Chapter 25: Random and Mixed Effects Models

In a linear model, we always assume independent variables are not random. If an independent variable is random, then we call it a random effect; otherwise, we call it a fixed effect. If a model contains both random and fixed effects, then we call it a mixed effect model. In general in a mixed effect model, we assume the random effect and the error term are independent.

25.1 Single-Factor Random effect model. Consider the one-way model

\[ Y_{ij} = \mu_i + \epsilon_{ij}, \]

where \( i = 1, \cdots, I \) and \( j = 1, \cdots, n_i \). Consider the factor effect model

\[ Y_{ij} = \mu + \alpha_i + \epsilon_{ij}. \]

If we assume \( \mu_i \sim N(0, \sigma^2_\mu) \) and \( \epsilon_{ij} \sim N(0, \sigma^2) \), then the model is a one-way random effect model. This model contains parameters \( \mu, \sigma^2_\mu, \) and \( \sigma^2 \). Then, we have \( E(Y_{ij}) = \mu, \ V(Y_{ij}) = \sigma^2 + \sigma^2_\mu, \ V(\bar{Y}_i) = \sigma^2 + \sigma^2_\mu/n_i, \) and

\[ \text{Cov}(Y_{ij}, Y_{ij'}) = \sigma^2_\mu. \]

Consider the balanced case in which we have \( n_i = n \). Let

\[ SSE = \frac{1}{I(n - 1)} \sum_{i=1}^{I} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_i)^2 \]

and

\[ SSR = \frac{1}{I-1} \sum_{i=1}^{I} \sum_{j=1}^{n} (\bar{Y}_i - \bar{Y}_.)^2. \]

Then,

\[ E(MSE) = \sigma^2 \]

and

\[ E(MSR) = \sigma^2 + n\sigma^2_\mu. \]

In general, we have the following problems.

- Estimation of \( \mu, \sigma^2_\mu \) and \( \sigma^2 \): we may use \( \hat{\mu} = \bar{y}_., \hat{\sigma}^2 = MSE, \) and \( \hat{\sigma}^2_\mu = (MSR - MSE)/n. \)

- Test \( \sigma^2_\mu = 0 \): we can use either an F-test or a bootstrap method.
### 25.2 Two-factor studies

Consider the two-way model

\[ Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}, \]

for \( i = 1, \ldots, I, \ j = 1, \ldots, J, \) and \( k = 1, \ldots, n, \) where \( \alpha_i, \beta_j, \) \((\alpha\beta)_{ij}\) are iid random effects and distributed of \( N(0, \sigma^2_\alpha), \) \( N(0, \sigma^2_\beta), \) and \( N(0, \sigma^2_{\alpha\beta}) \).

This model may also be used as a mixed effect in which we assume \( \alpha_i \) are fixed subject to \( \sum_{i=1}^n \alpha_i = 0, \) \( \beta_j \) and \((\alpha\beta)_{ij}\) are iid \( N(0, \sigma^2_\beta) \) and \( N(0, \sigma^2_{\alpha\beta}) \), respectively. A nested effect model may also be proposed.

Then, we have the following table

<table>
<thead>
<tr>
<th>Mean Square</th>
<th>df</th>
<th>Expected value as</th>
<th>Fixed Effect</th>
<th>Random Effect</th>
<th>Mixed Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSA</td>
<td>( I - 1 )</td>
<td>( \sigma^2 + \frac{nJ}{I-1} \sum_{i=1}^I \alpha_i^2 )</td>
<td>( \sigma^2 + nJ\sigma^2_\alpha + n\sigma^2_{\alpha\beta} )</td>
<td>( \sigma^2 + \frac{nJ}{I-1} \sum_{i=1}^I \alpha_i^2 + n\sigma^2_{\alpha\beta} )</td>
<td></td>
</tr>
<tr>
<td>MSB</td>
<td>( J - 1 )</td>
<td>( \sigma^2 + \frac{nI}{J-1} \sum_{j=1}^J \beta_j^2 )</td>
<td>( \sigma^2 + nI\sigma^2_\beta + n\sigma^2_{\alpha\beta} )</td>
<td>( \sigma^2 + nI\sigma^2_{\alpha\beta} )</td>
<td></td>
</tr>
<tr>
<td>MSAB</td>
<td>( (I-1)(J-1) )</td>
<td>( \sigma^2 + \frac{n}{(I-1)(J-1)} \sum_{i=1}^I \sum_{j=1}^J (\alpha\beta)_{ij}^2 )</td>
<td>( \sigma^2 + n\sigma^2_{\alpha\beta} )</td>
<td>( \sigma^2 + n\sigma^2_{\alpha\beta} )</td>
<td></td>
</tr>
<tr>
<td>MSE</td>
<td>( (n-1)IJ )</td>
<td>( \sigma^2 )</td>
<td>( \sigma^2 )</td>
<td>( \sigma^2 )</td>
<td></td>
</tr>
</tbody>
</table>

Therefore, we can use the moment method to estimate those parameters.