1. 6.2.12. 

**Solution:** The loglikelihood function is

\[ \ell(\sigma^2) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu_0)^2. \]

Then,

\[ \ell'(\sigma^2) = -\frac{n}{2\sigma^2} - \frac{1}{2\sigma^4} \sum_{i=1}^{n} (X_i - \mu_0)^2 \Rightarrow \hat{\sigma}^2(\mu_0) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2. \]

Comparing it with \( \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \), they are not identical.

2. 6.2.13. 

**Solution:** It cannot be the likelihood function since its value can be negative at \( \theta < 0 \).

3. 6.2.21. 

**Solution:** The loglikelihood function is

\[ \ell(\mu) = -\frac{n}{2} \log(2\pi) - \frac{1}{2} [n(\bar{X} - \mu)^2 + \sum_{i=1}^{n} (X_i - \bar{X})^2]. \]

It is maximized at \( \mu = \bar{X} \) if \( \bar{X} > 0 \) and 0 if \( \bar{X} = 0 \). Thus, the MLE is \( \hat{\mu} = \max(0, \bar{X}) \).

4. 6.2.22. 

**Solution:** For the model of \( x = (X_1, \cdots, X_n) \), the likelihood function is

\[ L(\theta|x) = h(x)g_\theta(T). \]

If the model for \( T \) is used such that \( T(x_1) = T(x_2) \), then \( L(\theta|x_1) = h(x_1)g_\theta(T) \) and \( L(\theta|x_2) = h(x_2)g_\theta(T) \), indicating that they have the same maximum. Therefore, \( \hat{\theta} \) is also the MLE for the model for \( T \).

5. 6.2.24. 

**Solution:** The PDF is

\[ f_\theta(x) = \frac{1}{\theta_2 - \theta_1} I(\theta_1 \leq x \leq \theta_2). \]

The likelihood function is

\[ \ell(\theta) = \frac{1}{(\theta_2 - \theta_1)} I(\min_{i \leq n} X_i \leq \max_{i \leq n} X_i \leq \theta_2). \]

To make \( \ell(\theta) \) large, we need to increase \( \theta_1 \) and decrease \( \theta_2 \). Therefore, we have \( \hat{\theta}_1 = \min_{i \leq n} (X_i) \) and \( \hat{\theta}_n = \max_{i \leq n} (X_i) \).
6. 6.3.1.

Solution: (Students only need to answer the confidence interval problem, but I provide answers for both). We have \( \bar{x} = 4.88 \). For \( H_0 : \mu = 5 \) against \( H_1 : \mu \neq 5 \), the value of the test statistic is

\[
Z = \left| \frac{4.88 - 5}{\sqrt{0.5/10}} \right| = 0.5366 < 1.96.
\]

Therefore, we accept \( H_0 \) and conclude \( \mu = 5 \) at 0.05 significance level. The \( P \)-value of the test is \( 2\Psi(-0.5366) = 0.5915 \). The 95\% confidence interval is

\[
4.88 \pm 1.96 \times \sqrt{0.5/10} = [4.4417, 5.3183].
\]

7. 6.3.23.

(a) Note that

\[
\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu).
\]

Using \( \bar{X} \sim N(0, \sigma^2/n) \) and \( X_i \sim N(0, \sigma^2) \), we have

\[
E[\sum_{i=1}^{n}(X_i - \bar{X})^2] = n\sigma^2 - \sigma^2 = (n - 1)\sigma^2.
\]

Thus,

\[
E(S^2) = \frac{1}{n-1} E\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sigma^2,
\]

implying that \( S^2 \) is an unbiased estimator of \( \sigma^2 \).

(b)

\[
E\left(\frac{n-1}{n}S^2\right) = \frac{n-1}{n} E(S^2) = \frac{n-1}{n} \sigma^2 \rightarrow \sigma^2
\]

as \( n \to \infty \). Therefore, the bias goes to zero as \( n \to \infty \).