_1,j_2\in S_2} E\left[\left(\mu_n^3(i+1,j_1) + \frac{3}{4}\mu_n(i+1,j_1) + \frac{1}{3^{1/2}4}\right)\mu_n(i+1,j_2)\middle|\bar{\mathcal{F}}_i\right]\right]. \end{aligned}$$

Then

$$\begin{aligned}
&E\left[\mu_n^3(i+1,j_1)\mu_n(i+1,j_2)\middle|\bar{\mathcal{F}}_i\right] \\
&\leq E^{3/4}\left[\mu_n^4(i+1,j_1)\middle|\bar{\mathcal{F}}_i\right] E^{1/4}\left[\mu_n^4(i+1,j_2)\middle|\bar{\mathcal{F}}_i\right] \\
&\leq \exp\left(\frac{6M\|F\|^2}{n}\right).
\end{aligned}$$

The other terms in (6) can be treated the same way, and using the induction hypothesis we get

$$A_2 \leq 2c'E\left[\kappa_n(i)(\kappa_n(i)-1)/2\right] \leq 2c_2c'n(n-1) = cn\,(n-1)\,.$$

Similarily, using the fact that $Card\,(S_p) = \binom{\kappa_n(i)}{p} \leq \kappa_n\,(i)^p\,/p!$, we obtain

$$\begin{aligned}
A_3 &\leq cn(n-1) \\
A_4 &\leq cn(n-1)(n-2) \\
A_5 &\leq d_i \exp\left(\frac{6M\|F\|^2}{n}\right) n(n-1)(n-2)(n-3) \leq d_i\left(1+\frac{c}{n}\right)n^4.
\end{aligned}$$

Putting together the estimates on A_1, A_2, A_3, A_4 and A_5 yields

$$\begin{aligned}
&E\left[\kappa_n^4(i+1)\right] \\
&\leq c\,(n+n\,(n-1)+n\,(n-1)\,(n-2)) + d_i\left(1+\frac{c}{n}\right)n^4 \\
&\leq n^4\left(\frac{3c}{n}+d_i\left(1+\frac{c}{n}\right)\right) \\
&\leq n^4 d_i\left(1+\frac{4c}{n}\right) \\
&= n^4 d_{i+1}
\end{aligned}$$

as long as we choose $d_i = (1+4c/n)^i$. Therefore by induction, since $d_i \leq e^{4c}$, we get, for all $0 \leq i \leq n-1$,

$$E\left[\kappa_n^4(i+1)\right] \leq e^{4c}n^4,$$

which is the desired result, proving the lemma. □

Lemma 4.2 *Let $\varepsilon > 0$. Then, for some $c > 0$, for all $n \geq 1$, $t \in [0,1]$, $x, z \in S$, we have*

$$E\left[|v_{\varepsilon,n}(t,z) - v_{\varepsilon,n}(t,x)|^4\right] \leq \frac{c}{\varepsilon^8}|z-x|^4.$$

Proof: Assume first that $t = k/n$ for $1 \leq k \leq n$. Let D denote the gradient operator on S, which coincides with the derivative in the angular parameter. Then, invoking Remark 3.3 in [2, Remark 3.3] and its proof, we have

$$v_{\varepsilon,n}(t,x) = v_{\varepsilon,n}(0,x) + \sum_{j=1}^{4} N_j(k,x), \tag{7}$$

where

$$N_1(k,x) = \frac{1}{n}\sum_{l=1}^{k}\sum_{j=1}^{\kappa_n(l-1)} \int_{\frac{l-1}{n}}^{\frac{l}{n}} DH_\varepsilon^x(b_s^j)db_s^j$$

$$N_2(k,x) = \frac{1}{2n}\sum_{l=1}^{k}\sum_{j=1}^{\kappa_n(l-1)} \int_{\frac{l-1}{n}}^{\frac{l}{n}} D^2H_\varepsilon^x(b_s^j)ds$$

$$N_3(k,x) = \frac{1}{n}\sum_{l=1}^{k}\sum_{j=1}^{\kappa_n(l-1)} H_\varepsilon^x(b_{\frac{l}{n}}^j)\left(\mu_n(l,j)-1\right)$$

$$N_4(k,x) = \sum_{l=1}^{k}\left(\left(V_{\varepsilon,n}\left(\frac{l}{n}\right), H_\varepsilon^x\right) - E\left[\left(V_{\varepsilon,n}\left(\frac{l}{n}\right), H_\varepsilon^x\right)\Big|\bar{\mathcal{F}}_l\right]\right).$$

Using Burkholder's and Jensen's inequalities, Lemma 4.1, and the classical heat kernel bound $\sup_{x\in S}|D^m H(x)| \leq c_m\varepsilon^{-(m+1)/2}$, writing E_b for the expectation w.r.t. b only,

$$\begin{aligned}
&E\left[|N_1(k,x)-N_1(k,z)|^4\right] \\
&= E\left[E_b\left[|N_1(k,x)-N_1(k,z)|^4\right]\right] \\
&\leq \frac{c}{n^4}E\left[\left|\sum_{l=1}^{k}\int_{\frac{l-1}{n}}^{\frac{l}{n}}\left(\sum_{j=1}^{\kappa_n(l-1)} DH_\varepsilon(x-b_s^j) - DH_\varepsilon(z-b_s^j)\right)^2 ds\right|^2\right] \\
&\leq \frac{c|x-z|^4}{\varepsilon^6},
\end{aligned}$$

Similar techniques yield

$$E\left[|N_2(k,x)-N_2(k,z)|^4\right] \leq c|x-z|^4\varepsilon^{-8}.$$

Let $\eta_n^{l,j}(t)$ be the mean-one exponential martingale on $[\frac{l-1}{n}, \frac{l}{n})$ based on $F(v_{\varepsilon,n}(s, b_s^j))\, W(ds, b_s^j)$. Since b is independent of W, we can write

$$\begin{aligned}
&E\left[|N_3(k,x) - N_3(k,z)|^4\right] \\
&= E\left[E_W\left[|N_3(k,x) - N_3(k,z)|^4\right]\right] \\
&= E\left[E_W\left[\left|\frac{1}{n}\sum_{l=1}^{k}\sum_{j=1}^{\kappa_n(l-1)}\int_{\frac{l-1}{n}}^{\frac{l}{n}} \eta_n^{l,j}(s)F(v_{\varepsilon,n}(s,b_s^j))\left(H_\varepsilon^x(b_{\frac{l}{n}}^j) - H_\varepsilon^z(b_{\frac{l}{n}}^j)\right)W(ds,b_s^j)\right|^4\right]\right] \\
&\leq \frac{c}{n^4}E\left[\left|\sum_{l=1}^{k}\int_{\frac{l-1}{n}}^{\frac{l}{n}} E_W\left(\sum_{j=1}^{\kappa_n(l-1)} \alpha_n^{l,j}(s)\right)^2 ds\right|^2\right] \\
&\leq \frac{cM^2\|F\|^4|x-z|^4}{\varepsilon^4 n^4}\sum_{l=1}^{k}\int_{\frac{l-1}{n}}^{\frac{l}{n}} E\left[\left(\sum_{j=1}^{\kappa_n(l-1)}\eta_n^{l,j}(s)\right)^4\right] ds
\end{aligned}$$

where we denote by E_W the expectation w.r.t. W only, and

$$\alpha_n^{l,j}(s) :\equiv \eta_n^{l,j}(s)F(v_{\varepsilon,n}(s,b_s^j))Q(b_s^j)^{1/2}\left(H_\varepsilon^x\left(b_{\frac{l}{n}}^j\right) - H_\varepsilon^z\left(b_{\frac{l}{n}}^j\right)\right)$$

and where we used Burkhölder's and Jensen's inequalities and the fact that $\kappa_n(l-1) \in \bar{\mathcal{F}}_{l-1}$. To conclude that $E\left[|N_3(k,x) - N_3(k,z)|^4\right] \leq c|x-z|^4\varepsilon^{-4}$ one may now use the techniques and results of Lemma 4.1.

The family $\{N_4(k,x) - N_4(k,z); 1 \leq k \leq n-1\}$ is a $\bar{\mathcal{F}}_k$-square-integrable martingale, whose bracket is given by (see [2, p. 1575])

$$\sum_{l=1}^{k}\frac{1}{n^2}\sum_{j=1}^{\kappa_n(l)}\sigma_n^2(l,j)\left(H_\varepsilon^x\left(b_{\frac{l}{n}}^j\right) - H_\varepsilon^z\left(b_{\frac{l}{n}}^j\right)\right)^2.$$

Burkholder's inequality and Lemma 4.1 immediately yield $E\left[|N_4(k,x) - N_4(k,z)|^4\right] \leq c|x-z|^4\varepsilon^{-4}$, and the lemma for $t = k/n$.

The generalization to any $t \in [0,1]$ can be performed as follows. For any $t \in [0,1]$ such that t is not of the form k/n where k is an integer, note that

$$v_{\varepsilon,n}(t,x) = v_{\varepsilon,n}(0,x) + N_1(t,x) + N_2(t,x) + N_3([nt],x) + N_4([nt],x),$$

where the extensions of $N_1, \cdots, N_4$ to all of $[0,1]$ (abusively still using the notation N_i for these extensions) are given as follows:

$$\begin{aligned}
N_1(t,x) &= N_1([nt],x) + \frac{1}{n}\sum_{j=1}^{\kappa_n([nt])}\int_{[nt]}^t DH_\varepsilon^x(b_s^j)db_s^j;\\
N_2(t,x) &= N_2([nt],x) + \frac{1}{2n}\sum_{j=1}^{\kappa_n([nt])}\int_{[nt]}^t D^2H_\varepsilon^x(b_s^j)ds;\\
N_3\,(t,x) &= N_3([nt],x);\\
N_4\,(t,x) &= N_4([nt],x).
\end{aligned}$$

It is trivial to check that $|v_{\varepsilon,n}\,(0,x) - v_{\varepsilon,n}\,(0,z)| \le c\,|x-z|\,/\varepsilon$. Thus, our relation for general t easily follows from the same arguments as in the previous steps of this proof for $t = k/n$.□

Proposition 4.3 *For any $\varepsilon > 0$, $\{(V_{\varepsilon,n}, v_{\varepsilon,n}); n \ge 1\}$ is tight in $D([0,1];\mathcal{M}(S)\times C(S))$.*

Proof: The main difference between our $V_{\varepsilon,n}$ and the measure-valued process in [2] is that our function $F(v_{\varepsilon,n}\,(s,x))$ replaces the "observation function" $h(x)$ in [2]. Because both functions are bounded, the tightness of $V_{\varepsilon,n}$ is obtained exactly as in [2, Theorem 4.4]. Since tightness of our pair of sequences is equivalent to tightness of its components, we only need to prove that $\{v_{\varepsilon,n}; n \ge 1\}$ is tight in $D([0,1];C(S))$.

Let us define $\mathcal{S}_1$ as the space of $\mathcal{F}_t$-adapted $\mathcal{M}(S)$-valued càdlàg processes U (i.e. $U \in D([0,1];\mathcal{M}(S))$) such that

$$\sup_{t\in[0,1]} \|U(t)\|_1^2 := \sup_{t\in[0,1]} \sup\{E[(U(t),\varphi)^2]; \|\varphi\| \le 1, \|\Delta\varphi\| \le 1\} < \infty. \tag{8}$$

Using the same techniques as in Lemma 4.2, it can also be proved that,

$$\sup_{t\in[0,1];n\ge 0} \|V_{\varepsilon,n}\,(t)\,\|_1 \equiv L < \infty. \tag{9}$$

Denote by $\mathbb{F}$ the family of finite linear combinations of elements of $\{e_l; l \ge 1\}$. It separates points in $C(S)$, and is closed under addition. For $p \ge 1$, $s > 0$, let also $W^{s,p}(S)$ be the Sobolev space defined as the completion of $C^\infty(S)$ under the norm (see Adams [1])

$$\|\psi\|_{s,p}^p = \int_S |\psi(x)|^p dx + \int_{S^2} \frac{|\psi(x) - \psi\,(z)\,|^p}{|x-z|^{1+sp}} dxdz,$$

and recall that $W^{s,p}(S)$ is compactly imbedded in $C(S)$ when $sp>1$. To prove the tightness of $v_{\varepsilon,n}$ in $D([0,1];C(S))$, according to Jakubowski's Theorem [8, Theorem 3.1], we only need to show the following claims:

1. For all $n\geq 1$ and $s<1$,
$$E\left[\sup_{t\in[0,1]}\|v_{\varepsilon,n}(t)\|_{s,4}^4\right]<\infty.$$
2. For any $\varphi\in\mathbb{F}$, the sequence $\{\int_S v_{\varepsilon,n}(t,x)\varphi(x)dx;n\geq 1\}$ is tight in $D([0,1];\mathbf{R})$.

To prove Point 2, notice that $v_{\varepsilon,n}(t)=V_{\varepsilon,n}(t)*H_\varepsilon$. Hence, by symmetry of H_ε, for any $\varphi\in\mathbb{F}$,
$$\int_S v_{\varepsilon,n}(t,x)\varphi(x)dx=\left(V_{\varepsilon,n}(t),\varphi*H_\varepsilon\right).$$
The function $\varphi*H_\varepsilon$ being smooth on S, the tightness of Point 2 can be derived as in [2, Propositions 4.1 and 4.2].

To prove Point 1, we detail only the proof of the fact that, for any $s<1$,
$$E\left[\sup_{0\leq k\leq n}\int_{S^2}\frac{|v_{\varepsilon,n}(k/n,x)-v_{\varepsilon,n}(k/n,z)|^4}{|x-z|^{1+4s}}dxdz\right]<\infty;$$
the proof of the $L^4(S)$-bound is easier, and given the proof of Lemma 4.2, using a supremum over all t, not just of the form $t=k/n$, does not introduce additional difficulties. Taking up the notations of Lemma 4.2, using the decomposition given in (7), reserving the right to also use the notational extension for N_i on $[0,1]$ as in the last paragraph of the proof of Lemma 4.2, and letting $N_0\left(t,x\right)=v_{\varepsilon,n}\left(0,x\right)$, it is then sufficient to show that
$$E\left[\sup_{0\leq k\leq n}\int_{S^2}\frac{|N_i(k,x)-N_i(k,z)|^4}{|x-z|^{1+4s}}dxdz\right]<\infty, \tag{10}$$
for $i=0,1,2,3,4$.

We have by Jensen's inequality
$$\begin{aligned}&\int_{S^2}\frac{\left|v_{\varepsilon,n}\left(0,x\right)-v_{\varepsilon,n}\left(0,z\right)\right|}{|x-z|^{1+4s}}dxdz\\ &\leq\frac{1}{n}\sum_{j=1}^{n}\int_{S^2}\frac{\left|H_\varepsilon\left(b_0^j-x\right)-H_\varepsilon\left(b_0^j-z\right)\right|}{|x-z|^{1+4s}}dxdz\\ &=\int_{S^2}\frac{\left|H_\varepsilon\left(x\right)-H_\varepsilon\left(z\right)\right|}{|x-z|^{1+4s}}dxdz=\|H_\varepsilon\|_{s,4}^4<\infty.\end{aligned}$$

which deals with the $i = 0$ term. Since the process

$$M_1(t) = \int_{S^2} \frac{|N_1(t,x) - N_1(t,z)|^4}{|x-z|^{1+4s}} dxdz$$

is easily shown to be a $\mathcal{F}_t$-continuous positive submartingale, we get, by the proof of Lemma 4.2,

$$\begin{aligned} E\left[\sup_{t\in[0,1]} M_1(t)\right] &\leq cE[M_1(1)] \\ &\leq \int_{S^2} \frac{c}{\varepsilon^6} |x-z|^{3-4s} dxdz \end{aligned}$$

which is finite as soon as $s < 1$. It is also clear that

$$E\left[\sup_{t\in[0,1]} |N_2(t,x) - N_2(t,z)|^4\right] \leq E\left[\left|\frac{1}{2n}\sum_{l=1}^{n}\sum_{j=1}^{\kappa_n(l-1)} \int_{\frac{l-1}{n}}^{\frac{l}{n}} |D^2 H_\varepsilon^x(b_s^j) - D^2 H_\varepsilon^z(b_s^j)| ds\right|^4\right],$$

which can be estimated along the same lines as in Lemma 4.2, proving inequality (10) for $i = 2$.

For a fixed path of b, the process $\{N_3(k,x) - N_3(k,z); 0 \leq k \leq n\}$ is a $\mathcal{F}_k^W$-martingale in L^4, where

$$\mathcal{F}_k^W = \sigma\left\{W(s,x);\, 0 \leq s \leq \frac{k}{n}, x \in S\right\},$$

as the following argument shows: we can decompose N_3 into

$$\begin{aligned} N_3(k+1,x) &= \frac{1}{n}\sum_{l=1}^{k}\sum_{j=1}^{\kappa_n(l-1)} H_x^\varepsilon\left(b_{l/n}^j\right)\{\mu_n(l,j) - 1\} \\ &+ \frac{1}{n}\sum_{j=1}^{\kappa_n(k)} H_x^\varepsilon\left(b_{(k+1)/n}^j\right)\{\mu_n(k+1,j) - 1\}, \end{aligned}$$

where the first term on the right-hand side is measurable w.r.t. $\mathcal{F}_k^W$, and coincides with $N_3(k,x)$; in the second term, $\kappa_n(k)$ is also measurable w.r.t. $\mathcal{F}_k^W$, while $\mu_n(k+1,j)$ is independent of $\mathcal{F}_k^W$ and is a mean-one random variable; thus taking the conditional expectation given $\mathcal{F}_k^W$ kills the second term, and leaves the first one unchanged.

Thus

$$E\left[\sup_{0\le k\le n}|N_3(k,x)-N_3(k,z)|^4\right]=E\left[E_W\left[\sup_{0\le k\le n}|N_3(k,x)-N_3(k,z)|^4\right]\right]$$
$$\le cE\left[|N_3(n,x)-N_3(n,z)|^4\right]\le c_\varepsilon|x-z|^4,$$

which shows (10) for $i=3$. The case $i=4$ is also easily deduced from the fact that $\{N_4(k,x)-N_4(k,z);0\le k\le n\}$ is a $\bar{\mathcal{F}}_k$-martingale. □

5 Identification of the limit

Before identifying the limit of $\{V_{\varepsilon,n};n\ge 1\}$, we need an existence and uniqueness result for a mollified and measure-valued version of (1). Recall in (8) the subspace $\mathcal{S}_1$ of $\mathcal{M}(S)$-valued processes.

Proposition 5.1 *Fix F Lipschitz and $\varepsilon>0$. There exists a unique $\mathcal{S}_1$-valued solution to*

$$(U(t),\varphi)=(v_0,\varphi)+\int_0^t(U(s),\Delta\varphi)\,ds+\int_0^t\int_S F([U(s)*H_\varepsilon](x))\varphi(x)W(ds,x)U(s,dx)\quad(11)$$

for $t\in[0,1]$, $\varphi\in C_b^2(S)$. Let us call V_ε this solution. Then V_ε has a density w_ε which satisfies

$$\forall p\in\mathbf{N}:\sup\{E[|w_\varepsilon(t,x)|^p];t\in[0,1],x\in S,\varepsilon>0\}<\infty.$$

Proof:

In order to prove the uniqueness part of the claim, by standard stopping time arguments, we can suppose that a solution to (11) satisfies

$$\sup\{(U(s),\varphi);s\le 1,\|\varphi\|\le 1,\|\Delta\varphi\|\le 1\}\le c<\infty.\quad(12)$$

Let now Φ be the map defined on $\mathcal{S}_1$ by

$$([\Phi(U)](t),\varphi)=\int_0^t\int_S F([U(s)*H_\varepsilon](x))\varphi(x)W(ds,x)U(s,dx),\quad t\in[0,1]$$

We will show that Φ satisfies the following property on $\mathcal{S}_1$: if U_1,U_2 are two elements of $\mathcal{S}_1$, then, for all $t\in[0,1]$,

$$E\left[[(\Phi(U_1)](t)-[\Phi(U_1)](t),\varphi)^2\right]\le 2\left(D_1(t)+D_2(t)\right),$$

where

$$D_1(t) = \int_0^t \left(\int_S (F([U_1(s) * H_\varepsilon](x) - F([U_2(s) * H_\varepsilon](x))) \right. \\ \left. \varphi(x)Q(x)U_1(s,dx) \right)^2 ds$$

$$D_2(t) = \int_0^t \left(\int_S F([U_2(s) * H_\varepsilon](x)\varphi(x)Q(x)\left(U_1(s,dx) - U_2(s,dx)\right) \right)^2 ds.$$

Then, using (12), we get

$$\begin{aligned} & E\left[|D_1(t)|^2\right] \\ & \leq M\|\varphi\|^2\|DF\|^2 \int_0^t E\left[\left(\int_S [(U_1(s) - U_2(s)) * H_\varepsilon](x)U_1(s,dx)\right)^2 ds\right] \\ & \leq M\left(\|\varphi\|\|DF\|\|H_\varepsilon\|\right)^2 \int_0^t E\left[|(U_1(s) - U_2(s), \mathbf{1})(U_1(s), \mathbf{1})|^2\right] ds \\ & \leq c\int_0^t E\left[(U_1(s) - U_2(s), \mathbf{1})^2\right] ds. \end{aligned}$$

Using the boundedness of F, we can also prove that

$$E\left[|D_1(t)|^2\right] \leq c\int_0^t E\left[(U_1(s) - U_2(s), \mathbf{1})^2\right] ds.$$

Hence,

$$\|[\Phi(U_2)](t) - [\Phi(U_1)](t)\|_1^2 \leq c\int_0^t \|U_2(s) - U_1(s)\|_1^2 ds.$$

The same kind of argument can be used for the term $\int_0^t (U(s), \Delta\varphi)ds$, which yields

$$\|U_2(t) - U_1(t)\|_1^2 \leq c\int_0^t \|U_2(s) - U_1(s)\|_1^2 ds.$$

The uniqueness result is then easily obtained by standard methods.

The existence of a density as well as its integrability are now a direct application of Kurtz and Xiong's results [9, Section 3]. □

We can now prove the main result of this article, which allows to say that our particle system is an approximation to (1).

Theorem 5.2 *Let $V_{\varepsilon,n}$ be the particle system defined in section 3. Then*

1. *The limit in law of* $\{V_{\varepsilon,n}, v_{\varepsilon,n}; n \geq 1\}$ *in* $D([0,1]; \mathcal{M}(S) \times C(S))$ *exists and is* $(V_\varepsilon, v_\varepsilon)$ *where* V_ε *is the solution to (11) and* $v_\varepsilon(t) = V_\varepsilon(t) * H_\varepsilon$.

2. *Let* v *be the unique solution to (4). Let* $v_\varepsilon^1 = v_\varepsilon$ *and* $v_\varepsilon^2 = w_\varepsilon$ *Then for* $i = 1, 2$,

$$\lim_{\varepsilon \to 0} E\left[\sup_{t \in [0,1]} \left|v_\varepsilon^i(t) - v(t)\right|^2_{L^2(S)}\right] = 0.$$

Proof:

Take a subsequence of $\{(V_{\varepsilon,n}, v_{\varepsilon,n}); n \geq 1\}$, that we will denote again by $(V_{\varepsilon,n}, v_{\varepsilon,n})$, converging in law to a couple $(\tilde{V}_\varepsilon, \tilde{v}_\varepsilon)$, in $D([0,1]; \mathcal{M}(S) \times C(S))$. By a Theorem of Skorokhod (see [6, Theorem 4.4.2]), we can assume that $(V_{\varepsilon,n}, v_{\varepsilon,n})$ and $(\tilde{V}_\varepsilon, \tilde{v}_\varepsilon)$ are defined on the same probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ (whose expectation will be denoted by $\tilde{E}$), and that $\{V_{\varepsilon,n}, v_{\varepsilon,n} : n \geq 1\}$ converges to $(\tilde{V}_\varepsilon, \tilde{v}_\varepsilon)$ almost surely on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. We will work now on that new probability space. Notice also that, by inequality (9), we have $\tilde{v}_\varepsilon \in \mathcal{S}_1$. It is now enough to show that $(\tilde{V}_\varepsilon(t), \varphi)$ satisfies equation (11) for a fixed $t \in [0,1]$, and for any φ in a countable system of smooth functions dense in $C(S)$, containing $\varphi_0 = \mathbf{1}$.

We now prove, using the basic properties of convolution and Fourier series, that $\tilde{v}_\varepsilon(t) = \tilde{V}_\varepsilon(t) * H_\varepsilon$ for all $t \in [0,1]$. Let $\{\mu_n; n \geq 1\}$ be a sequence of $\mathcal{M}(S)$ converging weakly to a measure μ, and such that $\mu_n * H_\varepsilon$ converges uniformly on S to a function m. Then $m = \mu * H_\varepsilon$. Indeed, let $\hat{\mu}_n$ be the Fourier transform of μ_n, defined by

$$\hat{\mu}_n(l) = (\mu_n, e_l), \qquad l \geq 1.$$

Then $\lim_{n\to\infty} \hat{\mu}_n(l) = \hat{\mu}(l)$ for all $l \geq 1$. On the other hand, if $\mu_n * H_\varepsilon$ converges uniformly to m, then it also converges in $L^2(S)$, and hence $\ell^2\text{-lim}\,(\mu_n \hat{*} H_\varepsilon) = \hat{m}$. But

$$\lim_{n\to\infty} (\mu_n \hat{*} H_\varepsilon)(l) = \lim_{n\to\infty} \hat{\mu}_n(l)\hat{H}_\varepsilon(l) = \hat{\mu}(l)\hat{H}_\varepsilon(l),$$

which gives $\hat{m} = \hat{\mu}\hat{H}_\varepsilon$ and hence $m = \mu * H_\varepsilon$. Applying this result to our sequence $\{V_{\varepsilon,n}, v_{\varepsilon,n}; n \geq 1\}$ yields

$$\tilde{v}_\varepsilon(t) = \tilde{V}_\varepsilon(t) * H_\varepsilon, \qquad t \in [0,1].$$

Pick a function $\varphi \in C_b^2(S)$. According to [2, Remark 3.3], the evolution of $(V_{\varepsilon,n}, \varphi)$ between 0 and t is given by

$$(V_{\varepsilon,n}(t), \varphi) = (V_n(0), \varphi) + \hat{M}_{1,n}(t) + \hat{M}_{2,n}(t) + M_{3,n}([nt]) + M_{4,n}([nt]), \tag{13}$$

where $\hat{M}_{1,n}$ is a square integrable $\mathcal{F}_t$-martingale with $\left\langle \hat{M}_{1,n} \right\rangle (t) = n^{-1} \int_0^t (V_{\varepsilon,n}(s), (D\varphi)^2)\, ds$, we define $\hat{M}_{2,n}\,(t) = \int_0^t (V_{\varepsilon,n}(s), \Delta\varphi) ds$, and $M_{3,n}$ is defined by

$$M_{3,n}(k) = \frac{1}{n} \sum_{l=1}^{k} \sum_{j=1}^{\kappa_n(l-1)} \varphi(b^j_{\frac{l}{n}})\,(\mu_n(l,j) - 1)$$

and for all $1 \leq k \leq n$,

$$M_{4,n}(k) = \sum_{l=1}^{k} \left(V_{\varepsilon,n}\left(\frac{l}{n}\right), \varphi \right) - \tilde{E}\left[\left(V_{\varepsilon,n}\left(\frac{l}{n}\right), \varphi \right) \Big| \bar{\mathcal{F}}_l \right].$$

The convergence of $V_n(0)$ is trivial. That of the remaining terms in (13) is now given.

Using the same techniques as in the estimate for N_1 in Lemma 4.2, we can show that the sequence $\{\langle \hat{M}_{1,n} \rangle (1); n \geq 1\}$ is uniformly integrable. Furthermore, $V_{\varepsilon,n}$ converges almost surely to $\tilde{V}_\varepsilon$ in $D([0,1]; \mathcal{M}(S))$, and $(D\varphi)^2$ is a bounded function. Therefore $\left\langle \hat{M}_{1,n} \right\rangle (1)$ converges to zero in $L^1(\tilde{\Omega})$, proving the convergence of $\hat{M}_{1,n}$ to zero in $L^2(\tilde{\Omega} \times [0,1])$. The same kind of arguments also show that $\hat{M}_{2,n}$ converges in $L^2(\tilde{\Omega} \times [0,1])$ to $\int_0^{\cdot} \left(\tilde{V}_\varepsilon(s), \Delta\varphi \right) ds$. It can be proved, as in the estimate for N_4 in Lemma 4.2, and using Lemma 4.1, that

$$\begin{aligned}
&\tilde{E}\left[(M_{4,n}([nt]))^2 \right] \\
&\leq \frac{c\|\varphi\|^2}{n^2} E\left[\sum_{l=1}^{[nt]} \sum_{j=1}^{\kappa_n(l-1)} \sigma_n^2(l,j) \right] \\
&\leq \frac{c}{n^{1/2}} \longrightarrow 0.
\end{aligned}$$

Using the exponential martingale η introduced in the proof of Lemma 4.2, we have

$$M_{3,n}(k) = \frac{1}{n} \sum_{l=1}^{k} \sum_{j=1}^{\kappa_n(l-1)} \int_{\frac{l-1}{n}}^{\frac{l}{n}} \eta_n^{l,j}(s) F(v_{\varepsilon,n}(s, b^j_s)) \varphi(b^j_{\frac{l}{n}}) W(ds, b^j_s).$$

Let $\tilde{M}_{3,n}(k)$ be defined by replacing $b^j_{\frac{l}{n}}$ by b^j_s in $M_{3,n}(k)$. Since φ is a Lipshitz function, it is easy to show that $\tilde{E}\left[\left(M_{3,n}([nt]) - \tilde{M}_{3,n}([nt]) \right)^2 \right]$ converges to zero as $n \to \infty$. For $t \in [0,1]$, now set

$$M_3(t) = \int_0^t \left(F(\tilde{V}_\varepsilon(s) * H_\varepsilon) \varphi W(ds, .), \tilde{V}_\varepsilon(s) \right).$$

We only need to show the convergence to 0 of $\tilde{E}\left[\left(M_3(t)-\tilde{M}_{3,n}([nt])\right)^2\right]$ as $n\to\infty$. This quantity is bounded above by the sum of the following four terms:

$$R_{1,n}(t) = \frac{M}{n^2}\tilde{E}\left[\sum_{l=1}^{[nt]}\int_{\frac{l-1}{n}}^{\frac{l}{n}}\left(\sum_{j=1}^{\kappa_n(l-1)}\varphi(b_s^j)(\eta_n^{l,j}(s)-1)F(v_{\varepsilon,n}(s,b_s^j))\right)^2 ds\right]$$
$$R_{2,n}(t) = M\int_0^t \tilde{E}\left[\left(\int_S \varphi(x)[F(v_{\varepsilon,n})-F(\tilde{v}_\varepsilon)](s,x)V_{\varepsilon,n}(s,dx)\right)^2\right] ds$$
$$R_{3,n}(t) = M\int_0^t \tilde{E}\left[\left(\int_S \varphi(x)[F(\tilde{v}_\varepsilon)](s,x)[V_{\varepsilon,n}-\tilde{V}_\varepsilon](s,dx)\right)^2\right] ds$$
$$R_{4,n}(t) = M\int_{[nt]/n}^t \tilde{E}\left[\left(\int_S \varphi(x)[F(\tilde{v}_\varepsilon)](s,x)\tilde{V}_\varepsilon(s,dx)\right)^2\right] ds.$$

The first term is controlled by using the easy fact that $\tilde{E}\left[(\eta_n^{l,j}(s)-1)^2|\bar{\mathcal{F}}_{l-1}\right]\le C/n$. The term $R_{2,n}(t)$ can be estimated as follows:

$$R_{2,n}(t)\le \|\varphi\|^2\|F\|^2 M\int_0^t \tilde{E}^{1/2}\left[\sup_{x\in S}\left([F(v_{\varepsilon,n})-F(\tilde{v}_\varepsilon)](s,x)\right)^4\right]\tilde{E}^{1/2}\left[(V_{\varepsilon,n},\mathbf{1})^4\right] ds.$$

By Lemma 4.1 point 2, $\tilde{E}[(V_{\varepsilon,n},\mathbf{1})^4]$ is bounded uniformly in n,s. Since $v_{\varepsilon,n}(s,.)$ converges to $\tilde{v}_\varepsilon(s,.)$ in $C(S)$ for all $s\in[0,1]$, we get $\lim_{n\to\infty}R_{2,n}(t)=0$ by dominated convergence. The sequence $(V_{\varepsilon,n}(s),\mathbf{1})$ converges almost surely to $(\tilde{V}_\varepsilon(s),\mathbf{1})$, and is uniformly square integrable. The convergence $\lim_{n\to\infty}R_{3,n}(t)=0$ follows. Finally, $\lim_{n\to\infty}R_{4,n}(t)=0$ is trivial. Point 1 is proved.

Since we know that V_ε has a density v_ε, point 2 of the theorem is easily shown, using the evolution forms of equations (11) and (1), by standard stability results in SPDEs (see [3]).□

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