There are totally 37 points in the exam. The students with score higher than or equal to 35 points will receive 35 points. Please write down your name, student ID number below.

NAME: 
ID: 

This is the distribution of the exam:
16.5 17.5 21.0 26.0 26.5 26.5 27.5 28.0 28.5 29.0 30.0 30.0 30.5
31.0 31.5 32.0 32.0 32.5 33.0 33.0 34.0 34.0 34.0 34.5 34.5 35.0
35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0

This is the summary of the exam:
Min. 1st Qu.  Median  Mean  3rd Qu.  Max.
16.50   29.25   33.00   31.29   35.00   35.00
(19 points, 1 point each). Fill in the blanks. Your answer should be real numbers.

(a) (4 points). Suppose a bag has 8 red balls and 12 blue balls. Random choose 4 balls. The probability of the event that all of them are red is \[ \frac{8}{20} = 0.4 \]. The probability of the event that all of them are blue is \[ \frac{12}{20} = 0.6 \]. The probability of the event that at least two blue balls appear is \[ \frac{12}{20} \times \binom{12}{2} = 0.8469 \]. Given three of them are blue, the conditional probability of the event that all of them are blue is \[ \frac{\binom{12}{3} \times \binom{8}{1}}{\binom{20}{4}} = 0.2195 \].

(b) (4 points). Suppose a box has 3 bags. The first bag has 4 yellow balls and 3 blue balls. The second bag has 5 red balls and 3 yellow balls. The third bag has 2 red balls, 3 yellow balls, and 6 blue balls. Random choose one bag and then choose one ball from the bag. If the selected bag is the third bag, the probability of the event that the ball is yellow is \[ \frac{3}{6} = 0.5 \]. The probability of the event that the ball is yellow is \[ \frac{3}{24} = 0.125 \]. Given the ball is yellow, the probability of the event that the selected bag is the first bag is \[ \frac{4}{24} = 0.1667 \]. Given the ball is red, the probability of the event that the selected bag is the first bag is \[ \frac{3}{24} = 0.125 \].

(c) (4 points). Assume events \( A, B, C, \) and \( D \) occur independently with \( P(A) = 0.3, P(B) = 0.25, P(C) = 0.7, \) and \( P(D) = 0.4 \). Then, \( P(\bar{A} \cap B \cap \bar{C} \cap D) = P(A) \times P(B) \times P(C) \times P(D) = 0.027 \). \( P(A \cup B \cup C) = 1 - P(A^c \cap B^c \cap C^c) = 1 - P(A^c) \times P(B^c) \times P(C^c) = 0.8425 \). \( P(A \cup C | B \cap D) = \frac{P(A \cup C \cap B \cap D)}{P(B \cap D)} = \frac{P(A) \times P(C) \times P(B) \times P(D)}{P(B) \times P(D)} = 0.8425 \). \( P(B | B \cup D) = \frac{P(B \cap (B \cup D))}{P(B \cup D)} = \frac{P(B)}{P(B) + P(D) - P(B)} = 0.7 \).

(d) (4 points). Flip a balanced coin 20 times. Let \( X \) be the total number of heads. Then, \( X \sim \text{Bin}(20, 0.5) \). \( P(7 \leq X \leq 13) = 0.8847 \), \( E(X) = 10 \), and \( V(X) = 5 \).

(e) (3 points). Suppose \( X \sim N(1.25, 2.5) \). Let \( Y = -2X + 5 \). Then, \( E(Y) = -2 \times 1.25 + 5 = 1.5 \), \( V(Y) = 2.5 \times 2.5 = 6.25 \), and \( Y \sim N(1.5, 6.25) \).

Answer: (a) \( \binom{8}{4} = 70, \binom{12}{4} = 330 \), \( \binom{12}{3} \times \binom{8}{1} = 26880 \), and \( \binom{12}{3} \times \binom{8}{1} = 0.336 \). (b) 0.2727, 0.4064, 0.4687, and 0. (c) 0.0315, 0.8425, 0.79, and 0.4545 (d) \( \text{Bin}(20, 0.5) \), 0.8847, 10, and 5 (e) 2.5, 10, and \( N(2.5, 10) \).
2. (8 points). Suppose the joint PMF of $X$ and $Y$ is

\[
\begin{array}{cccc}
 x & 1 & 2 & 3 \\
 1 & 0.05 & 0.1 & 0.1 \\
 2 & 0.2 & 0.15 & 0.1 \\
 3 & 0.1 & 0.1 & 0.05 \\
\end{array}
\]

(a) (2 points). Compute the marginal PMFs of $X$ and $Y$, respectively.

*Solution:* The marginal PMF of $X$ is

\[
\begin{array}{cccc}
 x & 1 & 2 & 3 \\
 p_X(x) & 0.25 & 0.5 & 0.25 \\
\end{array}
\]

and the marginal PMF of $Y$ is

\[
\begin{array}{cccc}
 y & 1 & 2 & 3 \\
 p_Y(y) & 0.35 & 0.35 & 0.3 \\
\end{array}
\]

(b) (2 points). Compute the CDF of $X$.

*Solution:* The CDF of $X$ is

\[
F(x) = \begin{cases} 
0 & \text{if } x < 1 \\
0.25 & \text{if } 1 \leq x < 2 \\
0.75 & \text{if } 2 \leq x < 3 \\
1 & \text{if } x \geq 3 
\end{cases}
\]

(c) (2 points). Compute Cov($X,Y$) and Corr($X,Y$).

*Solution:* Based on the marginal PMFs, we have $E(X) = 2$, $V(X) = 0.5$, $E(Y) = 1.95$ and $V(Y) = 0.6475$. Based on the joint PMF, we have $E(XY) = 3.8$. Thus, Thus,

\[
\text{Cov}(X,Y) = 3.8 - 2 \times 1.95 = -0.1
\]

and

\[
\text{Corr}(X,Y) = -0.1 \frac{\sqrt{0.5 \times 0.6475}} = -0.1757.
\]

(d) (2 points) Compute the conditional PMF of $Y$ given $X = 1$.

*Solution:* The conditional PMF of $Y$ given $X = 1$ is

\[
\begin{array}{cccc}
 y & 1 & 2 & 3 \\
 p_{Y|X=1}(y) & 0.2 & 0.4 & 0.4 \\
\end{array}
\]
3. (4 points). Let $X_1, \cdots, X_n$ be identically independently distributed (iid) random variables with common expected value $\mu$ and variance $\sigma^2$. Denote $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $T = \sum_{i=1}^{n} X_i$. Use the Central Limit Theorem (CLT) to compute the following probabilities.

(a) (2 points). Flip a die 2000 times. Let $T$ be the total of the numbers. Compute $P(6920 \leq T \leq 7080)$.

Solution: Flip a die once, we have $\mu = 7$ and $\sigma^2 = 2.9167$. Then, 

$$\bar{X} \approx N\left(3.5, \frac{2.9167}{2000}\right) = N(3.5, 0.001458).$$

Therefore 

$$P(6920 \leq T \leq 7080) = P(3.46 \leq \bar{X} \leq 3.54) \approx P(3.46 \leq N(3.5, 0.001458) \leq 3.54) = \Phi\left(\frac{3.54 - 3.5}{\sqrt{0.001458}}\right) - \Phi\left(\frac{3.46 - 3.5}{\sqrt{0.001458}}\right) = \Phi(1.05) - \Phi(-1.05) = 0.7063.$$ 

(b) (2 points). Assume $X_i$ is a continuous random variable with $\mu = 100$ and $\sigma = 20$. Compute $P(|\bar{X} - 100| \geq 1)$ for $n = 100$, $n = 1000$, and $n = 10000$, respectively.

Solution: Since $\bar{X} \approx N(100, 400/n)$, there is 

$$P(|\bar{X} - 100| \geq 1) = 1 - P(99 \leq \bar{X} \leq 101) \approx 1 - \left[ P(99 \leq N(100, \frac{400}{n}) \leq 101) \right]$$

$$= 1 - \left[ \Phi\left(\frac{101 - 100}{\sqrt{400/n}}\right) - \Phi\left(\frac{99 - 100}{\sqrt{400/n}}\right) \right]$$

$$= 1 - \left[ \Phi(0.05\sqrt{n}) - \Phi(-0.05\sqrt{n}) \right]$$

$$= \begin{cases} 
0.6171 & \text{when } n = 100 \\
0.1138 & \text{when } n = 1000 \\
0 & \text{when } n = 10000 
\end{cases}$$
4. (6 points). Compute the following probabilities by using properties of the linear combination of normal distribution. Suppose $X_1, X_2 \sim N(0.5, 4)$ and $X_3, X_4 \sim N(1.3, 8)$, independently.

(a) (2 points). Compute $P(2X_1 + 2X_2 - X_3 - X_4 \leq 10)$.

**Solution:** Let $Y = 2X_1 + 2X_2 - X_3 - X_4$. Then, $E(Y) = -0.6$ and $V(Y) = 48$. Thus $Y \sim N(-0.6, 48)$ and

$$P(2X_1 + 2X_2 - X_3 - X_4 \leq 10) = P(N(-0.6, 48) \leq 10)$$

$$= \Phi\left(\frac{10 + 0.6}{\sqrt{48}}\right)$$

$$= \Phi(1.53)$$

$$= 0.9370$$

(b) (2 points). Compute $P(|1.5X_1 - 1.5X_2 + 1.2X_3 - 1.2X_4| \leq 8)$.

**Solution:** Let $Y = 1.4X_1 - 1.5X_2 + 1.2X_3 - 1.2X_4$. Then, $E(Y) = 0$ and $V(Y) = 41.04$. Thus, $Y \sim N(0, 41.04)$ and

$$P(|1.4X_1 - 1.5X_2 + 1.2X_3 - 1.2X_4| \leq 8) = P(-8 \leq N(0, 41.04) \leq 8)$$

$$= \Phi\left(\frac{8}{\sqrt{41.04}}\right) - \Phi\left(-\frac{8}{\sqrt{41.04}}\right)$$

$$= \Phi(1.25) - \Phi(-1.25)$$

$$= 0.7888$$

(c) (2 points). Compute $P(X_1 + X_2 \leq X_3 + X_4)$.

**Solution:** Let $Y = X_1 + X_2 - X_3 - X_4$. Then, $E(Y) = -1.6$ and $V(Y) = 24$. Thus, $Y \sim N(-1.6, 24)$ and

$$P(X_1 + X_2 \leq X_3 + X_4) = P(Y \leq 0)$$

$$= P(N(-1.6, 24) \leq 0)$$

$$= \Phi\left(\frac{1.6}{\sqrt{24}}\right)$$

$$= \Phi(0.33)$$

$$= 0.6293$$