1. Problem 5.1.

(a) From the table, we have

\[ P(X = 1, Y = 1) = 0.20. \]

(b) There are four possible cases for such event: \( X = 0 \) and \( Y = 0 \), \( X = 1 \) and \( Y = 0 \), \( X = 0 \) and \( Y = 1 \), and \( X = 1 \) and \( Y = 1 \). Thus

\[ P(X \leq 1, Y \leq 1) = P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 1) \]
\[ = 0.10 + 0.04 + 0.08 + 0.20 \]
\[ = 0.42. \]

(c) At least of hose is used at the both islands. The event has four cases: \( X = 1 \) and \( Y = 1 \), \( X = 2 \) and \( Y = 1 \), \( X = 1 \) and \( Y = 2 \), and \( X = 2 \) and \( Y = 2 \).

\[ P(x \neq 0, y \neq 0) = 0.20 + 0.06 + 0.14 + 0.30 = 0.70. \]

(d) The marginal pmf of \( X \) is

\[ P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2) \]
\[ = 0.10 + 0.04 + 0.02 \]
\[ = 0.16; \]

\[ P(X = 1) = P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2) \]
\[ = 0.08 + 0.20 + 0.06 \]
\[ = 0.34; \]

\[ P(X = 2) = P(X = 2, Y = 0) + P(X = 2, Y = 1) + P(X = 2, Y = 2) \]
\[ = 0.06 + 0.14 + 0.30 \]
\[ = 0.50. \]

The marginal pmf of \( Y \) is

\[ P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0) \]
\[ = 0.10 + 0.08 + 0.06 \]
\[ = 0.24; \]

\[ P(Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) + P(X = 2, Y = 1) \]
\[ = 0.04 + 0.20 + 0.14 \]
\[ = 0.38; \]

\[ P(Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) + P(X = 2, Y = 2) \]
\[ = 0.02 + 0.06 + 0.30 \]
\[ = 0.38. \]

Then,

\[ P(X \leq 1) = P(X = 0) + P(X = 1) = 0.16 + 0.34 = 0.50. \]
Table 1: Joint Mass Function of $X$ and $Y$ for Problem 5.2

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.30</td>
<td>0.05</td>
<td>0.025</td>
<td>0.025</td>
<td>0.10</td>
</tr>
<tr>
<td>1</td>
<td>0.18</td>
<td>0.03</td>
<td>0.015</td>
<td>0.015</td>
<td>0.06</td>
</tr>
<tr>
<td>2</td>
<td>0.12</td>
<td>0.02</td>
<td>0.010</td>
<td>0.010</td>
<td>0.04</td>
</tr>
</tbody>
</table>

(e) Look at

$$P(X = 0, Y = 0) = 0.1 \neq P(X = 0)P(Y = 0) = 0.16 \times 0.24 = 0.0384.$$  

Thus, $X$ and $Y$ are not independent. You may select other points to show that the equality is not true.

2. Problem 5.2.

(a) The possible values of $X$ and $Y$ are 0, 1, 2 and 0, 1, 2, 3, 4 respectively. Because $X$ and $Y$ are independent,

$$P(X = i, Y = j) = P(X = i)P(Y = j)$$

for $i = 0, 1, 2$ and $j = 0, 1, 2, 3, 4$. Thus, we have Table 1.

(b) By joint pmf, we have

$$P(X \leq 1, Y \leq 1) = p(0, 0) + p(0, 1) + p(1, 0) + p(1, 1) = 0.30 + 0.05 + 0.18 + 0.02 = 0.56;$$

by marginal pmf, we have

$$P(X \leq 1, Y \leq 1) = P(X \leq 1)P(Y \leq 1) = (0.5 + 0.3)(0.6 + 0.1) = 0.56.$$  

It shows that the two methods give the same result.

(c) $P(X + Y = 0) = P(X = 0, Y = 0) = 0.30.$

(d) $P(X + Y \leq 1)$

$$= P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 0, Y = 1)$$

$$= 0.30 + 0.18 + 0.05$$

$$= 0.53.$$  

3. Problem 5.3.

(a) $P(X_1 = 1, X_2 = 1) = p(1, 1) = 0.15.$

(b) $P(X_1 = X_2) = p(0, 0) + p(1, 1) + p(2, 2) + p(3, 3) = 0.08 + 0.15 + 0.10 + 0.07 = 0.40.$
(c)  \( A = \{|X_1 - X_2| \geq 2\} \)

\[
P(A) = p(0, 2) + p(0, 3) + p(1, 3) + p(2, 0) + p(3, 0) + p(3, 1) + p(4, 0) + p(4, 1) + p(4, 2)
= 0.04 + 0.00 + 0.04 + 0.05 + 0.00 + 0.03 + 0.00 + 0.01 + 0.05
= 0.22.
\]

(d)  \( P(X_1 + X_2 = 4) = p(1, 3) + p(2, 2) + p(3, 1) + p(4, 0) \)
\[
= 0.04 + 0.10 + 0.03 + 0.00
= 0.17.
\]

and

\[
P(X_1 + X_2 \geq 4) = P(X_1 + X_2 = 4) + p(2, 3) + p(3, 2) + p(3, 3) + p(4, 1) + p(4, 2) + p(4, 3)
= 0.17 + 0.06 + 0.04 + 0.07 + 0.01 + 0.05 + 0.06
= 0.46.
\]

4. Problem 5.4.

(a) The marginal PMF of \( X_1 \) is

\[
\begin{array}{cccc}
x_1 & 0 & 1 & 2 & 3 & 4 \\
p_{X_1}(x_1) & 0.19 & 0.30 & 0.25 & 0.14 & 0.12 \\
\end{array}
\]

\[
E(X_1) = (0)(0.19) + (1)(0.30) + (2)(0.25) + (3)(0.14) + (4)(0.12) = 1.7.
\]

(b) The marginal PMF of \( X_2 \) is

\[
\begin{array}{ccc}
x_2 & 0 & 1 & 2 & 3 \\
p_{X_2}(x_2) & 0.19 & 0.30 & 0.28 & 0.23 \\
\end{array}
\]

(c) Since

\[
P(X_1 = 4)P(X_2 = 0) = (0.19)(0.12) = 0.0228 \neq P(X_1 = 4, X_2 = 0) = 0.08,
\]

\( X_1 \) and \( X_2 \) are not independent.

5. Problem 5.6.

(a) It is clear that \( Y \) follows a Binomial. The first parameter \( n \) is \( X \) because it is the total number of choice. Clearly \( p = 0.60 \). Thus, we have

\[
P(Y = 2|X = 4) = \binom{4}{2}(0.6^2)(0.4^2) = 0.3456.
\]

Thus, we have

\[
P(X = 4, Y = 2) = P(Y = 2|X = 4)P(X = 4) = 0.3456 \times 0.15 = 0.05184.
\]
(b) The possible $X$ values are 0, 1, 2, 3, 4. Given $X = x$, we have
\[ P(Y = X|X = x) = 0.6^x. \]

Thus,
\[
P(Y = X) = \sum_{x=0}^{4} 0.6^x p_x(x) = 0.1 + 0.2(0.6) + 0.3(0.6)^2 + 0.25(0.6)^3 + 0.15(0.6)^4 = 0.4014.
\]

(c) When $X = 0$, $Y$ can only be 0. Thus,
\[ P(X = 0, Y = 0) = P(Y = 0|X = 0)P(X = 0) = (1)(0.1) = 0.1. \]

When $X = 1$, $Y$ can be either 0 and 1. Thus,
\[
P(X = 1, Y = 0) = P(Y = 0|X = 1)P(X = 1) = (0.4)(0.2) = 0.08;
P(X = 1, Y = 1) = P(Y = 1|X = 1)P(X = 1) = (0.6)(0.2) = 0.12.
\]

When $X = 2$, $Y$ can be 0, 1 and 2. Thus,
\[
P(X = 2, Y = 0) = P(Y = 0|X = 2)P(X = 2) = (0.4)^2(0.3) = 0.048;
P(X = 2, Y = 1) = P(Y = 1|X = 2)P(X = 2) = (2)(0.6)(0.4)(0.3) = 0.144;
P(X = 2, Y = 2) = P(Y = 2|X = 2)P(X = 2) = (0.6)^2(0.3) = 0.108.
\]

When $X = 3$, $Y$ can be 0, 1, 2 and 3. Thus,
\[
P(X = 3, Y = 0) = P(Y = 0|X = 3)P(X = 3) = (0.4)^3(0.25) = 0.016;
P(X = 3, Y = 1) = P(Y = 1|X = 3)P(X = 3) = (3)(0.6)(0.4)^2(0.25) = 0.072;
P(X = 3, Y = 2) = P(Y = 2|X = 3)P(X = 3) = (3)(0.6)^2(0.4)(0.25) = 0.108;
P(X = 3, Y = 3) = P(Y = 3|X = 3)P(X = 3) = (0.6)^3(0.25) = 0.054.
\]

When $X = 4$, $Y$ can be 0, 1, 2, 3, 4. Thus,
\[
P(X = 4, Y = 0) = P(Y = 0|X = 4)P(X = 4) = (0.4)^4(0.15) = 0.00384;
P(X = 4, Y = 1) = P(Y = 1|X = 4)P(X = 4) = (4)(0.6)(0.4)^3(0.15) = 0.02304;
P(X = 4, Y = 2) = P(Y = 2|X = 4)P(X = 4) = (6)(0.6)^2(0.4)^2(0.15) = 0.05184;
P(X = 4, Y = 3) = P(Y = 3|X = 4)P(X = 4) = (4)(0.6)^3(0.4)(0.15) = 0.05184;
P(X = 4, Y = 4) = P(Y = 4|X = 4)P(X = 4) = (0.6)^4(0.15) = 0.01944.
\]

Thus, we have the table 2 Therefore, we have
\[
p_y(0) = 0.1 + 0.08 + 0.048 + 0.016 + 0.00384 = 0.24784;
p_y(1) = 0.12 + 0.144 + 0.072 + 0.02304 = 0.35904;
p_y(2) = 0.108 + 0.108 + 0.05184 = 0.26784;
p_y(3) = 0.054 + 0.05184 = 0.10584;
p_y(4) = 0.01944.
\]
Table 2: Joint Mass Function of $X$ and $Y$ for Problem 5.6

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>0.08</td>
</tr>
<tr>
<td>2</td>
<td>0.048</td>
</tr>
<tr>
<td>3</td>
<td>0.016</td>
</tr>
<tr>
<td>4</td>
<td>0.00384</td>
</tr>
</tbody>
</table>

6. Problem 5.22.

(a) 

$$E(X + Y) = (0 + 0)(0.02) + (0 + 5)(0.06) + (0 + 10)(0.02) + \cdots + (10 + 15)(0.01) = 14.01.$$

(b) 

$$E(\max(X, Y)) = (\max(0, 0))(0.02) + (\max(0, 5))(0.06) + \cdots + (\max(10, 15))(0.01) = 9.60.$$


$$E(3X + 10Y) = (3(0) + 10(0))(0.025) + (3(0) + 10(1))(0.015) + \cdots + (3(5) + 10(2)) = 15.4.$$  

8. Problem 5.52. Here, $X \sim N(10, 1), n = 4$. We want to find an $k$, such that

$$P\left(\sum_{i=1}^{4} X_i > K\right) = 1 - 0.05.$$

Let $Y = \sum_{i=1}^{4} X_i$. Then $E(Y) = 40$ and $V(Y) = 4$. Thus, $Y \sim N(40, 4)$. Therefore,

$$P\left(\sum_{i=1}^{4} X_i > K\right) = 1 - F(K) = 1 - \Phi\left(\frac{K - 40}{\sqrt{4}}\right) = 0.05$$

$$\Rightarrow \frac{K - 40}{\sqrt{4}} = 1.645 \Rightarrow K = 43.29.$$  


(a) Here $E(X) = V(X) = \lambda = 50$. Thus,

$$X \sim \text{approx } N(50, 50).$$

Thus,

$$P(35 \leq X \leq 70) = P(34.5 \leq X \leq 70.5)$$

$$= P(34.5 \leq N(50, 50) \leq 70.5)$$

$$= \Phi\left(\frac{70.5 - 50}{\sqrt{50}}\right) - \Phi\left(\frac{34.5 - 50}{\sqrt{50}}\right)$$

$$= \Phi(2.90) - \Phi(-2.19) = 0.9839.$$
(b) Here \( E(X) = V(X) = \lambda = 50 \), and \( n = 5 \). Thus,
\[
\bar{X} \sim \text{approx} \ N(50, \frac{50}{5}).
\]
Thus,
\[
\begin{align*}
P(225 \leq \sum_{i=1}^{5} X \leq 275) &= P(224.5 \leq \sum_{i=1}^{5} X \leq 275.5) \\
&= P(44.9 \leq \bar{X} \leq 55.1) \\
&= P(44.9 \leq N(50, \frac{50}{5}) \leq 55.1) \\
&= \Phi(\frac{275.5 - 50}{\sqrt{50/5}}) - P(\frac{224.5 - 50}{\sqrt{50/5}}) \\
&= \Phi(1.61) - \Phi(-1.61) \\
&= 0.8926.
\end{align*}
\]


(a) Let \( Y = X_1 + X_2 + X_3 \). Then
\[
E(Y) = E(X_1) + E(X_2) + E(X_3) = 3 \times 60 = 180
\]
and since they are independent.
\[
V(Y) = V(X_1) + V(X_2) + V(X_3) = 3 \times 15 = 45.
\]
Thus, \( Y \sim N(180, 45) \) and
\[
P(150 \leq Y \leq 200) = \Phi\left(\frac{200 - 180}{\sqrt{45}}\right) - \Phi\left(\frac{150 - 180}{\sqrt{45}}\right) = \Phi(2.98) - \Phi(-4.47) = 0.9986.
\]

(b) \[
P(\bar{X} \geq 55) = 1 - \Phi\left(\frac{55 - \mu}{\sqrt{\sigma^2/3}}\right) = 1 - \Phi\left(\frac{55 - 60}{\sqrt{5}}\right) = 0.9873
\]
and
\[
P(58 \leq \bar{X} \leq 62) = \Phi\left(\frac{62 - 60}{\sqrt{5}}\right) - \Phi\left(\frac{58 - 60}{\sqrt{5}}\right) = 0.6289.
\]

(c) Let \( Y = X_1 - 0.5X_2 - 0.5X_3 \). Then, \( E(Y) = 0 \) and
\[
V(Y) = V(X_1) + 0.5^2 V(X_2) + 0.5^2 V(X_3) = 22.5.
\]
Thus, \( Y \sim N(0, 22.5) \) and
\[
P(-10 \leq Y \leq 5) = \Phi\left(\frac{5}{\sqrt{22.5}}\right) - \Phi\left(\frac{-10}{\sqrt{22.5}}\right) = 0.8357.
\]

(d) Let \( Y_1 = X_1 + X_2 + X_3 \). Then, \( E(Y_1) = 40 + 50 + 60 = 150 \) and
\[
V(Y_1) = 10 + 12 + 14 = 36.
\]
Thus \( Y_1 \sim N(150, 36) \) and
\[
P(Y_1 \leq 160) = \Phi\left(\frac{160 - 150}{\sqrt{36}}\right) = 0.9522.
\]
Let $Y_2 = X_1 + X_2 - X_3$. Then $E(Y) = 40 + 50 - 2(60) - 30$ and

$$V(Y_2) = V(X_1) + V(X_2) + 4V(X_3) = 10 + 12 + 4(14) = 78.$$ 

Thus, $Y_2 \sim N(-30, 78)$ and

$$P(X_1 + X_2 \geq 2X_3) = P(Y_2 \leq 0) = \Phi\left(\frac{-30}{\sqrt{78}}\right) = \Phi(-3.4) = 0.0003.$$ 

11. Problem 5.60. $E(Y) = 20 - 21 = -1$ and

$$V(X) = \frac{1}{4}(V(X_1) + V(X_2)) + \frac{1}{9}(V(X_3) + V(X_4) + V(X_5)) = \frac{4}{2} + \frac{3.5}{3} = 3.1667.$$ 

So $Y \sim N(-1, 3.1667)$,

$$P(Y \geq 0) = 1 - \Phi\left(\frac{1}{\sqrt{3.1667}}\right) = 0.2870,$$ 

and

$$P(-1 \leq Y \leq 1) = \Phi\left(\frac{1 - (-1)}{\sqrt{3.1667}}\right) - \Phi\left(\frac{-1 - (-1)}{\sqrt{3.1667}}\right) = 0.3695.$$