Section 8.1: Hypotheses and Testing Procedure
Concepts and Formulae:

- Suppose there is a claim or a statement.
- The null hypothesis is an answer in which the claim or the statement is true, and denoted by $H_0$.
- The alternative hypothesis is an answer in which the claim or the statement is NOT true, and denoted by $H_a$ ($H_A$ or $H_1$).
- If we conclude $H_0$, we say “we accept $H_0$ or fail to reject $H_0$”; if we conclude $H_a$, we say “we reject $H_0$ and conclude $H_a$.”
A testing procedure is specified by the following steps:

- **Choose a test statistic.** A test statistic is a function of data which makes the decision about the rejection or acceptance of $H_0$.

- **A rejection range is a set in which we reject $H_0$** if the test statistic is in the set and accept $H_0$ if the test statistic is outside of the set.

- **Let $T$ be the test statistic and $R$ be the rejection range.** Then, we conclude

\[
\begin{cases} 
H_0 & \text{if } T \notin R \\
H_a & \text{if } T \in R
\end{cases}
\]
In general, a test can be summarized into the following table.

<table>
<thead>
<tr>
<th>Conclude</th>
<th>Truth</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H_0$</td>
<td>$H_a$</td>
</tr>
<tr>
<td>$H_0$</td>
<td>Correct</td>
<td>Type II Error</td>
</tr>
<tr>
<td>$H_a$</td>
<td>Type I Error</td>
<td>Correct</td>
</tr>
</tbody>
</table>

- Type I error rate (probability) is defined by
  \[ P(\text{Conclude } H_a|\text{Truth is } H_0) \]
  and type II error rate (probability) is defined by
  \[ P(\text{Conclude } H_0|\text{Truth is } H_a) \].

- We want to make both probabilities small. However, this is usually impossible in real applications. Thus classically, we only control type I error probabilities.

- The significance level denoted by $\alpha$ is the maximum of type I error probabilities. Therefore, if we choose $\alpha = 0.05$, we guarantee the type I error probability is less than or equal to 0.05.
First example of Section 8.1: Examples 8.1 and 8.3 on textbook. Suppose old rate of “no visible damage” is 25%. An experiment took 20 samples and want to know whether the rate increases with a new method.

- Null hypothesis is

\[ H_0 : \text{rate does not increase} \]

versus the alternative

\[ H_a : \text{rate increases.} \]

- Under \( H_0 \), the count \( X \) of visible damage follows \( \text{Bin}(20, 0.25) \). Under \( H_a \), \( X \sim \text{Bin}(20, p) \) with \( p > 0.25 \). Then, we can write

\[ H_0 : p = 0.25 \leftrightarrow H_a : p > 0.25. \]

- Assume the rejection range is

\[ R = \{8, 9, 10, 11, \cdots, 20\}. \]
• Then, the type I error probability is
\[ P(X \geq 8 | p = 0.25) = 1 - B(7; 20, 0.25) = 0.102. \]

• The significance level \( \alpha = 0.102. \)

• The type II error probability is
\[ P(X < 8 | p > 0.25) = B(7; 20, p) \]
which is a function of \( p. \)

• Suppose we change the testing problem as
\[ H_0 : p \leq 0.25 \leftrightarrow H_a : p > 0.25. \]
Then, the type I error probability is
\[ 1 - B(7; 20, p) \]
for \( p \leq 0.25 \) and the type II error probability is
\[ B(7; 20, p) \]
for \( p > 0.25. \) In this case, we still have the significance level \( \alpha = 0.102. \)
Graph of Type I and Type II error probabilities

The significance level $\alpha$ is the maximum of type I error probabilities, which gives

$$\alpha = 0.102.$$
Second example of Section 8.1: Examples 8.2 and 8.4 on textbook. Assume we choose 25 samples from $N(\mu, 81)$ and suppose we test

$$H_0 : \mu = 75 \leftrightarrow H_a : \mu < 75.$$ 

Suppose we use $R = \bar{X} < 70.8$, i.e., we conclude

$$\begin{cases} H_0 & \text{if } \bar{X} \geq 70.8 \\ H_a & \text{if } \bar{X} < 70.8 \end{cases}$$

Then, the type I error is

$$P(\bar{X} \leq 70.8 \text{ if } \mu = 75) = \Phi\left(\frac{70.8 - 75}{\sqrt{81/25}}\right) = \Phi(-2.33) = 0.01.$$ 

Then, the significance level is $\alpha = 0.01$ and type II error probability is

$$\beta(\mu) = P(\bar{X} \geq 70.8) = 1 - \Phi\left(\frac{70.8 - \mu}{1.8}\right).$$

for $\mu < 75$, which is a function of $\mu$ for $\mu < 75$. For example, we have

$$\beta(72) = 1 - \Phi(-0.67) = 0.7486.$$
The significance level $\alpha$ is the maximum of type I error probabilities, which gives

$$\alpha = 0.01.$$