Section 3.4: The Binomial Probability Distribution
A binomial experiment is derived if

- the experiment consists of $n$ trials (small experiments),
- each trial results two possible outcomes, say success (S) or failure (F),
- trials are independent
- the probability of success is the same for all trials

In summary, a binomial experiment is just the independent $n$-replication of a simple success/failure experiment.
• Let \(X\) be the total number of successes among the \(n\) trials of a binomial experiment.

• Then, we call \(X\) follows a binomial distribution, and we write

\[
X \sim Bin(n, p),
\]

where \(p\) is the probability of success and \(1 - p\) is the probability of failure.

• The PMF of \(X\) is

\[
p(x) = P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}
\]

for \(x = 0, 1, \ldots, n\).

• The PMF can be derived by the modified product rule:

  – Step 1: Among \(n\) positions, choose \(x\) positions for successes: \(\binom{n}{x}\) ways;

  – Step 2: Given positions, the probability is computed by independence: probability is \(p^x(1 - p)^{n-x}\).

  – Then, the final result is their product:

\[
\binom{n}{x} p^x (1 - p)^{n-x}.
\]
• In general, we write the PMF of $Bin(n, p)$ distribution as

$$b(x; n, p)$$

and the CDF as

$$B(x; n, p).$$

Then, we have

$$B(x; n, p) = \sum_{k=0}^{x} b(k; n, p).$$

• In addition, we have

$$E(X) = np$$

and

$$V(X) = np(1 - p).$$
First example of Section 3.4: Example 3.27 on textbook.

• A coin is tossed $n$ times.
• Let $X$ be the total number of heads.
• This is a binomial experiment and the PMF of $X$ is

$$p(x) = \binom{n}{x} \left( \frac{1}{2} \right)^n.$$ 

• Since

$$\sum_{x=0}^{n} \binom{n}{x} \left( \frac{1}{2} \right)^n = 1,$$

we have

$$\sum_{x=0}^{n} \binom{n}{x} = 2^n.$$
Second example of Section 3.4: Example 3.30 on textbook.

- 500000 drivers, 400000 are insured.

- Randomly choose 10 of them. Let $X$ be the total number of insured drivers.

- Then, the exactly PMF of $X$ is

  $$p(x) = \binom{400000}{x} \binom{100000}{n-x} \binom{500000}{10}.$$  

- It can be approximated by a binomial distribution as

  $$p(x) \approx \binom{10}{x} 0.8^x 0.2^{10-x}.$$
Third example of Section 3.4: Example 3.31 on textbook. Each of six randomly selected cola drinkers is given a glass containing cola $S$ or containing cola $F$. The glasses are identical. Let $X$ be the number of those who prefer $S$. Suppose the preference of $S$ is 0.5.

- Then, $X \sim \text{Bin}(6, 0.5)$.

- Then, the probability of exactly three is
  \[
P(X = 3) = b(3; 6, 0.5) = \binom{6}{3} 0.5^3 0.5^3 = 0.31.
  \]

- The probability of at least three is
  \[
P(X \geq 3) = \sum_{x=3}^{6} \binom{6}{x} 0.5^6 = 0.656.
  \]

- The probability of at most one is
  \[
P(X \leq 1) = \sum_{x=0}^{1} \binom{6}{x} 0.5^6 = 0.109.
  \]
Fourth example of Section 3.4: Example 3.32 on textbook. Suppose

\[ X \sim Bin(15, 0.2). \]

Use the Binomial table (Table A.1 on page 736(v6) or 664(v7)) to compute the following probabilities.

- The probability of at most 8 is
  \[ P(X \leq 8) = B(8; 15, 0.2) = 0.999. \]

- The probability of exactly 8 is
  \[ P(X = 8) = B(8; 15, 0.2) - B(7; 15, 0.2) = 0.999 - 0.996 = 0.003. \]

- The probability of at least 8 is
  \[ P(X \geq 8) = 1 - B(7; 15, 0.2) = 1 - 0.996 = 0.004. \]

- The probability of between 4 and 7 is
  \[ P(4 \leq X \leq 7) = B(7; 15, 0.2) - B(3; 15, 0.2) = 0.996 - 0.648 = 0.348. \]
Fifth example of Section 3.4: Example 3.34 on textbook. Suppose $X \sim Bin(10, 0.75)$. Then, we have

$$E(X) = 10 \times 0.75 = 7.5$$

and

$$V(X) = 10 \times 0.75 \times (1 - 0.75) = 1.875.$$
Sixth example of Section 3.4. Flip a die 10 times. Let \( X \) be the total number of 5 or 6. Let \( Y \) be the total number of 1.

- Then,
  \[
  X \sim Bin(10, \frac{1}{3})
  \]
  and
  \[
  Y \sim Bin(10, \frac{1}{6})
  \]

- Therefore, we have
  \[
  E(X) = 10 \times \frac{1}{3} = \frac{10}{3}
  \]
  and
  \[
  V(X) = 10 \times \frac{1}{3} \times (1 - \frac{1}{3}) = \frac{20}{9};
  \]
  
  \[
  E(Y) = 10 \times \frac{1}{6} = \frac{5}{3}
  \]
  and
  \[
  V(Y) = 10 \times \frac{1}{6} \times (1 - \frac{1}{6}) = \frac{25}{18}.
  \]